UNIT: 4.

Defintion Residue at Infinity :- If $f(z)$ has an isolated singularity at $z=\infty$: or is analytic there, then the residue at $z=\infty$ is defined as,

$$Res(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz$$

$$Res(z=-\infty) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

where C is any closed contour which encloses all the finite singularities of $f(z)$. The integral is taken in positive direction (anticlockwise direction).

Remark :- The function may be regular at infinity, yet has a residue there.

Consider the function

$$f(z) = \frac{b}{(z-a)}$$

For this function :-

$$Res(z=\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz = -\frac{1}{2\pi i}\int_C \frac{b}{z-a}\,dz$$

Put $z-a = re^{i\theta}$, $dz = rie^{i\theta}d\theta$

$$= -\frac{b}{2\pi \cancel{i}}\int_0^{2\pi}\frac{\cancel{r}\cancel{i}e^{i\theta}}{\cancel{r}e^{i\theta}}d\theta = -\frac{b}{2\pi}\int_0^{2\pi}d\theta = -b$$

where C is the circle $|z-a|=r$

$\therefore Res(z=\infty) = -b$

Also $z=a$ is a simple pole of $f(z)$ and its residue there is,

$$\frac{1}{2\pi i}\int_C f(z)\,dz = b \quad \text{or}$$

$$\lim_{z\to a}(z-a)f(z) = b$$

$\therefore \operatorname{Res}(z=a) = b = -\operatorname{Res}(z=\infty)$

Residue At A Pole :-

Suppose a single valued function $f(z)$ has a pole of order m at $z=a$, then by definition of pole the principle part of Laurent's expansion of $f(z)$ contains only 'm' terms so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \text{--- (1)}$$

where $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{n+1}}$, $b_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^{-n+1}}$

c being a circle $|z-a| = r$.

Evidently, $b_1 = \frac{1}{2\pi i}\int_c f(z)\,dz$ --- (2)

The coefficient b_1 is called the residue of $f(z)$ at the pole $z=a$, and is denoted by $\operatorname{Res}(z=a) = b_1 = \lim_{z\to a} (z-a) f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$$(z-a) f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1 + \frac{b_2}{(z-a)} + \cdots + \frac{b_m}{(z-a)^{m-1}}$$

$$\Rightarrow \boxed{b_1 = \lim_{z\to a} (z-a) f(z)}$$

Imp

Cauchy's Residue Theorem :-

If $f(z)$ is analytic within and on a closed contour C, except at a finite number of poles $z_1, z_2, z_3, \ldots, z_n$ within C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{m=1}^{n} \text{Res}\,(z = z_m)$$

where R.H.S. denotes sum of residues of $f(z)$ at its poles lying within C.

Proof :- Suppose small $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the circles with centres at $z_1, z_2, \ldots, z_n$ respectively and radii so small that they lie within closed curve C and do not overlap.
$f(z)$ is analytic within the annulus bounded by these circles and the curve C.
By corollary to Cauchy's theorem,

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz \quad \text{---} (1)$$

But $\frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz$ = residue of $f(z)$ at $z = z_1$.

$$= \text{Res}\,(z = z_1)$$

$$\therefore \int_{\gamma_1} f(z)\,dz = 2\pi i\,\text{Res}\,(z = z_1)$$

Using this in (1), we get

$$\int_C f(z)\,dz = 2\pi i\,\text{Res}\,(z = z_1) + \cdots + 2\pi i\,\text{Res}\,(z = z_n)$$

$$= 2\pi i \sum_{m=1}^{n} \text{Res}\,(z = z_m)$$

Theorem :- If a function $f(z)$ is analytic except at finite number of

singularities (including that at infinity), then the sum of residues of these singularities is zero.

Proof:- Let C be a closed contour which encloses all the singularities of $f(z)$ except that at infinity. The sum ΣR of residues at all these singularities within C is given by,

$$\int_C f(z)\, dz = 2\pi i \, \Sigma R$$

(This follows from Cauchy's residue theorem)

This $\Rightarrow \frac{1}{2\pi i}\int_C f(z)\, dz = \Sigma R$

Also, $\frac{-1}{2\pi i}\int_C f(z)\, dz = \text{Res}\,(z=\infty)$

Adding these two equations, we get

$\text{Res}\,(z=\infty) + \Sigma R = 0.$

This proves the required result.

Ques:- Evaluate the residues of $\frac{z^2}{(z-1)(z-2)(z-3)}$ at 1, 2, 3 and infinity and show that their sum is zero.

Solution:-

Let $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

$\text{Res}\,(z=1) = \lim_{z\to 1} (z-1) f(z)$

$= \lim_{z\to 1} \frac{z^2}{(z-2)(z-3)} = \frac{1}{(1-2)(1-3)} =$

$$Res(z=2) = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} \frac{z^2}{(z-1)(z-3)} = \frac{2^2}{(2-1)(2-3)} = \frac{4}{(1)(-1)}$$

$$= -4$$

$Res(z=3)$

$$= \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \frac{z^2}{(z-1)(z-2)}$$

$$= \frac{3^2}{(3-1)(3-2)} = \frac{9}{(2)(1)} = \frac{9}{2}$$

$$Res(z=+\infty) = \lim_{z\to\infty} -z f(z)$$

$$= \lim_{z\to\infty} \frac{-z^3}{(z-1)(z-2)(z-3)}$$

$$= -1$$

Sum of residues $= \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0$

Ques: Evaluate the residues of $f(z)$ where

$$f(z) = \frac{e^z}{z^2(z^2+9)} \quad \text{at } z = 0, 3i, -3i.$$

Solution:

$$f(z) = \frac{e^z}{z^2(z^2+9)} = \frac{e^z}{(z-0)^2(z+3i)(z-3i)}$$

It's poles are $z = 0, 3i, -3i$

Since $z=0$ is the pole of second order, so

$$Res(z=0) = \lim_{z\to 0} (z-0)^2 f(z) = \lim_{z\to 0} \frac{1}{1!}\left(\frac{e^z}{z^2+9}\right)'$$

$$= \lim_{z\to 0} \frac{e^z.(z^2+9) - e^z(2z)}{(z^2+9)^2}$$

$$= \frac{1}{9}$$

$z = 3i$ is a simple pole,

$$\text{Res}(z = 3i) = \lim_{z \to 3i} (z - 3i) f(z)$$

$$= \lim_{z \to 3i} \frac{e^z}{z^2(z+3i)} = \frac{e^{3i}}{(3i)^2(3i+3i)}$$

$$= \frac{e^{3i}}{(-9)(6i)} = \frac{i}{54} e^{3i}$$

$z = -3i$ is a simple pole,

$\text{Res}(z = -3i)$

$$= \lim_{z \to -3i} (z+3i) f(z)$$

$$= \lim_{z \to 3i} \frac{e^z}{z^2(z-3i)} = \frac{e^{-3i}}{(-3i)^2(-6i)}$$

$$= \frac{e^{-3i}}{(-9)(-6i)} = \frac{e^{-3i}}{54i} = -\frac{i}{54} e^{-3i}$$

Evaluate by the method of calculus of residues : $\int_C \frac{dz}{(z-1)(z+1)}$ where C is circle $|z| = 3$.

:-

Let $f(z) = \frac{1}{(z-1)(z+1)}$

Poles of $f(z)$ are given by $(z-1)(z+1) = 0$

or $z = 1, -1$.

These are simple poles and lie within

$$\text{Res}(z = 1) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{1}{z+1} = \frac{1}{2}$$

$$\text{Res}(z = -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{1}{z-1}$$

$\text{Res}(z=1) + \text{Res}(z=-1) = \frac{1}{2} - \frac{1}{2} = 0$

$\int_C f(z)\,dz = 2\pi i$ (sum of residues)
$= 2\pi i(0) = 0$

Ques:- Evaluate by method of calculus of residues $\int_C \frac{dz}{(z^2+1)(z-4)}$, where C is a circle $|z| = 3$.

Solution:-

For poles $(z^2+1)(z-4) = 0$

$\Rightarrow z = i, -i, 4$

$z = 4$ lies outside C.

$\text{Res}(z=i) = \lim_{z\to i} (z-i) f(z)$

$= \lim_{z\to i} \frac{1}{(z+i)(z-4)} = \frac{1}{2i(i-4)}$

$\text{Res}(z=-i)$

$= \lim_{z\to -i} (z+i) f(z)$

$= \lim_{z\to -i} \frac{1}{(z-i)(z-4)} = \frac{1}{-2i(-i-4)}$

$= \frac{1}{2i(i+4)}$

Sum of residues $= \frac{1}{2i}\left[\frac{1}{i-4} + \frac{1}{i+4}\right]$

$= \frac{\cancel{2}i}{\cancel{2i}(i^2-16)} = \frac{1}{-1-16} = -\frac{1}{17}$

$\int_C f(z)\,dz = 2\pi i\left(-\frac{1}{17}\right) = -\frac{2\pi i}{17}.$

Ques:- Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

Solution :- Take C as unit circle $|z|=1$

so , $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \left(\frac{dz}{iz}\right) \cdot \frac{1}{a+\frac{b}{2}(e^{i\theta}+e^{-i\theta})}$

$$= \int_C \left(\frac{dz}{iz}\right) \cdot \frac{1}{a+\frac{b}{2}(z+\frac{1}{z})} = \int_C \frac{2\cancel{z}\, dz}{(i\cancel{z})\{2az+b(z^2+1)\}}$$

$$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$

$I = \frac{2}{i}\int_C f(z)\, dz$ where $f(z) = \frac{1}{bz^2+2az+b}$

For poles ; $bz^2+2az+b=0$ —— (1)

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Take $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

[$\alpha+\beta = -\frac{2a}{b}$ [from (1)] , then $\alpha\beta = \frac{b}{b} = 1$

$\alpha\beta = 1$]

Let $|\alpha| < 1$, $|\beta| > 1$

Then α is singular (Pole) point lies within C and β lies outside the C because $\alpha\beta = 1$.

Res ($z=\alpha$)

$$I = \frac{2}{i}\int_C \frac{dz}{b(z-\alpha)(z-\beta)}$$

$$Res\,(z=\alpha) = \lim_{z\to\alpha} (z-\alpha)\, f(z)$$

$$= \lim_{z\to\alpha} \frac{1}{b(z-\beta)} = \frac{1}{b(\alpha-\beta)}$$

$$\boxed{\begin{aligned} \therefore \alpha-\beta &= \frac{-\alpha+\sqrt{a^2-b^2}}{b} + \frac{\alpha+\sqrt{a^2-b^2}}{b} \\ &= \frac{2}{b}\sqrt{a^2-b^2} \end{aligned}}$$

$$Res\,(z=\alpha) = \frac{1}{\cancel{b}(\alpha-\beta)} = \frac{\cancel{b}}{2\sqrt{a^2-b^2}} \times \frac{1}{\cancel{b}} = \frac{1}{2\sqrt{a^2-b^2}}$$

$$\int_C f(z)\,dz = 2\pi i \times \frac{\cancel{b}\,1}{2\sqrt{a^2-b^2}} = \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$I = \frac{2}{i}\int_C f(z)\,dz = \frac{2}{i} \times \frac{\pi i}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

<u>Ques</u> :- Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$, $a^2<1$

<u>Solution</u> :-

Take C as unit circle $|z|=1$, and so $z = e^{i\theta}$, $dz = i e^{i\theta}\,d\theta$

$$d\theta = \frac{dz}{iz}$$

$$\text{Let } I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_C \left(\frac{dz}{iz}\right) \times \frac{1}{1+\frac{a}{2}\left(e^{i\theta}+e^{-i\theta}\right)}$$

$$= \int_C \left(\frac{dz}{iz}\right) \cdot \frac{1}{1+\frac{a}{2}\left(z+\frac{1}{z}\right)} = \int_C \frac{2\cancel{z}\,dz}{(i\cancel{z})(2z+az^2+a)}$$

$$= \frac{2}{i}\int_C \frac{dz}{az^2+2z+a}$$

$$I = \frac{2}{i}\int_C f(z)\,dz \quad , \quad \text{where } f(z) = \frac{1}{az^2+2z+a}$$

For poles,

$$az^2+2z+a = 0 \quad \text{——— (1)}$$

$$z = \frac{-2 \pm \sqrt{4-4a^2}}{2a}$$

$$z = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

Let $\alpha = \frac{-1+\sqrt{1-a^2}}{a}$, $\beta = \frac{-1-\sqrt{1-a^2}}{a}$

$\left[\because \text{ from eq}^n \text{ (1)}, \quad \alpha\beta = \frac{a}{a} = 1 \right]$

Let $|\alpha| < 1$, $|\beta| > 1$

Then α is a singular point lies in inside the C, & β lies outside the C. because $\alpha\beta = 1$.

$$I = \frac{2}{i}\int_C \frac{dz}{a(z-\alpha)(z-\beta)}$$

$$\text{Res}\,(z=\alpha) = \lim_{z\to\alpha} (z-\alpha)\,f(z)$$

$$= \lim_{z\to\alpha} \frac{1}{a(z-\beta)} = \frac{1}{a(\alpha-\beta)}$$

$$\because \alpha - \beta = -\frac{1}{a} + \frac{\sqrt{1-a^2}}{a} + \frac{1}{a} + \frac{\sqrt{1-a^2}}{a}$$

$$\alpha - \beta = \frac{2}{a} \times \frac{\sqrt{1-a^2}}{\sqrt{1-a^2}} = \frac{2\sqrt{1-a^2}}{a\sqrt{1-a^2}}$$

Note :- $\because e^{i6\theta} = \cos 6\theta + i\sin 6\theta$

$1 + e^{i6\theta} = 1 + \cos 6\theta + i\sin 6\theta$

$R.P(1 + e^{i6\theta}) = 1 + \cos 6\theta$

$$\operatorname{Res}(z=\alpha) = \frac{1}{a(\alpha-\beta)} = \frac{1}{\alpha} \times \frac{\alpha}{2\sqrt{1-a^2}}$$

$$= \frac{1}{2\sqrt{1-a^2}}$$

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{2\sqrt{1-a^2}} = \frac{\pi i}{\sqrt{1-a^2}}$$

$$I = \frac{2}{i}\int_C f(z)\,dz = \frac{2}{i} \times \frac{\pi i}{\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

Ques :- Prove that

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p}, \quad 0<p<1$$

Solution :-

Let C is the unit circle, $|z|=1$

$z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $d\theta = \frac{dz}{iz}$

$$I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \int_0^{2\pi} \frac{(1+\cos 6\theta)\, d\theta}{2(1-2p\cos 2\theta + p^2)}$$

$[\because e^{i6\theta} = \cos 6\theta + i\sin 6\theta]$

$$= R.P.\ \frac{1}{2}\int_0^{2\pi} \frac{(1+e^{i6\theta})\,d\theta}{\left[1 - \cancel{2}p\left(\frac{e^{i2\theta}+e^{-i2\theta}}{\cancel{2}}\right)+p^2\right]}$$

$$= R.P.\ \frac{1}{2}\int_C \frac{(1+z^6)\,dz}{iz[1-p(z^2+z^{-2})+p^2]}$$

$$= R.P.\ \frac{1}{2}\int_C \frac{(1+z^6)\,dz \cdot z^2}{iz[(1+p^2)z^2 - p(z^4+1)]}$$

$$= R.P.\ \frac{1}{2}\int_C \frac{2(1+z^6)\,dz}{-ip\left[z^4 - \left(\frac{1+p^2}{p}\right)z^2+1\right]}$$

$$= R.P.\left(\frac{-1}{2ip}\right)\int_C \frac{z(1+z^6)\,dz}{z^4-\left(\frac{1+p^2}{p}\right)z^2+1}$$

$$I = R.P.\left(\frac{-1}{2ip}\right)\int_C f(z)\,dz \quad \text{--- (1)}$$

$$f(z) = \frac{z(1+z^6)}{z^4-\left(\frac{1+p^2}{p}\right)z^2+1}$$

Poles of $f(z)$ are given by,

$$z^4-\left(\frac{1+p^2}{p}\right)z^2+1=0$$

$$pz^4-(1+p^2)z^2+p=0$$

$$z^2 = \frac{(1+p^2)\pm\left[(1+p^2)^2-4p^2\right]^{1/2}}{2p}$$

$$z^2 = \frac{(1+p^2)\pm(1-p^2)}{2p} = \frac{1}{p},\ p$$

so that

$$z = \pm\frac{1}{\sqrt{p}},\ \pm\sqrt{p}$$

The poles lying within the unit circle C are $\pm\sqrt{p}$ as $0<p<1$.

$$\text{Res}(z=\sqrt{p}) + \text{Res}(z=-\sqrt{p})$$

$$= \lim_{z\to\sqrt{p}}(z-\sqrt{p})f(z) + \lim_{z\to-\sqrt{p}}(z+\sqrt{p})f(z)$$

$$= \lim_{z\to\sqrt{p}}\frac{(z-\sqrt{p})\,z(1+z^6)}{(z^2-p)(z^2-1/p)} + \lim_{z\to-\sqrt{p}}\frac{(z+\sqrt{p})\,z(1+z^6)}{(z^2-p)(z^2-1/p)}$$

$$= \frac{\sqrt{p}\,(1+p^3)}{(\sqrt{p}+\sqrt{p})(p-\frac{1}{p})} + \frac{(-\sqrt{p})(1+p^3)}{(-\sqrt{p}-\sqrt{p})(p-1/p)}$$

[as $z^2-p=(z-\sqrt{p})(z+\sqrt{p})$

$$= \frac{1+p^3}{\frac{p^2-1}{p}}\left[\frac{\sqrt{p}-\sqrt{p}}{2\sqrt{p}-2\sqrt{p}}\right] = \frac{p(1+p^3)}{p^2-1}$$

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues within } C)$$
$$= 2\pi i\,\frac{p(1+p^3)}{p^2-1}$$

Now by (1),

$$I = R\cdot P\left(-\frac{1}{2ip}\right) \times \frac{2\pi i\,p(1+p^3)}{p^2-1}$$

$$= R\cdot P\left(\frac{1+p^3}{1-p^2}\right)\pi \qquad [\because a^3+b^3 = (a+b)(a^2+b^2-ab)]$$

$$= R\cdot P\left(\frac{\cancel{(1+p)}(1+p^2-p)}{(1-p)\cancel{(1+p)}}\right)\pi$$

$$\because a^2-b^2 = (a-b)(a+b)$$

$$\boxed{I = R\cdot P\,\frac{(1+p^2-p)\pi}{1-p}}$$

Ques:- Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

Solution:- Let $\displaystyle I = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = R.P.\int_0^{2\pi} \frac{e^{i2\theta}\,d\theta}{5+2(e^{i\theta}+e^{-i\theta})}$

$\because e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$R.P.$ of $e^{i2\theta} = \cos 2\theta$

Putting, $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$

$$\frac{dz}{iz} = d\theta$$

$$I = R.P.\ \frac{1}{i}\int_C \frac{z^2\,dz}{5+2(z+z^{-1})}\cdot\frac{1}{z},$$

where C is the circle $|z|=1$

$$I = R.P.\ \frac{1}{i}\int_C \frac{z^2\,dz}{2z^2+5z+2} = R.P.\ \frac{1}{2i}\int_C f(z)\,dz \qquad \text{—— (1)}$$

where $f(z) = \frac{z^2}{z^2 + \frac{5}{2}z + 1}$

For poles, $z^2 + \frac{5}{2}z + 1 = 0$

or, $2z^2 + 5z + 2 = 0$
$z = -2, -\frac{1}{2}$

$\therefore$ Only $z = -\frac{1}{2}$ lies within C.

$$\text{Res}\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{1}{2}} \frac{\left(z + \frac{1}{2}\right) z^2}{\left(z + \frac{1}{2}\right)(z+2)}$$

$$= \lim_{z \to -\frac{1}{2}} \frac{z^2}{(z+2)} = \frac{\frac{1}{4}}{\left(-\frac{1}{2} + 2\right)} = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \text{Res}\left(z = -\frac{1}{2}\right) = 2\pi i \times \frac{1}{6}$$

$$= \frac{\pi i}{3}$$

Put in (1),

$$I = R.P.\ \frac{1}{2i} \times \frac{\pi i}{3} = \frac{\pi}{6} \quad \text{Ans.}$$

- Prove that

$$\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta \, d\theta}{3 + 2\cos\theta} = \frac{2\pi}{\sqrt{5}} (3 - \sqrt{5})^n$$

n being positive integer.

Soln:-

Let $I = \int_0^{2\pi} \frac{(1 + 2\cos\theta)^n}{3 + 2\cos\theta} \cos n\theta \, d\theta$

Then

$$I = R.P. \int_0^{2\pi} \frac{(1 + e^{i\theta} + e^{-i\theta})^n}{3 + e^{i\theta} + e^{-i\theta}} e^{in\theta} \, d\theta$$

Putting $z = e^{i\theta}$, so that $dz = ie^{i\theta}\,d\theta$

$$d\theta = \frac{dz}{iz}$$

$$I = R.P. \int_C \frac{(1+z+z^{-1})^n z^n}{(3+z+z^{-1})} \cdot \frac{dz}{iz}$$

where C is unit circle.

$$I = R.P. \frac{1}{i}\int_C \frac{(z^2+z+1)^n\,dz}{z^2+3z+1}$$

so, $I = R.P. \frac{1}{i}\int_C f(z)\,dz$ —— (1)

where $f(z) = \frac{(z^2+z+1)^n}{(z^2+3z+1)}$

Poles of $f(z)$ are given by,

$$z^2+3z+1=0$$

This gives, $z = \frac{-3 \pm \sqrt{5}}{2}$

Let $\alpha = \frac{-3+\sqrt{5}}{2}$, $\beta = \frac{-3-\sqrt{5}}{2}$

Then $\alpha\beta = 1$, $|\alpha| < 1$, $|\beta| > 1$.

$\therefore$ $f(z)$ has only one simple pole $z = \alpha$ lying within C.

$$\text{Res}\,(z=\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha} \frac{\cancel{(z-\alpha)}(z^2+z+1)^n}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{(\alpha^2+\alpha+1)^n}{\alpha-\beta}$$

$\because$ $\alpha - \beta = -\cancel{\frac{3}{2}} + \frac{\sqrt{5}}{2} + \cancel{\frac{3}{2}} + \frac{\sqrt{5}}{2} = \frac{\cancel{2}\sqrt{5}}{\cancel{2}} = \sqrt{5}$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{-3+\sqrt{5}}{2}\right)^2 + \left(\frac{-3+\sqrt{5}}{2}\right) + 1\right]^n$$

$$= \frac{1}{\sqrt5}\left[\frac{9-6\sqrt5+5-6+2\sqrt5+4}{4}\right]^n = \frac{(3-\sqrt5)^n}{\sqrt5}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues within } C)$$

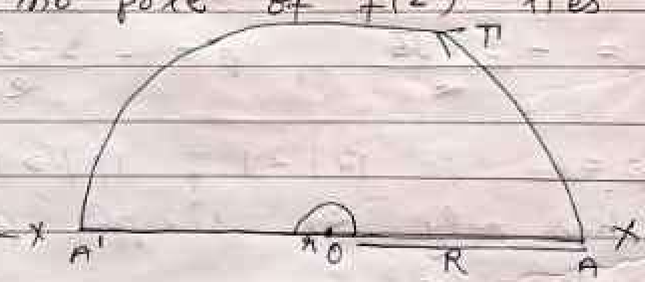
$$= \frac{2\pi i\,(3-\sqrt5)^n}{\sqrt5}$$

Using this in (1),

$$I = R.P.\ \frac{1}{i}\cdot\frac{2\pi i}{\sqrt5}(3-\sqrt5)^n$$

$$= \frac{2\pi}{\sqrt5}(3-\sqrt5)^n$$

$\int_{-\infty}^{\infty} f(z)\,dz$ where the function $f(z)$ is s.t. no pole of $f(z)$ lies on the real li



$$C = \left(\int_{+r}^{R} + \int_{T} + \int_{-R}^{-r} + \int_{\gamma}\right) f(z)\,dz$$

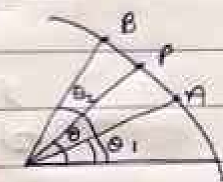
$$f(z)\,dz = \int_r^R f(z)\,dz + \int_T f(z)\,dz + \int_{-R}^{-r} f(z)\,dz + \int_\gamma f(z)$$

(Y)

Theorem :- If AB is the arc lies between $\theta_1 < \theta < \theta_2$ of the circle $|z-a| = r$ and if

$\lim_{z \to a} (z-a) f(z) = k$ given residues which is constant then

$\lim_{r \to 0} \int_{AB} f(z)\,dz = i(\theta_2 - \theta_1) k$



(T)

Theorem :- If AB is an arc where $\alpha \le \theta \le \beta$ of the circle $|z| = R$ and if

$\lim_{R \to \infty} z f(z) = k$ (constant)

then, $\lim_{R \to \infty} \int_{AB} f(z)\,dz = i(\beta - \alpha) k$

Example :- $\int_0^\infty \frac{dx}{1+x^2}$

Solution :- Consider the integral,

$I = \int_{-\infty}^{\infty} f(z)\,dz \quad \int_C f(z)\,dz$, where $f(z) = \frac{1}{1+z^2}$

$f(z) = \frac{1}{1+z^2} \quad \int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_T f(z)\,dz$ —— (1)

$\text{res}\, f(z) = \frac{1}{2i}$

Here C is the closed contour consisting of T, the upper half of the large circle $|z| = R$ and the real axis from $-R$ to R.

Poles of $f(z)$ are $z = \pm i$

$\text{Res}(z = i) = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{(z-i)}{(z-i)(z+i)}$

$= \lim_{z \to i} \frac{1}{(z+i)} = \frac{1}{2i}$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i(\text{ sum of residues within } C)$$
$$= \frac{2\pi i}{2i} = \pi$$

$$\lim_{|z|\to\infty} z f(z) = \lim_{|z|\to\infty} \frac{z}{1+z^2} = 0$$

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = i(\pi - 0)(0) = 0.$$

Making $R\to\infty$ in (1) and noting this,

$$\pi = \int_{-\infty}^{\infty} f(x)\,dx + 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_0^\infty \frac{dx}{1+x^2} = \pi$$

$$\text{or } \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

function inside the circle of convergence.

01/10/24

The Calculus of Residues :-

Ques:- By the method of contour integration, prove that $\int_0^\infty \frac{\cos mx \, dx}{x^2+a^2} = \frac{\pi e^{-ma}}{2a}$ where $m \geq 0, a >$

and deduce that $\int_0^\infty \frac{x \sin(mx)}{x^2+a^2} dx = \frac{\pi}{2} e^{-ma}$.

Solution:- Let $f(z) = \frac{e^{imz}}{z^2+a^2}$

Consider $\int_C f(z) dz$ where C is closed contour

Poles of $f(z)$ are given by $z^2+a^2=0$
or $z = \pm ia$

$\Rightarrow$ $f(z)$ has only simple one pole $z = ia$ inside C.

$$\text{Res}(z=ia) = \lim_{z\to ia} (z-ia) f(z) = \lim_{z \to ia} \frac{(z-ia) e^{imz}}{(z+ia)(z-ia)}$$

$$\text{Res}(z=ia) = \frac{1}{2ia} e^{-ma} \quad \text{——— (1)}$$

$$\lim_{|z|\to\infty} \frac{1}{z^2+a^2} = 0 \quad \text{(By Jordan's lemma)}$$

$$\lim_{R\to\infty} \int_\Gamma \frac{e^{imz}}{z^2+a^2} dz = \lim_{R\to\infty} \int_\Gamma f(z) dz = 0 \quad \text{——— (2)}$$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i \,(\text{sum of residues with } C)$$
$$= \frac{2\pi i e^{-ma}}{2ia} = \frac{\pi e^{-ma}}{a}$$

$$\int_{\Gamma} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{\pi}{a} e^{-ma}$$

Making $R \to \infty$ and noting (2), we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a} e^{-ma} \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-ma}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-ma} \quad \text{——— (3)}$$

or $$\boxed{\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-ma}} \quad \text{——— (4)}$$

Differentiating equation (4) w.r.t 'm',

$$\int_{0}^{\infty} -\frac{x \sin(mx)\,dx}{x^2+a^2} = \frac{\pi}{2a} e^{-ma}(-a)$$

$$\Rightarrow \boxed{\int_{0}^{\infty} \frac{x \sin(mx)}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}}$$

Ques:- If $a \geq 0$, then evaluate $\int_{0}^{\infty} \frac{\cos(ax)}{x^2+1}\,dx$.

Solution:- Let $f(z) = \frac{e^{iaz}}{z^2+1}$

Consider $\int_C f(z)\,dz$ where C is closed contour.

Poles of $f(z)$ are given by, $z^2+1=0$
or $z = \pm i$

$\Rightarrow$ $f(z)$ has only simple one pole $z = i$ inside C.

$$\text{Res}(z=i) = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{\cancel{(z-i)}\, e^{iaz}}{\cancel{(z-i)}(z+i)}$$

$$\text{Res}(z=i) = \frac{1}{2i} e^{-a} \quad \text{--- (1)}$$

$$\lim_{|z|\to\infty} \frac{1}{z^2+1} = 0 \quad \text{(By Jordan's lemma)}$$

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{iaz}}{z^2+1}\, dz = \lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0 \quad \text{--- (2)}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\,(\text{sum of residues with } C)$$
$$= 2\pi i \times \frac{e^{-a}}{2i} = \pi e^{-a}$$

Making $R \to \infty$ and noting (2), we get

$$\int_{\Gamma} f(z)dz + \int_{-R}^{R} f(x)\, dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} f(x)\, dx + 0 = \pi e^{-a}$$

or $$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\, dx = \pi e^{-a}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \pi e^{-a}$$

$$\boxed{\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\, dx = \frac{\pi}{2} e^{-a}}$$

Ques:- If $a>0$, $m>0$, then $\int_0^{\infty} \frac{\cos mx\, dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}(1+ma)e^{-ma}$

Solution:- $f(z) = \frac{e^{imz}}{(a^2+z^2)^2}$

Consider the integral $\int_C f(z)\, dz$, where C is the closed contour consisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R.

$$\lim_{|z|\to\infty} \frac{1}{(z^2+a^2)^2} = 0 \quad \text{(By Jordan's lemma)}$$

$$\lim_{R\to\infty} \int_\Gamma \frac{e^{imz}}{(z^2+a^2)^2} dz = 0 \quad \text{or} \quad \lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \qquad \text{—— (1)}$$

Poles of $f(z)$ are $z = \pm ia$ (repeated two times), $f(z)$ has only one pole order two at $z = ai$ within C.

$$f(z) = \frac{\phi(z)}{(z-ai)^2}, \quad \text{where } \phi(z) = \frac{e^{imz}}{(z+ai)^2}$$

$$\text{Res}(z = ia) = \lim_{z\to ia} \phi'(z)$$

$$= \lim_{z\to ia} \frac{e^{imz}[im(z+ai)^2 - 2(z+ia)]}{(z+ia)^4}$$

$$= \lim_{z\to ia} \frac{e^{imz}[im(z+ia) - 2]}{(z+ia)^3}$$

$$= \frac{e^{-ma}[im \cdot 2ia - 2]}{(2ia)^3} = \frac{e^{-ma}(1+ma)}{4a^3 i}$$

By Cauchy's residue theorem,

$$\int_C f(z) dz = 2\pi i\, [\text{Res}(z = ia)] = 2\pi i \times \frac{e^{-ma}(1+ma)}{4a^3 i}$$

$$\text{or} \quad \int_\Gamma f(z) dz + \int_{-R}^{R} f(x)\, dx = \frac{\pi(1+ma)}{2a^3} e^{-ma}$$

Making $R \to \infty$ and noting (1), we get

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2} dx = \frac{\pi}{2a^3}(1+ma)\, e^{-ma}$$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\cos mx \, dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}(1+ma)e^{-ma}$$

But $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx \, dx}{(x^2+a^2)^2} = 2\int_{0}^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx.$

$$\boxed{\int_{0}^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}}$$

<u>Ques:-</u> Apply calculus of residues to prove that

$$\int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2.$$

<u>Solution:-</u> Let $f(z) = \dfrac{\log(z+i)}{1+z^2}$

Consider the integral $\int_C f(z)\,dz$ where C is a closed contour consisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R.

$$k = \lim_{|z|\to\infty} z f(z) = \left[\lim_{|z|\to\infty} \frac{z}{z-i}\right]\left[\lim_{|z|\to\infty} \frac{\log(z+i)}{z+i}\right]$$

$$= 1(0) = 1$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)dz = i(\pi - 0)k = 0 \quad \text{——} (1)$$

Poles of $f(z)$ are $z = \pm i$.

$f(z)$ has only one simple pole at $z = i$ within

$$\text{Res}(z=i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{\log(z+i)}{z+i}$$

$$= \frac{\log 2i}{2i} = \frac{\log(2e^{i\pi/2})}{2i}$$

$$= \frac{\log 2 + (\pi i/2)}{2i}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res}\,(z=i)]$$

$$= \frac{2\pi i}{2i}\left[\log 2 + \left(\frac{i\pi}{2}\right)\right]$$

$$\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \pi\left[\log 2 + \frac{\pi i}{2}\right]$$

Making $R \to \infty$ and noting (1),

$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}\,dx = \pi\left[\log 2 + \frac{\pi i}{2}\right]$$

Equating real parts from both sides,

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2$$

$$\boxed{\int_0^{\infty} \frac{\log(1+x^2)}{x^2+1}\,dx = \pi \log 2}$$

Ques:- Apply the calculus of residues to evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$, $(a > b > 0)$

Solution:- Let $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

Consider the integral $\int_C f(z)\,dz$, where C is a closed contour.

$$\lim_{|z| \to \infty} \frac{1}{(z^2+a^2)(z^2+b^2)} = 0.$$

Hence, by Jordan's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} \frac{e^{iz}\, dz}{(z^2+a^2)(z^2+b^2)} = \lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0 \quad \text{--- (1)}$$

Poles of $f(z)$ are $z = \pm ia, \pm ib$.

Only two simple poles at $z = ia, ib$ lie within C.

$$\operatorname{Res}(z = ia) = \lim_{z\to ia} (z - ia) f(z)$$

$$= \lim_{z\to ia} \frac{\cancel{(z-ia)}\, e^{iz}}{\cancel{(z-ia)}(z+ia)(z^2+b^2)}$$

$$= \frac{e^{-a}}{2ia(-a^2+b^2)}$$

$$\operatorname{Res}(z = ib) = \lim_{z\to ib} (z - ib) f(z)$$

$$= \lim_{z\to ib} \frac{\cancel{(z-ib)}\, e^{iz}}{\cancel{(z-ib)}(z+ib)(z^2+a^2)}$$

$$= \frac{e^{-b}}{2ib(-b^2+a^2)} = \frac{e^{-b}}{2ib(a^2-b^2)}$$

$$\operatorname{Res}(z = ia) + \operatorname{Res}(z = ib) = \frac{1}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

By Cauchy's residues theorem,

$$\int_C f(z)\, dz = 2\pi i \,(\text{sum of residues within } C)$$

or $$\int_{\Gamma} f(z)\, dz + \int_{-R}^{R} f(x)\, dx = \frac{2\pi i}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Making $R\to\infty$ and noting (1), we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Equating real parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$2\int_{0}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\boxed{\int_{0}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]}$$

Ques:- Show that $\int_0^{\infty} \frac{x \sin ax}{x^2+k^2} dx = \frac{\pi}{2} e^{-ak}$,

where $a>0$, $k>0$.

Solution:- Let $f(z) = \frac{z e^{iaz}}{z^2+k^2}$.

Consider the integral $\int_C f(z) dz$, where C is a closed contour.

$$\lim_{|z|\to\infty} \frac{z}{z^2+k^2} = 0. \quad \text{(By Jordan's lemma)}$$

$$\lim_{R\to\infty} \int_{\Gamma} \frac{z}{z^2+k^2} e^{iaz} dz = \lim_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0 \quad \text{——(1)}$$

Poles of $f(z)$ are given by $z^2+k^2=0$

or $z = \pm ik$.

Now, $z = ik$ is the only simple pole within C.

$$\text{Res}(z=ik) = \lim_{z\to ik} (z-ik) f(z) = \lim_{z\to ik} \frac{z e^{iaz}}{z+ik} = \frac{e^{-ak}}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \,(\text{sum of residues within } C)$$
$$= \pi i e^{-ak}$$

or $\int_{\Gamma} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \pi i e^{-ak}$

Making $R \to \infty$ and noting (i), we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{x e^{iax}}{x^2+k^2}\,dx = \pi i e^{-ak}$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+k^2}\,dx = 2\int_{0}^{\infty} \frac{x \sin ax}{x^2+k^2}\,dx = \pi e^{-ak}$$

$$\boxed{\int_{0}^{\infty} \frac{x \sin ax\,dx}{x^2+k^2} = \frac{\pi}{2} e^{-ak}}$$

Ques:- Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^2(b+c)^2}$

where $b>0$, $c>0$.

Solution:- Let $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2}$

Consider the integral $\int_C f(z)\,dz$, where C is the closed contour consisting of Γ, the upper half of large circle $|z| = R$ and real axis from $-R$ to R.

$$h = \lim_{|z|\to\infty} z f(z) = \lim_{|z|\to\infty} \frac{z}{(z^2+b^2)(z^2+c^2)^2} = 0.$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = i(\pi - 0)h = 0$$

Poles of $f(z)$ are $z = \pm ib$, $\pm ic$, $f(z)$ has only two poles within C, & $z = ib$ (order 1) & $z = ic$ (order 2).

$$\text{Res}(z=ib) = \lim_{z\to ib} (z-ib)\, f(z)$$

$$= \lim_{z\to ib} \frac{\cancel{(z-ib)}}{\cancel{(z-ib)}(z+ib)(z^2+c^2)^2} = \frac{1}{2ib(-b^2+c^2)^2}$$

$$f(z) = \frac{\phi(z)}{(z-ic)^2}, \quad \text{where } \phi(z) = \frac{1}{(z^2+b^2)(z+ic)^2}$$

$$\phi'(z) = -2z(z^2+b^2)^{-2}(z+ic)^{-2} - 2(z+ic)^{-3}(z^2+b^2)^{-1}$$

$$\phi'(ic) = \frac{-2ic}{(-c^2+b^2)^2(-4c^2)} - \frac{2}{-8ic^3(-c^2+b^2)}$$

$$= \frac{i}{2c(b^2-c^2)^2} - \frac{i}{4c^3(b^2-c^2)} = \frac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}$$

$$\text{Res}(z=ic) = \frac{\phi'(ic)}{1!} = \phi'(ic)$$

$$\text{Res}(z=ib) + \text{Res}(z=ic) =$$

$$= \frac{1}{2ib(b^2-c^2)^2} + \frac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}$$

$$= \frac{i}{4(b^2-c^2)^2}\left[\frac{-2}{b} + \frac{3c^2-b^2}{c^3}\right] = \frac{i[-2c^3+b(3c^2-b^2)]}{4bc^3(b^2-c^2)^2}$$

$$= \frac{i[(c^3-b^3) - 3c^2(c-b)]}{4bc^3(b^2-c^2)^2}$$

$$= \frac{i(c-b)[c^2+b^2+cb-3c^2]}{4bc^3(b^2-c^2)^2}$$

$$= \frac{i(c-b)(b-c)(b+2c)}{4bc^3(b^2-c^2)^2} = -\frac{i(b+2c)}{4bc^3(b+c)^2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\ (\text{sum of residues within } C)$$

$$\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{2\pi i\,[-i(b+2c)]}{4bc^3(b+c)^2}$$

Making $R \to \infty$ and noting (1),

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^3(b+c)^2}}$$

Ques:- Prove that $\int_0^\infty \frac{\cos mx\,dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}} e^{-\frac{m}{2}\sqrt{3}} \cdot \sin\left(\frac{m}{2}\right.$

Solution:- Let $f(z) = \frac{e^{imz}}{z^4+z^2+1}$

Consider the integral $\int_C f(z)\,dz$ where C is closed contour.

$\Rightarrow \lim_{|z|\to\infty} \frac{1}{z^4+z^2+1} = 0$. Hence, by Jordan's lemma,

$$\lim_{R\to\infty} \int_\Gamma \frac{e^{imz}\,dz}{z^4+z^2+1} = \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0 \quad \text{—(1)}$$

Poles of $f(z)$ are given by,

$z^4+z^2+1 = 0$ or $(z^2-1)(z^4+z^2+1) = 0$

i.e. $(z^6-1) = 0 \Rightarrow z^6 = 1 = e^{2n\pi i}$

$\Rightarrow z = e^{2n\pi i/6}$

$\Rightarrow z = e^{n\pi i/3}$

$(n = 0, 1, 2, 3, 4, 5)$

The values $e^{i\pi/3}$, $e^{i2\pi/3}$, $e^{i4\pi/3}$, $e^{i5\pi}$

are the roots of $z^4+z^2+1=0$.

The poles lying within C are $e^{\pi i/3}, e^{2\pi i/3}$.

Let $\alpha = e^{\pi i/3}$, then $\beta = \alpha^2 = e^{2\pi i/3}$

Thus $f(z)$ has two simple poles at $z=\alpha, \alpha^2$ within C.

$\text{Res}(z=\alpha) = \lim_{z\to\alpha}$

Ques:- Prove that $\int_0^\infty \frac{\log x \, dx}{(1+x^2)^2} = \frac{\pi}{4}$

$(\frac{\pi}{4})$.

Solution:- Let $f(z) = \frac{\log z}{(1+z^2)^2}$;

Consider the integral $\int_C f(z)dz$, where C is the closed contour consisting of Γ, the upper half of large circle $|z|=R$ and real axis from $-R$ to R idented at $z=0$ by a small semi circle γ of radius r. $z=0$ is a branch point of $f(z)$. Poles of $f(z)$ are given by

$$(1+z^2)^2=0$$

$$\text{or} \quad z=i,-i$$

$\Rightarrow$ $f(z)$ has only one pole within C of order 2 at $z=i$.

$\text{Res}(z=i) = \frac{\phi'(i)}{1!}$, where $\phi(z) = \frac{\log z}{(z+i)^2}$ so that

$$f(z) = \frac{\phi(z)}{(z-i)^2}$$

$$\phi'(z) = \frac{[z^{-1}(z+i)^2 - 2(z+i)\log z]}{(z+i)^4}$$

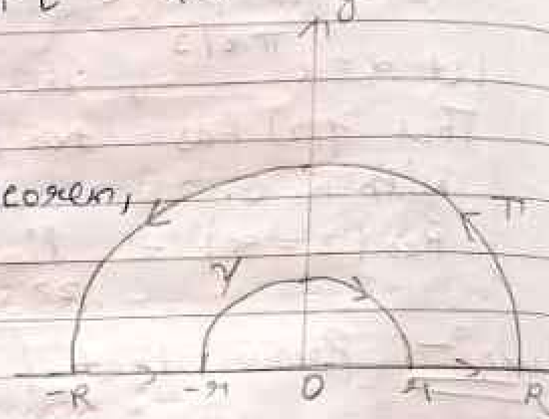
$$= \frac{z^{-1}(z+i) - 2\log z}{(z+i)^3}$$

$$\phi'(i) = \frac{-i(2i) - 2\log i}{-8i} = \frac{-1}{4i}\left[1 - \log e^{i\pi/2}\right]$$

$$= \frac{i}{4}\left[1 - \frac{i\pi}{2}\right] = \frac{1}{4}\left[i + \frac{\pi}{2}\right]$$

$$Res(z=i) = \frac{1}{4}\left(\frac{\pi}{2} + i\right)$$

By Cauchy's residue theorem,



$$\int_{-R}^{-r} f(x)\,dx + \int_{\gamma} f(z)\,dz + \int_{r}^{R} f(x)\,dx + \int_{T} f(z)\,dz$$

$$= 2\pi i\left[Res(z=i)\right]$$

$$= 2\pi i \cdot \frac{1}{4}\left(\frac{\pi}{2} + i\right) = \frac{\pi}{2}\left(i\frac{\pi}{2} - 1\right) \quad \text{--- (1)}$$

$$\Rightarrow k = \lim_{z\to\infty} z f(z) = \lim_{z\to\infty}\left[\frac{z^2}{(1+z^2)^2}\right]\left[\frac{\log z}{z}\right]$$

$$= (0)\cdot(0) = 0$$

$\Rightarrow$ ∴ By theorem,

$$\lim_{R\to\infty}\int_{T} f(z)\,dz = i(\pi - 0)k = 0 \quad \text{--- (2)}$$

$$\Rightarrow k_1 = \lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{z\log z}{(1+z^2)^2}$$

$$= \lim_{t\to\infty}\frac{(1/t)\log(1/t)\,t^4}{(1+t^2)^2}$$

$$= \lim_{t\to\infty}\frac{t^3(\log 1 - \log t)}{(1+t^2)^2}$$

$$= \lim_{t\to\infty}\left[\frac{t^4}{(1+t^2)^2}\right]\left[-\frac{\log t}{t}\right] = (1)\cdot(0) = 0.$$

$\therefore$ By theorem,

$$\lim_{r\to 0} \int_{\gamma} f(z)\,dz = -i(\pi - 0)k = 0 \quad \text{as } k_1 = 0 \qquad \text{--- (3)}$$

The negative sign is taken because the semicircle γ is negatively oriented.

Making $r\to 0$, $R\to\infty$ in (1) and noting (2), (3),

$$\int_{-\infty}^{0} f(x)\,dx + 0 + \int_{0}^{\infty} f(x)\,dx + 0 = \frac{\pi}{2}\left[i\frac{\pi}{2} - 1\right]$$

$$\int_{-\infty}^{0} \frac{\log x}{(1+x^2)^2}\,dx + \int_{0}^{\infty} \frac{\log x}{(1+x^2)^2}\,dx = \frac{\pi}{2}\left[i\frac{\pi}{2} - 1\right] \qquad \text{--- (4)}$$

$$\int_{-\infty}^{0} \frac{\log x\,dx}{(1+x^2)^2} = \int_{\infty}^{0} \frac{\log(-y)(-dy)}{(1+y^2)^2}, \quad \text{where } -y = x$$

$$= \int_{0}^{\infty} \frac{\log(ye^{i\pi})\,dy}{(1+y^2)^2} = \int_{0}^{\infty} (\log y + i\pi)\frac{dy}{(1+y^2)^2}$$

$$= \int_{0}^{\infty} (\log x + i\pi)\frac{dx}{(1+x^2)^2}$$

Using this in (4),

$$\int_{0}^{\infty} (\log x + i\pi)\frac{dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{\log x\,dx}{(1+x^2)^2} = \frac{\pi}{2}\left(\frac{i\pi}{2} - 1\right)$$

Equating real parts,

$$\int_{0}^{\infty} \frac{\log x\,dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{\log x\,dx}{(1+x^2)^2} = -\frac{\pi}{2}$$

or $$\boxed{\int_{0}^{\infty} \frac{\log x\,dx}{(1+x^2)^2} = \frac{-\pi}{4}}$$

Assignment - 3

Multivalued function :-

Multivalued function are those function which have two or more than two values of w for some or all values of z in given domain.
Let us consider,

$$w = f(z) = z^{\frac{1}{2}}$$
$$f(z) = (re^{i\theta})^{\frac{1}{2}}$$
$$= \left(re^{i(2n\pi+\theta)}\right)^{\frac{1}{2}}$$

$$f(z) = r^{\frac{1}{2}} e^{i(n\pi + \frac{\theta}{2})}$$
$$= r^{\frac{1}{2}} \left[e^{i\frac{\theta}{2}} e^{in\pi}\right]$$
$$= r^{\frac{1}{2}} e^{i\theta/2} \{1, -1\} \qquad \because \left[e^{in\pi} = \{-1, 1\}\right]$$

$$w = f(z) = r^{\frac{1}{2}} e^{i\theta/2} = f_1(z)$$
$$= -r^{\frac{1}{2}} e^{i\theta/2} = f_2(z)$$

So multivalued function are the collection of single valued function.

Branch :-

If a function $f(z)$ of domain D is a multi-valued function then branch is a single valued function of domain D, which is defined in the sub-domain of $f(z)$.

Ex (1) $z^{\frac{1}{2}} \Rightarrow 2$ branches $\{f_1(z), f_2(z)\}$

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$$= r^{\frac{1}{2}} \left[e^{i\frac{\theta}{2}} e^{in\pi}\right]$$

$$= r^{\frac{1}{2}} e^{i\theta/2} \{1,-1\} \qquad \because \left[e^{in\pi} = \{-1,1\}\right]$$

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Ex ① $z^{\frac{1}{2}} \Rightarrow 2$ branches $\{f_1(z), f_2(z)\}$

(2) $z^{\frac{1}{3}}$ = 3 branches

$$\left[r^{\frac{1}{3}} e^{i\theta/3},\ \alpha\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right),\ \beta\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)\right.$$

$$r^{\frac{1}{3}} = \left(re^{i\theta}\right)^{\frac{1}{3}}$$
$$= r^{\frac{1}{3}} e^{i(\theta+2n\pi)\frac{1}{3}}$$
$$= r^{\frac{1}{3}} e^{i\theta/3} \cdot e^{i\frac{2n\pi}{3}}$$
$$= r^{\frac{1}{3}} e^{i\theta/3}\left[1, \frac{-1}{2}+\frac{\sqrt{3}}{2}i, \frac{-1}{2}-\ldots\right.$$

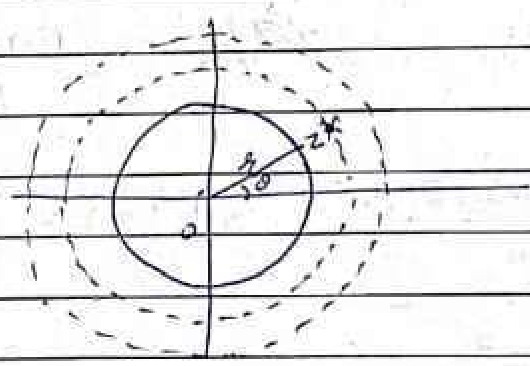
(3) $\log z$ = Infinite branches

Branch point

$$f(z) = z^{\frac{1}{2}}$$
$$= \left(re^{i\theta}\right)^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\theta/2} = f_1(z)$$

let $z=0$

$$f(z) = z^{\frac{1}{2}} = \left(re^{i(\theta+2\pi)}\right)^{\frac{1}{2}}$$
$$= r^{\frac{1}{2}} e^{i\frac{\theta}{2}} e^{\pi i}$$
$$= -r^{\frac{1}{2}} e^{i\theta/2} = f_2(z)$$



Branch point is a point ($z=0$) about which when ~~time~~ we take a rotation the value of the function changes.

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$$f(z) = -r^{\frac{1}{2}} e^{i(\theta+2\pi)\frac{1}{2}}$$
$$= -r^{\frac{1}{2}} e^{i\theta/2} e^{\pi i}$$
$$= r^{\frac{1}{2}} e^{i\theta/2} = f_1(z)$$

$(2\pi) \rightarrow$ Point $z \rightarrow f_1(z) \rightarrow f_2(z)$

$(4\pi) \rightarrow$ Point $z \rightarrow f_2(z) \rightarrow f_1(z)$

$$f_1(z) \rightleftharpoons f_2(z)$$

Branch point is a point about which branches of multivalued function $f(z)$ are interchanged when z describes closed path around that point.

Branch point is a type of singularity. Branch point is a point where function is discontinuous.

(1) $f(z) = z^{\frac{1}{2}}$
$f'(z) = \frac{1}{2} \frac{1}{z^{1/2}}$ Point $z=0$ is a branch point

(2) $f(z) = (z^2+1)^{\frac{1}{2}} \Rightarrow f'(z) = \frac{1}{2} \frac{\times 2z}{(z^2+1)^{\frac{1}{2}}}$

$z^2+1=0$

$z^2=-1$

$z = \pm i$

$f(z) = (z^2+1)^{\frac{1}{2}} = i, -i \rightarrow$ two branch point

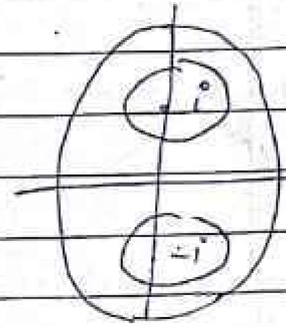
(b) $(z^2+1)^{\frac{1}{2}} = \{(z+i)(z-i)\}^{\frac{1}{2}}$

$z+i = r_1 e^{i\theta_1}$

$z-i = r_2 e^{i\theta_2}$

$(z^2+1)^{\frac{1}{2}} = \left(r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}\right)^{\frac{1}{2}}$

$= \sqrt{r_1 r_2} \cdot e^{i\frac{\theta_1}{2}} \cdot e^{i\frac{\theta_2}{2}}$



$= \sqrt{r_1 r_2}\; e^{i\frac{(\theta_1+\theta_2)}{2}}$

$\alpha_1 = \theta_1 \quad , \alpha_2 = \theta_1 + 2\pi$

$f(z) = \sqrt{r_1 r_2} \cdot e^{i\left(\frac{\theta_1+2\pi}{2}\right)} \cdot e^{i\left(\frac{\theta_2}{2}\right)}$

$= \sqrt{r_1 r_2} \cdot e^{i\frac{\theta_1}{2}} \cdot e^{i\frac{\theta_2}{2}} \cdot e^{\pi i}$

$= -\sqrt{r_1 r_2}\; e^{i\theta_1/2} \cdot e^{i\theta_2/2}$

$\theta_1 \to \theta_1 + 2\pi \qquad \theta_2 \to \theta_2 + 2\pi$

when we rotate z around

$f(z) = \sqrt{r_1 r_2}\; e^{i\frac{(\theta_1+2\pi)}{2}} \cdot e^{i\left(\frac{\theta_2+2\pi}{2}\right)}$

$= \sqrt{r_1 r_2}\; e^{\frac{i\theta_1}{2}} e^{\frac{i\theta_2}{2}} e^{\pi i} e^{\pi i}$

$= \sqrt{r_1 r_2}\; e^{i\theta_1/2}\, e^{i\theta_2/2}$

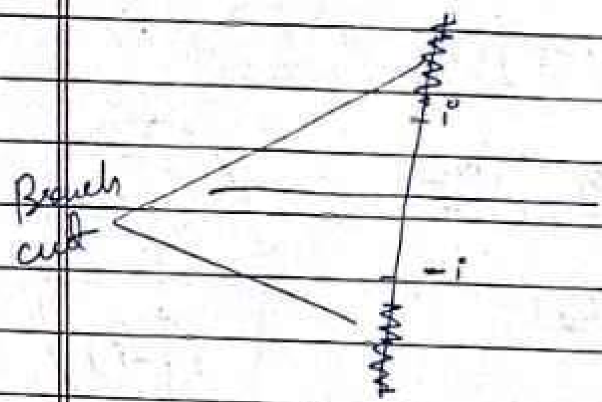
→ ~~Branch cu~~

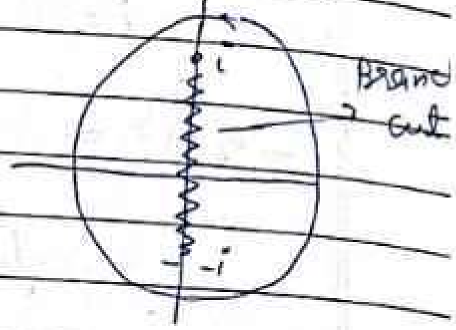
Branch Cut :-

Branch cut is a line/curve which reduces the multivalued function into single valued function.

$$0 \le \theta < 2\pi$$

$$f_1(z) = r^{\frac{1}{2}} e^{\frac{i\theta_1}{2}}$$



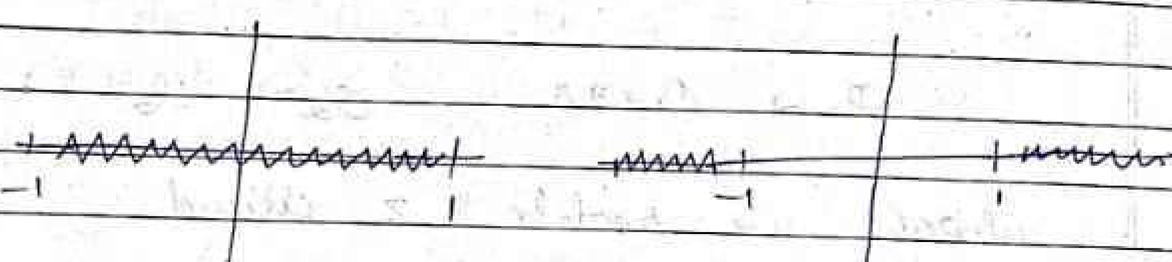


Ex (1) $f(z) = (z^2-1)^{\frac{1}{2}}$

$$[(z+1)(z-1)]^{\frac{1}{2}}$$

$$= (z+1)^{\frac{1}{2}} (z-1)^{\frac{1}{2}}$$

Branch point $= -1, 1$



(2) $f(z) = (z^3 - z)^{\frac{1}{3}}$

$$= [z(z^2-1)]^{\frac{1}{3}} = [z(z+1)(z-1)]^{\frac{1}{3}}$$

Branch point $= 0, -1, 1$

(3) $f(z) = (z^3-1)^{\frac{1}{2}}$ ...

$z^3 - 1 = 0$

$z^3 = 1$

$z = \{1, \omega, \omega^2\}$

so Branch points $= 1, \omega, \omega^2$

Infinite branches

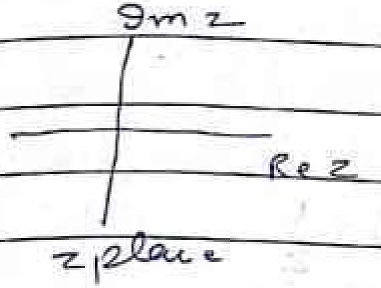
$f(z) = \ln z$

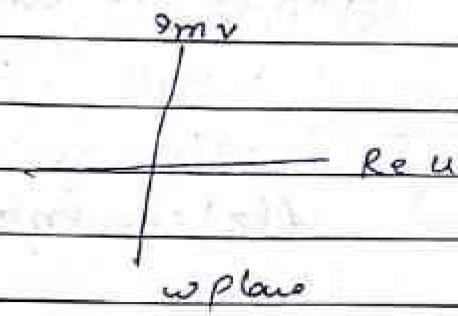
$f(z) = \ln(re^{i\theta})$

$= \ln r + \ln e^{i\theta}$

$= \ln r + \ln e^{i[2n\pi+\theta]}$

$= \ln r + i(\theta + 2n\pi) \qquad n \in \mathbb{Z}$



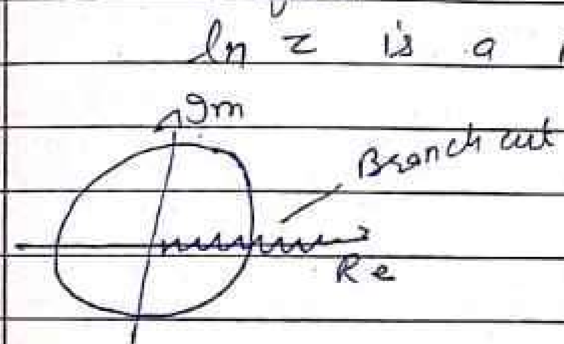


$w = f(z) = u + iv$

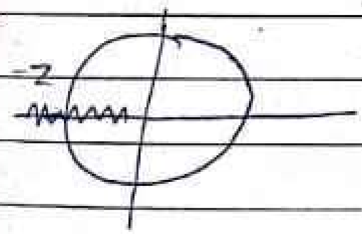
$u = \ln r$

$v = \theta + 2n\pi$

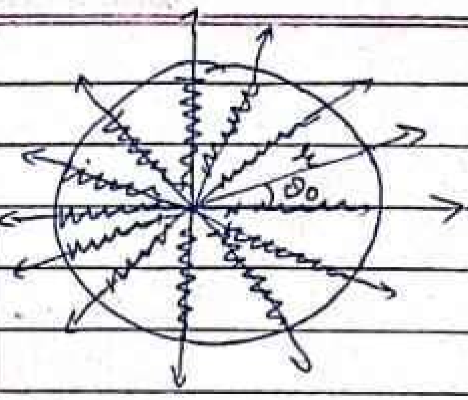
corresponding to particular value of z we have infinite value of f(z).
therefore.
ln z is a multivalued function



$\theta \in (0, 2\pi)$



$\theta \in (-\pi, \pi)$



$\theta \in (\theta_0, \theta_0 + 2\pi)$

$z_0 = re^{i\theta_0}$

<u>Infinite no. branches</u>

$f(z) = \ln z = f'(z) = \frac{1}{z}$

Branch point at $z=0$

<u>Ex</u> ②

$f(z) = \ln(z-3)$

$f'(z) = \frac{1}{z-3}$

$\therefore f(z)$ has branch point at $z=3$.

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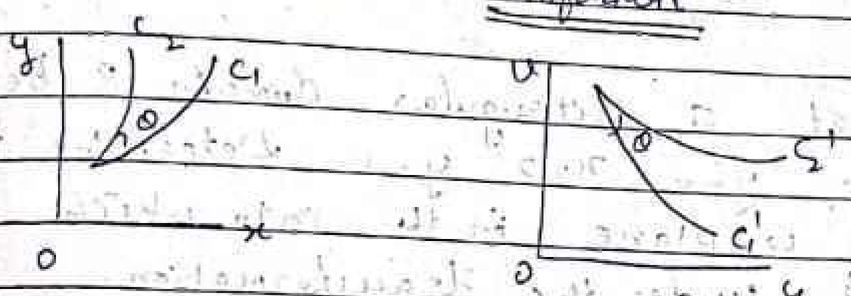
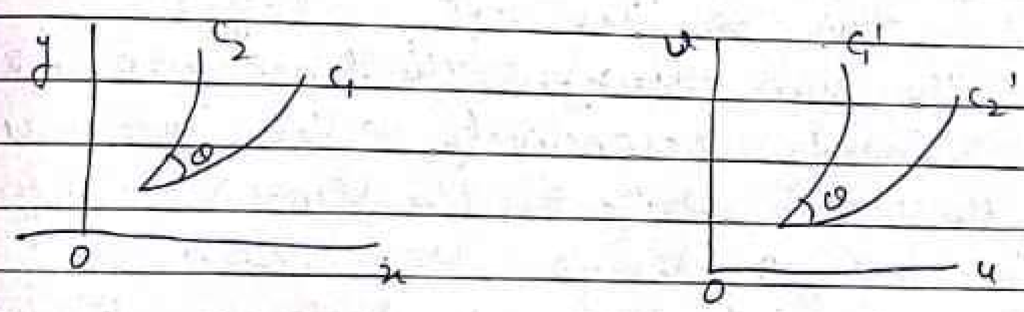
Conformal Transformation:

Suppose the transformation $u = u(x, y)$, $v = v(x, y)$ maps two curves C_1, C_2 of z plane & on the two curves C_1', C_2' of w plane

If the angle between C_1 & C_2 at z_0 is equal to the angle between C_1' & C_2' at w_0, then the transformation is called isogonal. If the sense of rotation as well as magnitude of the angle is preserved, the transformation is called conformal.



Conformal transformation



~~conf~~ Isogonal transformation

Some General transformation:

(1) Translation
(2) Rotation
(3) Magnification
(4) Inversion

(1) **Translation →**

The map $w = z + \beta$ corresponds to a translation for, by this transformation, the figure in w-plane is the same as figure in z-plane with a different origin.

Example → Let a rectangular domain R be bounded by $x=0$, $y=0$, $x=2$, $y=1$. Determine the region R' of w-plane into which R is mapped under the transformation.

$$w = z + (1-2i).$$

Sol^n

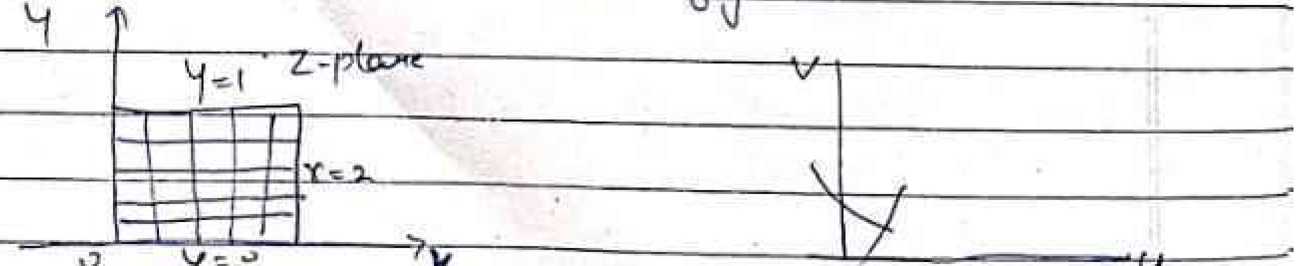
$$w = z + (1-2i) \text{ gives } u + iv = x + iy + (1-2i)$$

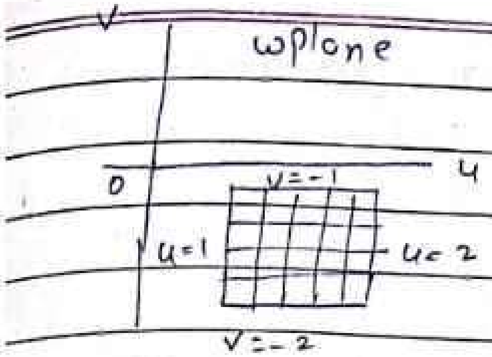
This $\Rightarrow$ $u = x+1$, $v = y-2$

By the map $u = x+1$, the lines $x=0$, $x=2$ are mapped respectively on the lines $u=1$, $u=3$.

By the map $v = y-2$, the lines $y=0$, $y=1$ are mapped on $v=-2$, $v=-1$ resp.

The required image is rectangle R' bounded by $u=1$, $u=3$, $v=-2$, $v=-1$ in w-plane as shown in the figure.





Similarly we can show that each point of R is mapped into one and only one point of R' and conversely. Hence the transformation is performs a translation of rectangle. for the given map is of the type $w = z+\beta$, where $\beta = 1-2i$, Here the rectangle R is translated in dir^n of vector $\beta = 1-2i$.

Rotation :.

By the transformation $w = ze^{i\theta_0}$, figures in z plane are rotated through angle θ_0 if $\theta_0 > 0$ the rotation is anti-clockwise & $\theta_0 < 0$ the rotation is clock-wise.

Ex.

Consider the transformation $w = ze^{i\pi/4}$ & determine the region R' in w-plane corresponding to the triangular region R bounded by the lines $x=0$, $y=0$ and $x+y=1$ in z-plane.

$w = ze^{i\pi/4}$ gives $u+iv = (x+iy)\left(\frac{1+i}{\sqrt{2}}\right)$

This $\Rightarrow$ $u = \frac{1}{\sqrt{2}}(x-y)$, $v = \frac{1}{\sqrt{2}}(x+y)$

Putting $x=0$ in the set (1)

$u = -\frac{1}{\sqrt{2}}y$ $v = \frac{1}{\sqrt{2}}y$ or $v = -u$

Putting $y=0$ in (1)

$u = \frac{1}{\sqrt{2}} x$, $v = \frac{1}{\sqrt{2}} x$ or $v = u$

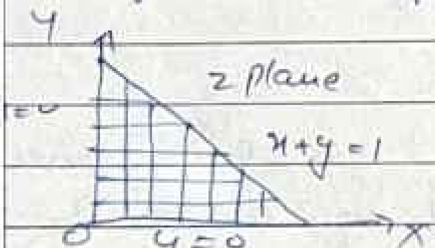
Putting $x + y = 1$ in (1),

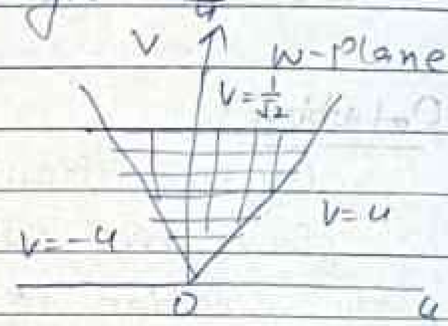
$$v = \frac{1}{\sqrt{2}}$$

Hence the triangular region R in z-plane is mapped in a triangular region R' of w-plane bounded by the lines

$$v = u \quad , \quad v = -u \quad , v = \frac{1}{\sqrt{2}}$$

the two regions are shown in both figure. the mapping $w = ze^{i\frac{\pi}{4}}$ performs a rotation of R through an angle $\frac{\pi}{4}$





Magnification (Stretching) →

Consider the mapping $w = az$, where a is real. the two figures in z-plane & w-plane are similar & similarly situated about their respective origin, but the fig in w-plane is a times the figure in z-plane. such map is called magnification or stretching or dilation.

Example Consider the transformation $w = 2z$ & determine the region R' of w-plane into which the triangular region R enclosed b

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lines $x=0$, $y=0$, $x+y=1$, in z-plane is mapped under the map.

Sol^n $w=2z$ gives $u+iv = 2x+i2y$

This $\Rightarrow u=2x,\ v=2y$

$x=0,\ u=2x \Rightarrow u=0$

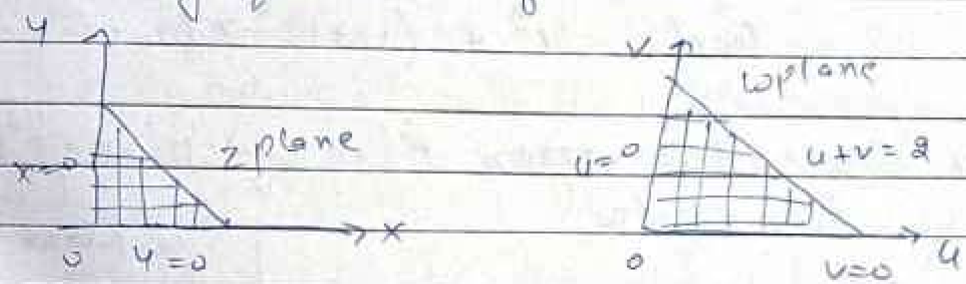
$y=0,\ v=2y \Rightarrow v=0$

$x+y=1,\ u=2x,\ v=2y \Rightarrow u+v=2(x+y) = 2(1)=2$

$\Rightarrow u+v=2$

Hence the region R' is triangular bounded by the lines $u=0,\ v=0,\ u+v=2$.

This transformation $w=2z$ performs a magnification of R into R'.



4 <u>Inversion</u> → By means of the transformation $w=\frac{1}{z}$, figures in z-plane are mapped upon the reciprocal figures in w-plane.

Ex. <u>Example</u> Consider the map $w=\frac{1}{z}$ and determine the region R' in w-plane of the infinite strip R bounded by $\frac{1}{4}<y<\frac{1}{2}$.

Sol^n $w=\frac{1}{z}$ gives $u+iv = \frac{x-iy}{x^2+y^2}$

This $\Rightarrow u=\frac{x}{x^2+y^2},\ v=\frac{-y}{x^2+y^2} \Rightarrow \frac{u}{v} = \frac{-x}{y} \Rightarrow x = \frac{-uy}{v}$

$$iv = \frac{-y}{x^2+y^2} \Rightarrow v = \frac{-y}{\frac{u^2y^2}{v^2}+y^2} = \frac{-v^2}{y(u^2+v^2)} \Rightarrow 1 = \frac{-v}{y(u^2+v^2)}$$

$$\Rightarrow y = \frac{-v}{u^2+v^2} \quad \text{--- (1)}$$

$$y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2} \Rightarrow -2v < u^2+v^2$$

$$\Rightarrow u+(v+1)^2 > 1 \quad \text{--- (2)}$$

$$y > \frac{1}{4} \Rightarrow \frac{-v}{u^2+v^2} > \frac{1}{4} \Rightarrow -4v > u^2+v^2$$

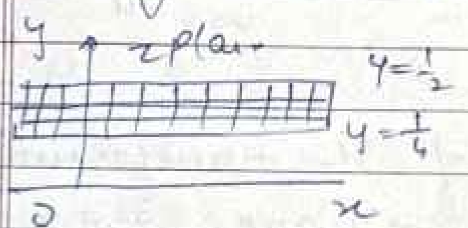
$$\Rightarrow u^2+(v+2)^2 < 2^2$$

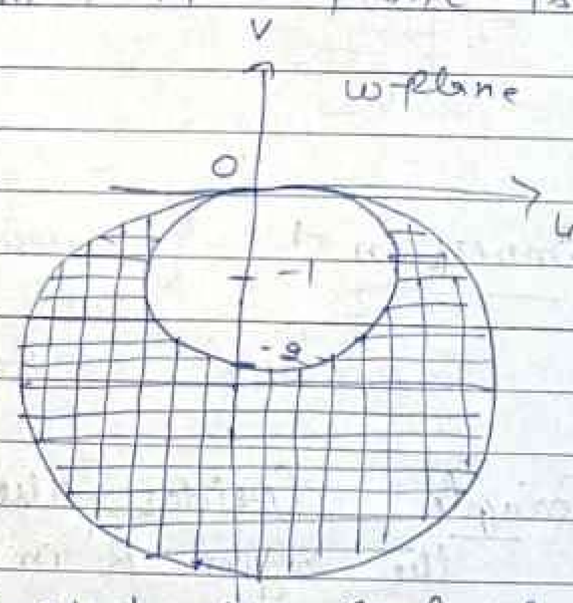
Finally,

$$\frac{1}{4} < y < \frac{1}{2} \Rightarrow u^2+(v+2)^2 < 2^2$$

$$\text{and} \quad u^2+(v+1)^2 > 1$$

This shows that region R' in w-plane is bounded by two circles





$u^2+(v+2)^2 = 2^2$

and $u^2+(v+1)^2 = 1$

S.t. the region is exterior to the circle $u^2+(v+1)^2=1$ and interior to the circle $u^2+(v+2)^2 = 2^2$.

Bilinear transformation or Mobius transformation

A transformation of the form

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

is called a bilinear transformation where a, b, c, d are complex constants. Also called Mobius transformation. (1) is expressible as

$$cwz + wd - az - b = 0$$

Evidently it is linear both in w & z. That is why it is called bilinear. Here we assume that $ad - bc \neq 0$ which is called determinant of the transformation. The transformation (1) is said to be normalised if $ad - bc = 1$. If the determinant vanishes, then w is merely a constant as shown below.

Let w_1 & w_2 be the values of w corresponding to z_1 & z_2 in (1) then

$$w_2 - w_1 = \frac{az_2+b}{cz_2+d} - \left(\frac{az_1+b}{cz_1+d}\right)$$

$$= \frac{(ad-bc)(z_2-z_1)}{(cz_1+d)(cz_2+d)} = 0 \quad \text{if } ad-bc=0$$

$$w_2 - w_1 = 0 \quad \text{if } ad-bc=0$$

This shows w is constant.

Cross Ratio:-

If z_1, z_2, z_3, z_4 are distinct points, then

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the ratio

$$\left(\frac{z_4-z_1}{z_4-z_3}\right)\left(\frac{z_2-z_3}{z_2-1}\right)$$

is called the cross ratio of z_1, z_2, z_3, z_4 this ratio is invariant under the billinear transformation.

Theorem

Billinear transformation can be considered as combination of the transformation of translation, rotation, stretching & inversion. Prove it.

Pf Consider the billinear transformation.

$$\omega = \frac{az+b}{cz+d} \quad \text{— (1)}$$

where $ad-bc \neq 0$, $c \neq 0$

from (1)

$$\omega = \frac{a}{c} \cdot \frac{z+\left(\frac{b}{a}\right)}{z+\left(\frac{d}{c}\right)} = \frac{a}{c}\left[1 + \frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right]$$

$$\omega = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$$

taking $z_1 = z+\frac{d}{c}$, $z_2 = \frac{1}{z_1}$, $z_3 = \frac{bc-ad}{c^2} z_2$

we obtain $\omega = \frac{a}{c} + z_3$ which is similar to

$$z_1 = z+\frac{d}{c}$$

the above three auxiliary transformation namely z_1, z_2, z_3 are of the form

$$\omega = z+\alpha, \quad \omega = \frac{1}{z}, \quad \omega = \beta z$$

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This proves that every general bilinear transformation is the resultant of the bilinear transformation

$$w = z+\alpha \ , \ w = \frac{1}{z} \ , \ w = \beta z$$

where (1) $w = z+\alpha$ represent translation
(2) $w = \frac{1}{z}$ represent inversion
(3) $w = \beta z$ represent dilation.

Proved

Theorem-2
To show that the resultant (or product) of two bilinear transformation is a bilinear transformation.

Proof Consider two bilinear transformations

$$w = \frac{az+b}{cz+d} \quad s.t. \quad ad-bc \neq 0 \quad \text{—} (1)$$

$$\xi = \frac{a_1 w + b_1}{c_1 w + d_1} \quad s.t. \quad a_1 d_1 - b_1 c_1 \neq 0 \quad \text{—} (2)$$

Putting the value of w from (1) in (2) we get

$$\xi = \frac{a_1\left(\frac{az+b}{cz+d}\right) + b_1}{c_1\left(\frac{az+b}{cz+d}\right) + d_1}$$

$$= \frac{(aa_1 + b_1 c)z + (b_1 d + a_1 b)}{(c_1 a + d_1 c)z + (d_1 d + c_1 b)}$$

$A = aa_1 + b_1 c \qquad B = b_1 d + a_1 b$
$C = c_1 a + d_1 c \qquad D = d_1 d + c_1 b$

$\xi = Az + B$

Here $AD-BC = (aa_1+b_1c)(d_1d+c_1b) - (b_1d+a_1b)(c_1a+d_1c)$

$= aa_1d_1d + bb_1cc_1 - b_1c_1ad - a_1d_1bc$

$= (a_1d_1 - b_1c_1)(ad-bc) \neq 0$ by ① & ②

$$\phi = \frac{Az+B}{Cz+D} \quad s.t. \quad AD-BC \neq 0$$

this is bilinear transformation this is called resultant or product of the transformation ① & ②.

<u>Theoram</u>

Preservance of cross ratio →

To prove that cross-ratio remains invariant under a bilinear transformation.

Proof Let w_1, w_2, w_3, w_4 be the images of four distinct points z_1, z_2, z_3, z_4 in z plane under a bilinear map then

$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

$$w = \frac{az+b}{cz+d} \quad -①$$

where $ad-bc \neq 0$

If we show that

$$\frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)} = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)} \quad -②$$

from ①

$$w_2 - w_3 = \frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d}$$

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$$w_2 - w_3 = \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)} \quad \text{--- (3)}$$

This $\Rightarrow (w_4 - w_1)(w_2 - w_3) =$

$$\frac{(ad - bc)(z_4 - z_1)}{(cz_4 + d)(cz_1 + d)} \cdot \frac{(ad - bc)(z_2 - z_3)}{(cz_2 + d)(cz_3 + d)}$$

or

$$(w_4 - w_1)(w_2 - w_3) = \frac{(ad - bc)^2 (z_4 - z_1)(z_2 - z_3)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

Similarly, again (3) we get

$$(w_2 - w_1)(w_4 - w_3) = \frac{(ad - bc)^2 (z_2 - z_1)(z_4 - z_3)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}$$

Dividing the last two eq^n, we get the required equation (2).

theorem

fixed point of a Bilinear transformation. $\rightarrow$

To prove that in general there are two values of z for which $w = z$, but there is only one if

$$(a - d)^2 + 4bc = 0$$

Show that if there are distinct invariant points p & q, the transformation may be put in the form (normal form.

$$\frac{w - p}{w - q} = k\left(\frac{z - p}{z - q}\right)$$

& that if there is only one invariant point p, the transformation may be put in the form (normal form)

$$\frac{1}{w-p} = \frac{1}{z-p} + k.$$

Proof Consider the bilinear transformation

$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

(i) The invariant points are given by $w=z$, i.e. $z = \frac{az+b}{cz+d}$

This gives $cz^2 - (a-d)z - b = 0$ --- (2)

solving,

$$z = \frac{(a-d) \pm \sqrt{(a-d)^2+4bc}}{2c}$$

Taking

$$\left.\begin{aligned} p &= \frac{(a-d) + \left((a-b)^2 + 4bc\right)^{1/2}}{2c} \\ q &= \frac{(a-d) - \sqrt{[(a-d)^2+4bc]}}{2c} \end{aligned}\right\} \text{--- (3)}$$

This shows that p, q are distinct points and are invariant points for (1)

If $(a-d)^2 + 4bC = 0$ then (3) declares that

$$p = q = \frac{a-d}{2c} \quad \text{--- (4)}$$

Here there is only one invariant point namely,

$$p = \frac{a-d}{2c} \text{ if } (a-d)^2 + 4bc = 0$$

(ii) suppose p & q are distinct invariant points of the transformation (1) so that they satisfy (2). As a result of which

$$cp^2 - (a-d)p - b = 0$$

$$cq^2 - (a-d)q - b = 0$$

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for which

$$\left.\begin{aligned} cp^2 - ap &= b - pd \\ cq^2 - aq &= b - qd \end{aligned}\right\} \quad -(5)$$

$$\frac{w-p}{w-q} = \frac{\left(\frac{az+b}{cz+d}\right) - p}{\left(\frac{az+b}{cz+d}\right) - p} = \frac{(az+b) - pcz - pd}{(az+b) - cqz - qd}$$

$$= \frac{(a-pc)z + b - pd}{(a-cq)z + b - qd} = \frac{(a-pc)z + cp^2 - ap}{(a-cq)z + cq^2 - aq}$$

$$= \frac{(a-cp)(z-p)}{(a-cq)(z-q)}$$

taking $k = \frac{a-cp}{a-cq}$, we get $\frac{w-p}{w-q} = k\frac{(z-p)}{z-q}$ -(6)

which is required.

(iii) Let there be only one invariant point p so that

$p = \frac{a-d}{2c}$, $cp^2 - ap = b - pd$ by (4) & (5)

Now $\frac{1}{w-p} = \frac{1}{\frac{az+b}{cz+d} - p} = \frac{cz+d}{(a-cp)(z)+(b-pd)}$

$$= \frac{cz + (2cp - a)}{(a-cp)z + cp^2 - ap}$$

$$\frac{cz - cp + a - cp}{(a-cp)(z-p)} = \frac{c}{a-cp} + \frac{1}{z-p}$$

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Taking $k = \frac{c}{a - cp}$ we get $\frac{1}{w-p} = k + \frac{1}{z-p}$

which is equivalent

(b)

Parabolic

A linear bilinear with one fixed points z_0 is called parabolic & is expressible as

$$\frac{1}{w - z_0} = \frac{1}{z - z_0} + h \quad \text{if} \quad z_0 \neq \infty$$

$$w = z + h \quad \text{if} \quad z_0 = \infty$$

or

A linear transformation with two different fixed points z_1 & z_2 is expressible as

$$\frac{w - z_1}{w - z_2} = \frac{k(z - z_1)}{(z - z_2)} \quad \text{if} \quad z_1, z_2 \neq \infty$$

If $z_2 = \infty$, then it becomes $w - z_1 = k(z - z_1)$

A transformation with two diff^n fixed points is called hyperbolic if $k > 0$, & elliptic if $k = e^{i\alpha}$ & $\alpha \neq 0$ & loxodromic if $k = ae^{i\alpha}$ where $a \neq 1$, $\alpha \neq 0$, α & a both are real numbers & $a > 0$.

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Q find fixed point & normal form of given bilinear transformation.

$$w = \frac{z-1}{z+1}$$

Sol^n

The fixed pt are given by $z = \frac{z-1}{z+1}$

or $z^2 + z - z + 1 = 0$ or $(z-i)(z+i) = 0$

or $z = i, -i$

$$w - i = \frac{z-1}{z+1} - i = \frac{(1-i)z - (1+i)}{z+1}$$

$$w + i = \frac{z-1}{z+1} + i = \frac{(1+i)z - (1-i)}{z+1}$$

dividing, we get

$$\frac{w-i}{w+i} = \frac{1-i}{1+i}\left[\frac{z - (1+i)/(1-i)}{z - (1-i)/(1+i)}\right]$$

$\frac{w-i}{w+i} = -i\left(\frac{z-i}{z+i}\right)$ which is normal form

Here $k = -i = e^{-i\pi/2}$. Hence transformation is elliptic.

(2) $w = \frac{3iz+1}{z+i}$

Sol^n fixed point are given by $z = \frac{3iz+1}{z+i}$

or $z^2 - 2iz - 1 = 0$

or $(z-i)^2 = 0$

or $z = i, i$

Hence, there is only one distinct fixed point namely i.

$$w-i = \frac{3iz+1}{z+i} - i = \frac{2iz+2}{z+i}$$

$$\frac{1}{w-i} = \frac{z+i}{2(iz+1)} = \frac{z+i}{2i(z-i)}$$

$$= \frac{1}{2i}\left[1+\frac{2i}{z-i}\right] = \frac{1}{z-i} + \frac{1}{2i}$$

or

$$\frac{1}{w-i} = \frac{1}{z-i} - \frac{i}{2}$$ which is the normal form

The transformation is parabolic. Ans