Complex Integration

UNIT :- 2.

Assignment = 1.

Domain :- An open set that is connected is called a domain. Every neighbourhood is a domain.

Region :- A domain together with some, none or all of its boundary points is reffered to as a region.
An open connected set is said to be a open domain or open region.
If the boundary points of S are also added to an open domain, then it is called closed domain.

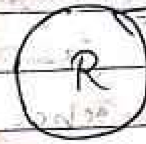
Jordan arc :- The arc C is simple arc or a Jordan arc if it does not cross it self.

~~Con~~ **Contour** :- It is a continuous chain of a finite number of regular arcs.
If the contour is closed and does not intersect itself, then it is called closed contour.

Examples :- Boundaries of triangles & rectangles.

Simply connected region :- A connected region R is said to be a simply connected region if all the interior points of a closed curve C drawn in the region R, are the

points of R.



Connected Region :- A region R is said to be connected if any two points of R can be connected by a curve which lies entirely within the region.

Multiply Connected Region :- If all the points of the area bounded by two or more closed curves drawn in the region R, are the points of R, then region R is said to be multiply - connected region.

e.g :-

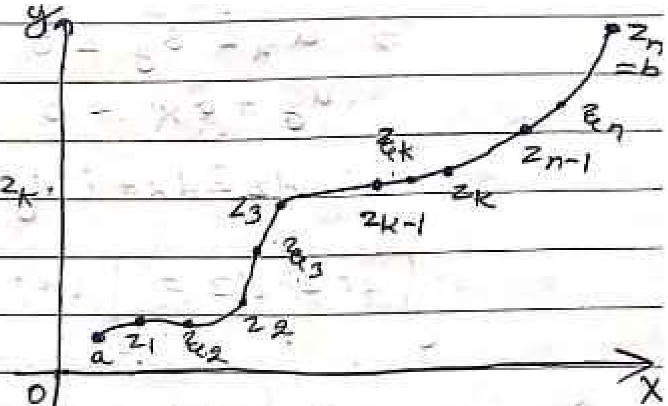
If a region is not simply connected then it is called multiply connected region (domain).

A uniformly convergent series of continuous functions can be integrated by term.

Complex Line Integral :- Suppose f(z) is continuous at every point of a curve C having a finite length, i.e. C is a rectifiable curve.

Divide C into n- Parts by means of points,
$z_0, z_1, z_2, — z_n$

Let $z_0 = a$, $b = z_n$
We choose a point ξ_k on each arc joining z_{k-1} & z_k.
Form the sum,



$$S_n = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

Suppose maximum value of $(z_r - z_{r-1}) \to 0$ as $n \to \infty$
Then, the sum (S_n) tends to a fixed limit which does not depend upon the mode of subdivision & denote this limit by

$$\int_a^b f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz$$

which is called the complex line integral or line integral of $f(z)$ along C.

(4-marks)

Cauchy's Theorem :- If $f(z)$ is analytic function of z and if $f'(z)$ is continuous at each point within and on a closed ~~cou~~ contour C, then $\int_C f(z)\,dz = 0$

Proof :- ~~$f(z) = u+iv$ is analytic function of z and if $f'(z)$ is continuous at each point within and~~

$f(z) = u+iv$ is analytic and so it is continuous on Contour C & also $f'(z)$ exists. It means that u, v, u_x, u_y, v_x, v_y all are continuous in C.

$f(z) = u + iv$ = analytic $\Rightarrow u_x = v_y$,
$u_y = -v_x$

$\Rightarrow u_x - v_y = 0$ —(1)
$u_y + v_x = 0$ —(2)

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Green's Theorem:- If $P(x,y)$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$

all are continuous functions of x & y in a closed contour C, then $\int_C (P\,dx + Q\,dy) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$\Rightarrow$ Using above theorem:-

$$\Rightarrow \iint_C \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_C \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx$$

From (1) & (2),

$$= -\iint_C 0\,dx\,dy + i\iint_C 0\,dx\,dy = 0$$

$$\boxed{\int_C f(z)\,dz = 0}$$

(12 marks)

Cauchy's Theorem:-

If a function $f(z)$ is analytic and single-valued inside and on a closed contour C, then

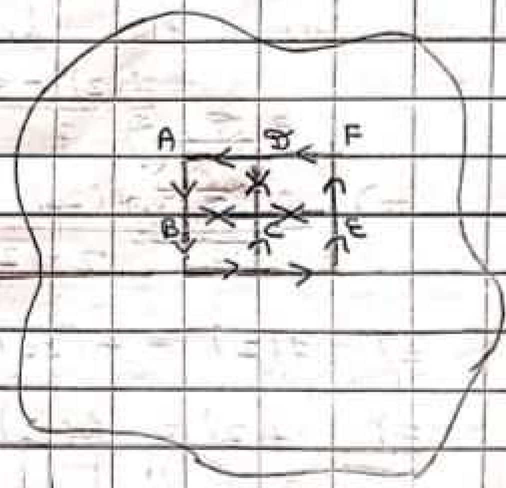
$\int f(z)\,dz = 0$

Proof :- To prove the theorem, we first state two lemmas :-

Lamma(1) :- If c is a closed contour, then
$$\int_c z\,dz = 0 \quad , \quad \int_c dz = 0$$

Lamma(2) :- Goursat's Lemma :- Given $\varepsilon > 0$, it is possible to divide the region inside c into finite no. of meshes either complete squares (c_n) or partial squares (D_m) s.t. within each mesh $\exists$ a point z_0 for which
$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \forall \ |z - z_0| < \delta$$

dxdy.



As in Lemma (2), divide the interior of C into complete squares $c_1, c_2, \ldots, c_n$ & partial squares $D_1, D_2, \ldots, D_m$.

Consider the integral,
$$\sum_{r=1}^{n} \int_{c_r} f(z)\,dz + \sum_{r=1}^{m} \int_{D_r} f(z)\,dz \quad \text{--- (1)}$$

where the path of every integral is in anticlockwise direction.
Consider the squares ABCD & CEFD with common side CD. In the square ABCD,

the side CD is described from C to D, where as in square $CEFD$, it is described from D to C. So the two integrals along the side get cancelled. By parallel reasoning, all the integrals along straight sides of squares cancel.

The integrals which remains are taken along curved boundaries of partial squares because they are described only once.

Clearly some of the remaining integrals gives rise to $\int_C f(z)\,dz$

$$\therefore \int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} f(z)\,dz + \sum_{r=1}^{m} \int_{D_r} f(z)\,dz \qquad (2)$$

Now the lemma (2),

$$\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \eta(z)$$

where $|\eta(z)| < \varepsilon$

$$\Rightarrow \frac{f(z)-f(z_0)-(z-z_0)f'(z_0)}{z-z_0} = \eta(z)$$

$$\Rightarrow f(z) - f(z_0) - (z-z_0)f'(z_0) = (z-z_0)\eta(z)$$

$$\Rightarrow f(z) = f(z_0) + (z-z_0)f'(z_0) + (z-z_0)\eta(z)$$

Put in (2),

So,

$$\int_{C_r} f(z)\,dz = \int_{C_r} \left[f(z_0) + (z-z_0)f'(z_0) + (z-z_0)\eta(z\right.$$

$$= f(z_0)\int_{C_r} dz + f'(z_0)\int_{C_r} z\,dz - z_0 f'(z_0)\int_{C_r} d$$

$$+ \int_{C_r} (z-z_0)\eta(z)\,dz$$

Using lemma (1),

$$\Rightarrow \int_{C_r} f(z)\,dz = \int_{C_r} (z-z_0)\,\eta(z)\,dz$$

$$\Rightarrow \left|\int_{C_r} f(z)\,dz\right| = \left|\int_{C_r} (z-z_0)\,\eta(z)\,dz\right| \le \int_{C_r} |z-z_0|\,|\eta(z)|\,|dz| \quad \text{— (3)}$$

If l_r is the edge of C_r, then $4l_r$ is the length of C_r and distance between z and z_0 is $|z-z_0| < \sqrt{2}\, l_r$

Further, $|\eta(z)| < \varepsilon$

So, from (3),

$$\left|\int_{C_r} f(z)\,dz\right| < (\sqrt{2}\, l_r)\,\varepsilon\,(4 l_r)$$
$$= 4\sqrt{2}\,\varepsilon\, l_r^2 \quad \text{— (4)}$$

In case of partial squares, the length is not greater than $4l_r' + \delta_r$, where δ_r is the length of curved part D_r and l_r' is maximum length of each side of D_r.

$$\left|\int_{D_r} f(z)\,dz\right| < \epsilon\,(\sqrt{2}\, l_r')\,(4l_r' + \delta_r) \quad \text{— (5)}$$

F Equation (2),

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} f(z)\,dz + \sum_{r=1}^{m} \int_{D_r} f(z)\,dz$$

dz

$$\Rightarrow \left|\int_C f(z)\,dz\right| \le \sum_{r=1}^{n} \left|\int_{C_r} f(z)\,dz\right| + \sum_{r=1}^{m} \left|\int_{D_r} f(z)\,dz\right|$$

$$< \sum_{r=1}^{n} 4\sqrt{2}\,\epsilon\, l_r^2 + \sum_{r=1}^{m} \epsilon\,(\sqrt{2}\, l_r')\,(4l_r' + \delta_r)$$

[Using (4) & (5)]

$$= 4\sqrt{2}\,\epsilon \sum_{r=1}^{n} l_r^2 + 4\sqrt{2}\,\epsilon \sum_{r=1}^{m} (l_r')^2 + \sqrt{2}\,\epsilon \sum_{r=1}^{m}$$

$$= 4\sqrt{2}\,\epsilon \left(\sum_{r=1}^{n} l_r^2 + \sum_{r=1}^{m} (l_r')^2 \right) + \sqrt{2}\,\epsilon \sum_{r=1}^{m} l_r' \delta_r$$

$$\boxed{\text{Put } A = \sum_{r=1}^{n} l_r^2 + \sum_{r=1}^{m} (l_r')^2}$$

$$= 4\sqrt{2}\,\epsilon A + \sqrt{2}\,\epsilon \sum_{r=1}^{m} l_r' \delta_r$$

$$= 4\sqrt{2}\,\epsilon A + \sqrt{2}\,\epsilon\, l L$$

[where $\sum l_r' = l$ (some constant length)
and $\sum_{r=1}^{m} \delta_r = L$ (length of the contour)]

$$\Rightarrow \left| \int_C f(z)\,dz \right| < \epsilon \left(4\sqrt{2}\,A + \sqrt{2}\,lL\right)$$

Making $\epsilon \to 0$ as ϵ is arbitrary,

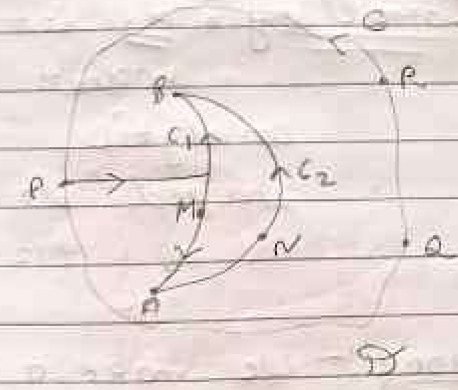
$$\left| \int_C f(z)\,dz \right| = 0$$

$$\Rightarrow \boxed{\int_C f(z)\,dz = 0}$$

Extension of Cauchy's Theorem:-

Corollary:- (1) Suppose $f(z)$ is analytic in a simply connected domain D. Then the integral along any rectifiable curve in D joining any two given points of D is same i.e it does not depend upon the curve joining the two curve

Sn **Proof:-** Suppose that the two points $A(z_1)$ & $B(z_2)$ of simply connected domain D are joined by curves c_1 & c_2.



Then by Cauchy theorem,

$$\int_{PQRP} f(z)\,dz = 0 \quad \text{——— (1)}$$

$$\Rightarrow \int_{PMANBMPQRP} f(z)\,dz = 0$$

$$\Rightarrow \int_{PM} f(z)\,dz + \int_{MANBM} f(z)\,dz + \int_{MP} f(z)\,dz + \int_{PQRP} f(z)\,dz = 0$$

$$\Rightarrow \int_{PM} \cancel{f(z)dz} + \int_{MANBM} f(z)\,dz - \int_{PM} \cancel{f(z)dz} + \underbrace{\int_{C} f(z)\,dz}_{0 \rightarrow \text{From (1)}} = 0$$

$$\Rightarrow \int_{MANBM} f(z)\,dz = 0$$

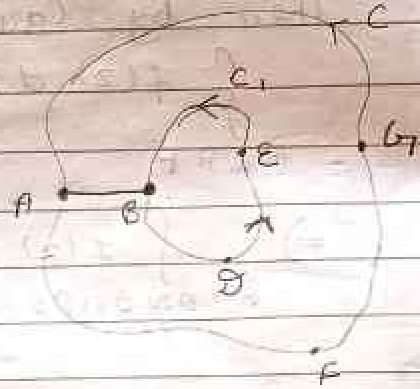
$$\Rightarrow \int_{ANB} f(z)\,dz + \int_{BMA} f(z)\,dz = 0$$

$$\Rightarrow \int_{c_2} f(z)\,dz + \int_{c_1} f(z)\,dz = 0$$

$$\Rightarrow \boxed{\int_{c_2} f(z)\,dz = \int_{c_1} f(z)\,dz}$$

Corollary :- 2. Let a closed contour C contain another closed contour C_1 and $f(z)$ be analytic at every point lying in the ring shaped domain bounded by C and C_1. Then
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

Proof :- We make a cross at joining a point A of contour C to a point B on C_1.



Then by Cauchy theorem,
$$\cancel{\int_{AFGA} f(z)\,dz = 0}$$
$$\cancel{\int_{ABEDBAFGA} f(z)\,dz = 0} \qquad \int_{AFGABEDBA} f(z)\,dz = 0$$
$$\Rightarrow \int_{AFGA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BEDB} f(z)\,dz + \int_{BA} f(z)\,dz$$
$$\Rightarrow \int_C f(z)\,dz + \cancel{\int_{AB} f(z)\,dz} - \int_{C_1} f(z)\,dz - \cancel{\int_{AB} f(z)\,dz}$$
$$\Rightarrow \int_C f(z)\,dz - \int_{C_1} f(z)\,dz = 0$$
$$\Rightarrow \boxed{\int_C f(z)\,dz = \int_{C_1} f(z)\,dz}$$

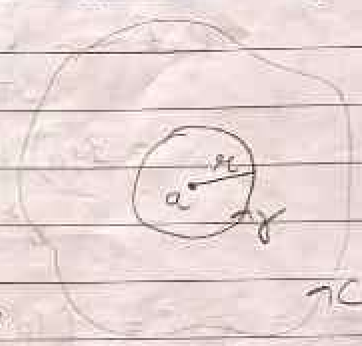
Deduction :- If the contour C contains non-intersecting contours $C_1, C_2, \ldots, C_n$ and $f(z)$ is analytic at every point in the region bounded by $C, C_1, -$

then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \text{---} + \int_{C_n} f(z)\,dz$$

Cauchy Integral Formula :- If $f(z)$ is analytic within and on a closed contour C and 'a' is any point within C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

Proof :- Suppose $f(z)$ is analytic within and on a closed contour C & 'a' is interior point of C.
Describe a circle γ (gamma) about the centre $z=a$ of small radius r.
$|z-a| = r$ does not intersect the curve C.



The f^n :- $F(z) = \frac{f(z)}{z-a}$ is analytic in the annulus bounded by C & γ.
Then by cauchy theorem and corollary of cauchy theorem,

$$\int_C F(z)\,dz = \int_\gamma F(z)\,dz$$

$$\Rightarrow \int_C \frac{f(z)\,dz}{z-a} = \int_\gamma \frac{f(z)\,dz}{(z-a)}$$

$$= \int_\gamma \left[\frac{f(z) - f(a) + f(a)}{(z-a)}\right] dz$$

$$\Rightarrow \int_C \frac{f(z)\,dz}{z-a} = f(a)\int_\gamma \frac{dz}{z-a} + \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz \quad \text{—— (1)}$$

Since, $f(z)$ is analytic within C, it is continuous at $z=a$.

So given $\epsilon > 0$, $\exists$ some $\delta > 0$ s.t.

$$|f(z) - f(a)| < \epsilon \quad \forall \ |z-a| < \delta = r$$

Since r is at ~~our~~ our choice,

$$|z-a| = r < \delta$$

Put $z - a = e^{i\theta}$

$$dz = i e^{i\theta} d\theta$$

Put in (1),

$$= f(a) \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} + \epsilon \int_0^{2\pi} \frac{i e^{i\theta}}{e^{i\theta}} d\theta$$

$$= f(a) \, i \, 2\pi + \underbrace{\epsilon \cdot (2\pi i)}_{=0}$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{z-a}}$$

Cauchy integral formula for the derivative of an analytic function:-

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{z-a}$$

$$f'(a) = \frac{1!}{2\pi i} \int_C \frac{f(z) \, dz}{(z-a)^2}$$

$$| \qquad | \qquad |$$

$$f^n(a) = \frac{n!}{2\pi i} \int \frac{f(z) \, dz}{(z-a)^{n+1}}$$

$$\Rightarrow f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-(a+h)} - \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)} \right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \frac{1}{2\pi i}\int_C f(z)\,dz \left[\frac{1}{z-(a+h)} - \frac{1}{z-a} \right] \right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)} \left[\frac{1}{\left(1-\frac{h}{z-a}\right)} - 1 \right] \right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} \left[\left(1-\frac{h}{(z-a)}\right)^{-1} - 1 \right] \right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} \left(1+\frac{h}{z-a} + \frac{h^2}{(z-a)^2} + \cdots - 1 \right) \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} \left(\frac{1}{z-a} + \frac{h}{(z-a)^2} + \cdots \right) \right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2}$$

$$\boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2}}$$

Cauchy integral formula for 2^{nd} derivative of an ~~al~~ analytic function:-

$$\boxed{f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^3}}$$

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h}$$

$\because \overline{a+h} = (a+h)$
(small bracket)

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-\overline{a+h})^2} - \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{2\pi i}\int_C f(z)dz\left[\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right]\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}\left\{\frac{1}{\left(1-\frac{h}{z-a}\right)^2} - 1\right\}\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2}\lim_{h\to 0}\frac{1}{h}\left\{\left(1-\frac{h}{z-a}\right)^{-2} - 1\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}\lim_{h\to 0}\frac{1}{h}\left\{1+2\frac{h}{z-a}+2\cdot 3\left(\frac{h}{z-a}\right)^2+\dots\right.$$

$$\boxed{f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^3}}$$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \quad \text{——} (*)$$

Let the result $(*)$ is true for $n=k$, then

$$f^k(a) = \frac{k!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{k+1}}$$

Put $n = k+1$ ~~in $(*)$~~,

$$f^{k+1}(a) = \lim_{h\to 0}\frac{f^k(a+h) - f^k(a)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left\{\frac{k!}{2\pi i}\left[\int_C \frac{f(z)dz}{(z-a-h)^{k+1}} - \frac{f(z)dz}{(z-a)^{k+1}}\right]\right\}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{k!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{k+1}}\left\{\left(1-\frac{h}{z-a}\right)^{-(k+1)} - 1\right\}\right]$$

$(k+1)! = (k+1)\,k!$

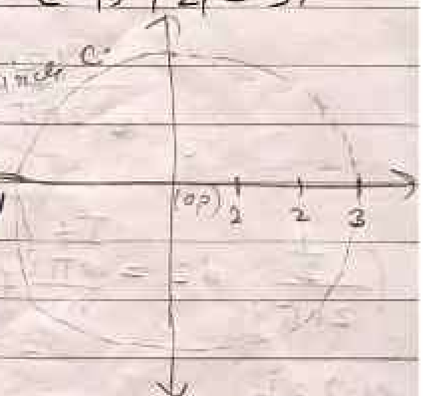
$$= \lim_{h\to 0}\left\{\frac{k!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{k+1}}\left[1 + \frac{\overline{k+1}\,h}{z-a} + \frac{(k+1)(k+2)h^2}{(z-a)^2} + \cdots\right]\right.$$

$$= \frac{(k+1)!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{k+2}}$$

$$\boxed{f^{k+1}(a) = \frac{(k+1)!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{k+2}}}$$

Ques:- Evaluate $\int_C \frac{e^{2z}\,dz}{(z+1)^4}$, where C is $|z| = 3$.

$z=-1$ lies in inside the circle C



Sol^n:- We know,

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow \int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

Compare both, we have

$a = -1$, $n = 3$, $f(z) = e^{2z}$, $f'''(z) = 8e^{2z}$, $f'''(-1) = 8/e^2$

$$\int_C \frac{e^{2z}\,dz}{(z+1)^4} = \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{6} \times \frac{8}{e^2} = \frac{8\pi i}{3e^2} \text{ Ans}$$

Ques:- Evaluate $\int_C \frac{1}{z(z+\pi i)}\,dz$ where C is $|z+3i| = 1$

Sol^n:- The point $z = -\pi i$ (3.14) lies inside C, & $f(z)$ is not analytic within and on C.

$$\Rightarrow \frac{1}{z(z+\pi i)} = \frac{A}{z} + \frac{B}{z+\pi i} \quad \text{---(1)}$$

$1 = A(z+\pi i) + B(z)$

Put $z=0$, $A(\pi i) = 1$

$$A = \frac{1}{\pi i}$$

Put $z = -\pi i$

$1 = 0 + B(-\pi i)$

$$B = -\frac{1}{\pi i}$$

Value of A & B put in ①, we get

$$\frac{1}{z(z+\pi i)} = \frac{1}{\pi i}\cdot\frac{1}{z} + \left(\frac{-1}{\pi i}\right)\cdot\frac{1}{z+\pi i}$$

$$\int_C \frac{1/z}{(z+\pi i)}\,dz = \frac{1}{\pi i}\left[\int_C \frac{dz}{z} - \int_C \frac{dz}{(z+\pi i)}\right]$$

It is a analytic so $\int_C f(z)\,dz = 0$

$$\int_C \frac{\frac{1}{z}}{z+\pi i}\,dz = \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{1}{\pi i}\int_C \frac{dz}{(z+\pi i)} \rightarrow$$

$f(z) = \frac{1}{z}$

$a = -\pi i$, $n = 0$

$$= \frac{2\pi i}{0!} f(-\pi i)$$

$$= 2\pi i \times \frac{1}{-\pi i}$$

$$= -2 \text{ Ans}$$

$$\int_C \frac{1}{z(z+\pi i)}\,dz = \frac{1}{\pi i}\int_C \frac{dz}{z+\pi i}$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$$

$$f(-\pi i) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z+\pi i)}$$

Sol^n :- $\displaystyle\int_C \frac{dz}{z(z+\pi i)} = I$ let

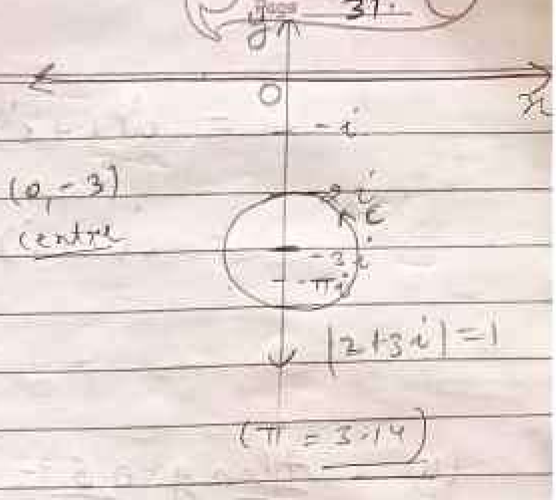
(centre $(0,1)$), $z = 0, -\pi i$

$n = 0$, $a = -\pi i$, $f(z)$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\int_C \frac{\frac{dz}{z}}{(z+\pi i)} = \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{2\pi i}{0!} \times f(-\pi i)$$



$$= 2\pi i \times \frac{1}{-\pi i}$$

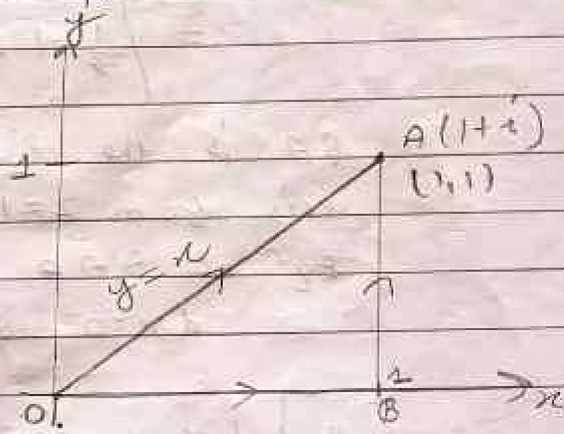
$$= -2$$

$$\int_C \frac{dz}{z(z+\pi i)} = -2$$

Ques:- Find the value of the integral,

$$\int_0^{1+i} (x - y + ix^2)\, dz$$

(i) along the straight line from $z=0$ to $z=1+i$
(ii) along the real axis from $z=0$ to $z=1$ and then parallel to imaginary axis from $z=1$ to $z=1+i$.



Sol^n:- $z = x + iy$
so, $dz = dx + i\,dy$

(i) Along OA :- OA is the line joining $z=0$ to $z=1+i$
Eq^n of OA is $y = x$
So, $dy = dx$ on OA.

$$\int_{OA} (x - y + ix^2)\, dz = \int_{OA} (x - y + ix^2)(dx + i\,dy)$$

$$= \int_0^1 (\cancel{x} - \cancel{x} + ix^2)(dx + i\,dx)$$

$$= \int_0^1 (ix^2\,dx + i^2 x^2\,dx)$$

$$= \int_0^1 ix^2\,dx(1+i)$$

$$= i(1+i)\int_0^1 x^2\,dx$$

$$= i(1+i)\left[\frac{x^3}{3}\right]_0^1$$

$$= i(1+i)\left[\frac{1}{3}-0\right] = \frac{i(1+i)}{3} = \frac{i+i^2}{3}$$

$$= \frac{i-1}{3}$$

(ii) **Along OB :-** OB is real axis from $z=0$ to $z=1$
Eq^n of OB is $y=0$, so that $z=x$
$dz = dx$ on OB.

$$\therefore \int_{OB} (x-y+ix^2)\,dz = \int_0^1 (x+ix^2)\,dx$$

$$= \left|\frac{x^2}{2} + i\frac{x^3}{3}\right|_0^1$$

$$= \left(\frac{1}{2} + \frac{i}{3} - 0\right) = \frac{1}{2} + \frac{i}{3} \quad \text{--- } (*)$$

BA is the line parallel to imaginary axis from $z=1$ to $z=1+i$
Eq^n of BA is $x=1$, so that
$dx=0$ & $dz = i\,dy$ on BA

$$\therefore \int_{BA} (x-y+ix^2)\,dz = \int_0^1 (1-y+i)\,i\,dy$$

$$= (1+i)\,i\int_0^1 dy - i\int_0^1 y\,dy$$

$$= (1+i)\,i\,[y]_0^1 - i\left[\frac{y^2}{2}\right]_0^1$$

$$= (1+i)\,i\,[1] - i\left[\frac{1}{2}\right]$$

$$= (1+i)\,i - \frac{i}{2} = i-1-\frac{i}{2}$$

$$= \frac{i}{2} - 1 \quad \text{--- } (**)$$

Hence,

$$\int_{OBA} (x-y+ix^2)\,dz = \int_{OB} (x-y+ix^2)\,dz + \int_{BA} (x-y+ix^2)\,dz$$

From eq^n (*) & (**), we have

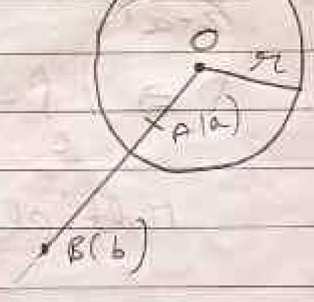
$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1 = \frac{-1}{2} + \frac{5i}{6}$$

$$\int_{OBA} (x-y+ix^2)\,dz = -\frac{1}{2} + \frac{5i}{6}$$

Inverse points w.r.t a circle :- Two points A(a) and B(b) are said to be inverse points w.r.t a circle with centre O & radius r if O, A, B are collinear and $OA \cdot OB = r^2$.

$\Rightarrow$ z_1, z_2 are inverse points w.r.t. to the circle $|z| = r$ if

$z_1 \bar{z}_2 = r^2$



Poisson's Integral formula :-

If $f(z)$ is analytic within and on a closed contour C (circle) or $\gamma(R^2)$ defined by $|z| = R$. Then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{(R^2 - a\bar{a})\, f(z)\, dz}{(z-a)(R^2 - \bar{a} z)}$$

Hence, Deduce the poisson's integral formula,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, d\phi$$

where $a = re^{i\theta}$, with in the circle $|z| = R$.

Proof:- Suppose $f(z)$ is analytic with in & on the circle C, defined by $|z| = R$ & $a = re^{i\theta}$ is a point A inside C.

~~Show~~ so that :- $0 < r < R$

The inverse point $A'(a')$ of $A(a)$ w.r.t the circle C is given by, $a' = \frac{R^2}{a} = \frac{R^2}{\bar{a}}$ which lies outside the circle C.

$\because |z| = R$
$|z|^2 = R^2$, $z^2 = R$
$z \cdot \bar{z} = R^2$
$\bar{a} \cdot a' = R^2$

By Cauchy integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz \quad \text{— (1)}$$

Since $f(z)$ is analytic with in & on a circle

$\Rightarrow \frac{f(z)}{z-a'}$ is also analytic with in & on a circle

$$\Rightarrow \int_C \frac{f(z)\, dz}{(z-a')} = 0 \quad \text{— (2)}$$ (By Cauchy theorem)

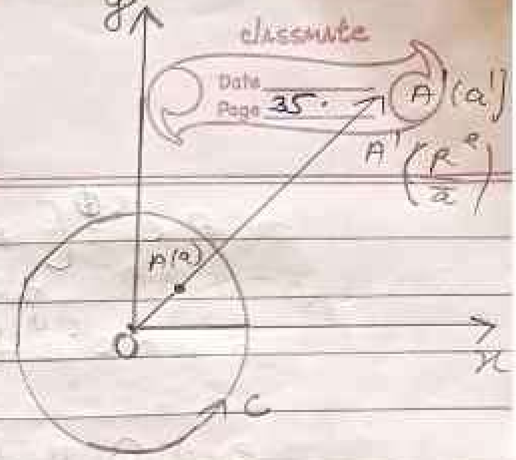
Multiply by $\frac{1}{2\pi i}$ on both sides

$$\frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{(z-a')} = 0 \quad \text{— (2)}$$

Subtracting (2) from (1), we get

$$f(a) - 0 = \frac{1}{2\pi i}\left\{\int_C \frac{f(z)\, dz}{z-a} - \int_C \frac{f(z)\, dz}{(z-a')}\right\}$$

$$= \frac{1}{2\pi i}\left\{\int_C \frac{f(z)\, dz\, (a-a')}{(z-a)(z-a')}\right\}$$



Using $a' = \frac{R^2}{\bar{a}}$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{\left(a - \frac{R^2}{\bar{a}}\right) f(z)\,dz}{(z-a)\left(z - \frac{R^2}{\bar{a}}\right)}$$

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2 - a\bar{a})\, f(z)\,dz}{(z-a)\,(R^2 - z\bar{a})}} \quad \text{--- } (3)$$

It's is a represented poisson Integral formula.

$\Rightarrow$ For the 2^{nd} part :-

Any point z on the circle C i.e $|z| = R$

$$z = Re^{i\phi}, \quad a = re^{i\theta}$$

$$a\bar{a} = r^2, \quad \bar{a} = re^{-i\theta}$$

$$dz = iRe^{i\phi}\,d\phi$$

$\therefore R^2 - a\bar{a} = R^2 - |a|^2 = R^2 - r^2$ --- (4)

$$(z-a)(R^2 - \bar{a}z) = \left(Re^{i\phi} - re^{i\theta}\right)\left(R^2 - re^{-i\theta}\cdot Re^{i\phi}\right)$$

$$= Re^{i\phi}\left(R - re^{i(\theta-\phi)}\right)\left(R - re^{-i(\theta-\phi)}\right)$$

$$= Re^{i\phi}\left(R^2 - 2rR\cos(\theta-\phi) + r^2\right)$$

$$\boxed{\therefore e^{i(\theta-\phi)} = \cos(\theta-\phi) + i\sin(\theta-\phi)}$$

$$= Re^{i\phi}\left[R - r\cos(\theta-\phi) - ir\sin(\theta-\phi)\right]\left[R - r\cos(\theta-\phi) + ir\sin(\theta-\phi)\right]$$

$$= Re^{i\phi}\Big[R^2 - rR\cos(\theta-\phi) + \cancel{irR\sin(\theta-\phi)} - rR\cos(\theta-\phi) + r^2\cos^2(\theta-\phi) - \cancel{ir^2\cos(\theta-\phi)\sin(\theta-\phi)} - \cancel{irR\sin(\theta-\phi)} + \cancel{ir^2\cos(\theta-\phi)\sin(\theta-\phi)} - (ir)^2\sin^2(\theta-\phi)\Big]$$

$$= Re^{i\phi}[R^2 - 2rR\cos(\theta-\phi) + r^2(\cos^2(\theta-\phi) + \ldots$$

$$= Re^{i\phi}[R^2 - 2rR\cos(\theta-\phi) + r^2] \quad \text{--- (5)}$$

eq^n (4) & (5) in (2), we get,

$$f(re^{i\theta}) = f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})\, iRe^{i\phi}\, d\phi}{Re^{i\phi}[R^2 + r^2 - 2rR\cos(\theta-\phi)]}$$

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})\, d\phi}{R^2 - 2rR\cos(\theta-\phi) + r^2}$$

Hence proved.

$$\Rightarrow \quad f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

$$f(Re^{i\phi}) = u(R,\phi) + i\,v(R,\phi)$$

Real,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2}\, u(R,\phi)\, d\phi$$

Imaginary parts,

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2}\, v(R,\phi)\, d\phi$$

Morera's Theorem (Converse of Cauchy theorem)

If $f(z)$ is continuous function in a domain D and C is any closed contour with in D

$$\int_C f(z)\, dz = 0$$

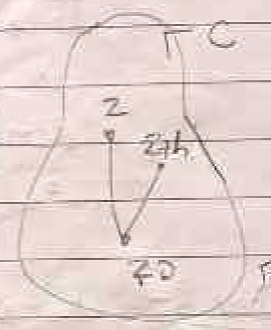
then prove that $f(z)$ is analytic. ~~within D~~

(a-ϕ)] **Proof** :- Let z_0 be a fixed point & z is a variable point inside the domain $\mathcal{D}$; Then the value of the integral $\int_{z_0}^{z} f(t)\,dt$ is independent of the curve joining z_0 to z & depends on z only.

$$F(z) = \int_{z_0}^{z} f(t)\,dt \quad \text{—} ①$$

$$F(z+h) = \int_{z_0}^{z+h} f(t)\,dt \quad \text{—} ②$$

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(t)\,dt - \int_{z_0}^{z} f(t)\,dt$$



$$= \int_{z_0}^{z+h} f(t)\,dt + \int_{z}^{z_0} f(t)\,dt$$

$$F(z+h) - F(z) = \int_{z}^{z+h} f(t)\,dt$$

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(t)\,dt$$

$$\Rightarrow \frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(t)\,dt - f(z)$$

$$= \frac{1}{h}\int_{z}^{z+h} \left(f(t) - f(z)\right) dt$$

$$\Rightarrow \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \le \frac{1}{|h|}\int_{z}^{z+h} |f(t) - f(z)|\,|dt|$$

Since $f(z)$ is continuous at z, given $\varepsilon > 0$, there exists a $\delta > 0$ s.t.

$|f(t) - f(z)| < \varepsilon$ whenever $|t - z| < \delta$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \frac{1}{|h|} \cdot \epsilon |h| < \varepsilon$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \varepsilon$$

Thus $\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} - f(z) = 0$

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

$$\boxed{F'(z) = f(z)}$$

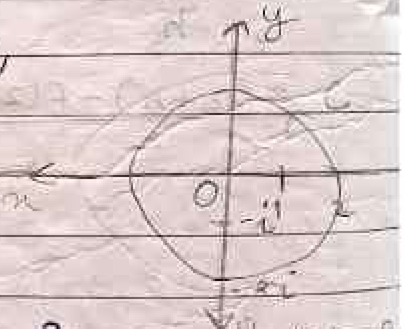
Thus, the derivative of $F(z)$ exists and so, $F(z)$ is analytic in D. But we know that the derivative of analytic function is analytic.

$\therefore$ $F'(z)$ i.e $f(z)$ is analytic in D.

Ques :- Using Cauchy integral formula, evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$

$z = -i, 3, -3$, 3 analytic

Sol^n :- By Cauchy integral formula,



$$\Rightarrow \quad f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Compare both, we get

$n = 0$, $a = -i$, $f(z) = \frac{z}{9-z^2}$

$$\int_C \frac{\frac{z}{9-z^2}\,dz}{(z+i)^1} = \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{2\pi i}{0!} \times f(-i)$$

$$= \frac{2\pi i}{1} \times \frac{(-i)}{(9+1)} = -\frac{2\pi i^2}{10}$$

$= \ldots \pi$ Ans

Ques:- Evaluate $\int_C \frac{e^{2z}}{(z-\pi i)} dz$, where C is the ellipse $|z-2|+|z+2|=6$

Ans:- Eq^n of ellipse is,

$$|z-2|+|z+2|=6$$

Substituting, $z = x+iy$,

$$|x+iy-2|+|x+iy+2|=6$$

$$\sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$$

$$\sqrt{(x-2)^2+y^2} = 6-\sqrt{(x+2)^2+y^2}$$

Squaring both sides, we get

$$\cancel{x^2}+\cancel{y^2}+4-4x = 36+\cancel{x^2}+\cancel{y^2}+4x+4-12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow \quad 12\sqrt{(x+2)^2+y^2} = 8x+36$$

$$3\sqrt{(x+2)^2+y^2} = 2x+9$$

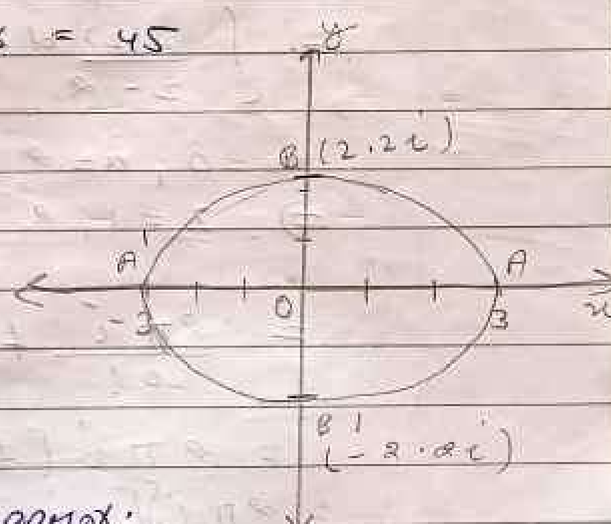
$$9(x^2+4+4x+y^2) = 4x^2+81+36x$$

$$9x^2+36+\cancel{36x}+9y^2-4x^2-\cancel{36x}=81$$

$$5x^2+9y^2 = 81-36 = 45$$

$$\frac{5x^2}{45}+\frac{9y^2}{45} = \frac{45}{45}$$

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$



Comparing, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$a^2 = 9, \quad b^2 = 5$$

$$a = 3, \quad b = \sqrt{5} = 2.2 \text{ approx.}$$

The point $z = \pi i = 3.14i$ lies outside C.

So, $\frac{e^{2z}}{z-\pi i}$ is analytic within and on C.

Hence by, Cauchy theorem,

$$\int_C \frac{e^{\alpha z}}{z-\pi i}\, dz = 0$$

Ques: Evaluate by Cauchy's integral formula:

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz$$

where c is circle $|z| = 3$

Ans: Let $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

Then $1 = A(z-2) + B(z-1)$

Put $z = 1$, $A = -1$

$z = 2$, $B = 1$

$$\boxed{\frac{2\pi i}{n!} f^n(a) = \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}}$$

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz$$

$f(z) = \sin(\pi z^2) + \cos(\pi z^2)$

$$= \int_C \frac{f(z)}{z-2}\, dz - \int_C \frac{f(z)}{z-1}\, dz$$

$n = 0, a = 2$, $\quad n = 0, a = 1$

$\Rightarrow z = 1, 2$ both lies inside the c.

$$= \frac{2\pi i}{0!} f(2) - \frac{2\pi i}{0!} f(1)$$

$$= 2\pi i [f(2) - f(1)]$$

$$= 2\pi i \{[\sin(\pi \cdot 2^2) + \cos(\pi \cdot 2^2)] - [\sin(\pi \cdot 1) + \cos(\pi \cdot 1)]\}$$

$$= 2\pi i [(0 + 1) - (0 - 1)]$$

$$= 2\pi i [1 + 1]$$

Ques:- Evaluate, $\int_C \frac{\tan(z/2)\,dz}{(z-x_0)^2}$

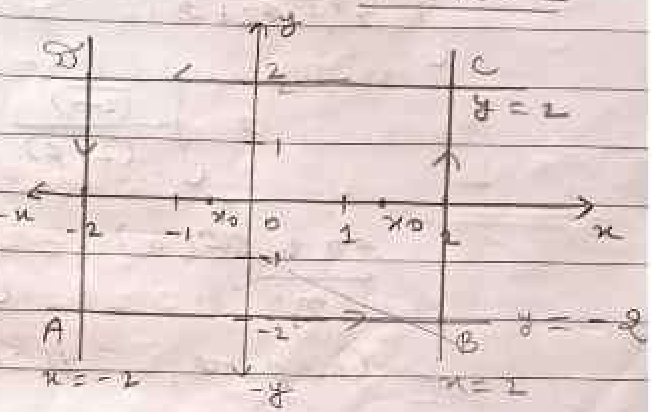
where C is the boundary of the square whose sides lie along the lines $x=\pm 2$, $y=\pm 2$ & it is described in positive sense, where $|x_0|<2$.

$-2 < x_0 < 2$

Sol^n:-

$$I = \int_C \frac{\tan(z/2)\,dz}{(z-x_0)^2}$$

where C is rectangle $ABCD$ & point $z = x_0$ lies within C.



$$\Rightarrow \int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

Let $f(z) = \tan\frac{z}{2}$, $a = x_0$, $n=1$

$$I = \frac{2\pi i}{1!} f'(x_0) = 2\pi i\, f'(x_0)$$

$$= 2\pi i \left\{\frac{d}{dz}\left(\tan\frac{z}{2}\right)\right\}_{z=x_0}$$

$$= 2\pi i \times \frac{1}{2}\left[\sec^2\frac{z}{2}\right]_{z=x_0}$$

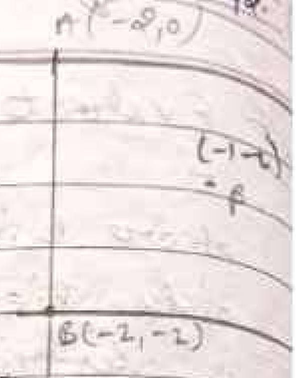
$$= \pi i \times \sec^2\left(\frac{x_0}{2}\right) = \pi i \sec^2\left(\frac{x_0}{2}\right)$$

Ques:- Evaluate $\int_C \frac{dz}{z^2+2z+2}$, where C is the square having vertices at $(0,0)$, $(-2,0)$, $(-2,-2)$, $(0,-2)$ oriented in anticlockwise direction.

Sol^n:- Here C is the square $OABC$.

Now, $z^2+2z+2=0$ gives

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$$



Thus, $z = -1+i, \; -1-i$

let $\alpha = -1+i$, & $\beta = -1-i$

The point $z=\alpha$ lies outside the C & $z=\beta$ lies inside C.

$$\int_C \frac{1}{z^2+2z+2}\,dz = \int_C \frac{1}{(z-\alpha)(z-\beta)}\,dz$$

$$= \int_C \frac{\frac{1}{(z-\alpha)}}{(z-\beta)}\,dz$$

$$\frac{2\pi i}{n!} f^n(a) = \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

Compare, $f(z)=\frac{1}{z-\alpha}$, $a=\beta$, $n=0$

$$= \frac{2\pi i}{0!} f(\beta)$$

$$= 2\pi i \times \frac{1}{\beta-\alpha} = \frac{2\pi i}{\beta-\alpha}$$

$$= \frac{2\pi i}{-1-i+1-i} = \frac{2\pi i}{-2i} = -\pi \quad \text{Ans}$$

Evaluate $\int_C \frac{\sin z\,dz}{(z-\frac{\pi}{4})^3}$, where C is $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$

Let $I = \int_C \frac{\sin z\,dz}{(z-\frac{\pi}{4})^3}$, $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$

$$\frac{2\pi i}{n!} f^n(a) = \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$f(z) = \sin z$, $a = \frac{\pi}{4}$, $n=2$

$$I = \frac{2\pi i}{2!} f^2\left(\frac{\pi}{4}\right)$$

$$= 2\pi i \; f^2($$

$f(z) = \sin z, \quad f'(z) = \cos z, \quad f''(z) = -\sin z$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$= \pi i \left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi i}{\sqrt{2}} \text{ Ans}$$

Ques 2: If C is unit circle about the origin, described in positive sense, show that

a) $\int_C \frac{e^{-z}}{z^2}\, dz = -2\pi i$

$\Rightarrow$ let $I = \int_C \frac{e^{-z}}{z^2}\, dz$, where C is circle $|z| = 1$
(centre - (0,0))
(radius - 1)

$$\frac{2\pi i}{n!} f^n(a) = \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}$$

$f(z) = e^{-z}, \quad a = 0, \quad n = 1$, $\boxed{f'(z) = -e^{-z}, \quad f'(0) = -e^{-0} = -1}$

as $z = 0$ lies inside the C.

$= -\frac{1}{e^0} = -\frac{1}{1} = -1$

$$I = \frac{2\pi i}{1!} f'(0)$$

$$= 2\pi i(-1) = -2\pi i \text{ Ans}$$

$\int_C \left(\frac{\sin z}{z}\right) dz = 0$

$\Rightarrow$ Let $I = \int_C \left(\frac{\sin z}{z}\right) dz$, where $|z| = 1$

$$\frac{2\pi i}{n!} f^n(a) = \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}$$

$f(z) = \sin z, \quad a = 0, \quad n = 0, \quad f(0) = \sin(0) = 0$

$$I = \frac{2\pi i}{0!} f(a) = 2\pi i\, f(0)$$

$$= 2\pi i \times 0$$

$$= 0 \text{ Ans}$$

Theorem :-

A necessary and sufficient condit for a function $f(z)$ to possess an indefinite integral in a simply connected domain D is that the function is analytic in D. Further, any two indefinite integrals differ by a constan

Solⁿ :- Let $f(z)$ possess an indefinite integral $F(z)$ so that

$$F(z) = \int_a^z f(t)\,dt \quad \text{——— (1)}$$

To prove that $f(z)$ is analytic.
By Morera's theorem,

$$F'(z) = f(z).$$

Showing thereby $F(z)$ possess a derivative $f(z)$. Also the derivative of an analytic function is analytic. It follows that $f(z)$ is analytic.

Conversely :- Suppose that $f(z)$ is analytic in a domain D. To prove that $f(z)$ possesses an indefinite integral. Let z_0 be any fixed poin and z an arbitrary point in D.

$$F(z) = \int_{z_0}^{z} f(t)\,dt \quad \text{——— (2)}$$

For the integral of $f(z)$ along any curve in D joining z_0 to z is the same. Prove as in theorem that $F'(z) = f(z)$.
This proves that $f(z)$ possesses indefinite integr given by (2).

Second part :- Let $F(G)$ and $G(z)$ be two indefinite integrals of the same function $\cancel{\epsilon}$ $f(z)$.

Then, $F'(z) = f(z) = G'(z)$

This $\Rightarrow F'(z) = G'(z) \Rightarrow \frac{d}{dz}(F-G) = 0$

Integrating, we get,
$$F - G = \text{constant}.$$

This completes the proof.

Theorem :- **Fundamental theorem** :- Let $f(z)$ be single valued analytic function in a simple connected domain D. If a, b are two points $\ni a, b \in D$, then
$$\int_a^b f(z)\,dz = F(b) - F(a),$$ where $F(z)$ is an ~~analytic~~ indefinite integral of $f(z)$.

Proof :- By definition of indefinite integral,
$$F(z) = \int_{z_0}^{z} f(t)\,dt$$
$$F(b) - F(a) = \int_{z_0}^{b} f(t)\,dt - \int_{z_0}^{a} f(t)\,dt$$
$$= \int_{z_0}^{b} f(t)\,dt + \int_{a}^{z_0} f(t)\,dt$$
$$= \int_a^b f(t)\,dt$$

$$\boxed{F(b) - F(a) = \int_a^b f(z)dz}$$ Hence, the Fundamental theorem.

Theorem :- Cauchy's inequality :- If $f(z)$ is analytic within and on a circle C, given by $|z-a| = R$ and if $|f(z)| \le M$ for every z on C, then

$$|f^n(a)| \le \frac{M\, n!}{R^n}$$

Proof :- Given that, $|z-a| = R$

$$\Rightarrow z-a = Re^{i\theta} \Rightarrow z = Re^{i\theta} + a$$

$$dz = iRe^{i\theta}\, d\theta \Rightarrow |dz| = R\, d\theta$$

$$\because |i| = 1$$

we know that, $f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}$

$$|f^n(a)| \le \frac{n!}{2\pi} \int_C \frac{|f(z)|\,|dz|}{|z-a|^{n+1}}$$

$$\le \frac{M n!}{2\pi} \int_0^{2\pi} \frac{R\, d\theta}{R^{n+1}} \le \frac{M n!}{2\pi R^n} \int_0^{2\pi} d\theta$$

$$\le \frac{M n!}{R^n \cancel{2\pi}} (\cancel{2\pi}) \le \frac{M n!}{R^n}$$

$$\boxed{|f^n(a)| \le \frac{M n!}{R^n}}$$

Definition :- Entire function :- A function f(z) is called an integral function or entire function if it is analytic in every finite region.

Liouville's theorem :- If an entire function is bounded for all values of z, then it is constant.

If a function $f(z)$ is analytic for finite values of z, and ~~it~~ is bounded, then $f(z)$ is constant.

OR

If f is regular in whole z-plane and if $|f(z)| < k \ \forall z$, then $f(z)$ must be constant.

Proof :- Let a & b be arbitrary distinct points in z-plane and let C be a large circle with centre $z=0$ and radius R such that C encloses a & b.

Equation of C is $|z| = R$

so that $z = Re^{i\theta}$, $dz = iRe^{i\theta}$

$$|dz| = R\,d\theta$$

It is given in statement, $f(z)$ is bounded $\forall z$

$\Rightarrow |f(z)| \leq M \ \forall z$ where $M > 0$.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)}, \quad f(b) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-b)}$$

$$f(a) - f(b) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) f(z)\,dz$$

$$= \left(\frac{a-b}{2\pi i}\right)\int_C \frac{f(z)\,dz}{(z-a)(z-b)}$$

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi}\int_C \frac{|f(z)|\cdot|dz|}{(|z|-|a|)(|z|-|b|)}$$

$$\leq \frac{M\,|a-b|\cdot 2\pi R}{2\pi(R-|a|)(R-|b|)}$$

$\leq \text{''} \int_0^{2\pi} |dz|$

$\leq \text{''} \int_0^{2\pi} R\,d\theta$

$\leq R[2\pi]$

$$|f(a) - f(b)| \leq \frac{MR\,|a-b|}{(R-|a|)(R-|b|)} \to 0 \text{ as } R \to \infty$$

$\therefore \quad f(a) - f(b) = 0$

$f(a) = f(b)$

Showing thereby $f(z)$ is constant.

Taylor's theorem:- If a function $f(z)$ is analytic within a circle C with its centre $z=a$ and radius R, then at every point z inside C,

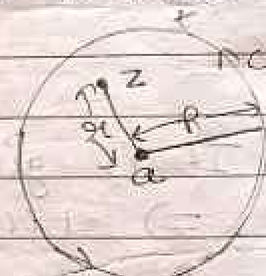
$$f(z) = \sum_{n=0}^{\infty} f^n(a) \frac{(z-a)^n}{n!}$$

OR

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where} \quad a_n = \frac{f^n(a)}{n!}$$

Proof:- Let $f(t)$ be analytic within a circle C whose eq^n is $|t-a| = R$. Let z be any point within C s.t $|z-a| = r < R$

$$\left[\frac{r}{R} < 1\right]$$



By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z} = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)-(z-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)}\left\{1 - \left(\frac{z-a}{t-a}\right)\right\}^{-1}$$

Using binomial,

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + \cancel{\frac{x}{1-x}} \; \frac{x^n}{(1-x)}$$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)}\left\{1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \left(\frac{z-a}{t-a}\right)^3 + \cdots + \frac{\left(\frac{z-a}{t-a}\right)^n}{\left\{1 - \frac{z-a}{t-a}\right\}}\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)} + \frac{(z-a)}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^2} + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)^3}$$

Page 49.

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + U_{n+1} \quad \text{—— (1)}$$

$$U_{n+1} = \left(\frac{z-a}{t-a}\right)^n \left(1 - \left(\frac{z-a}{t-a}\right)\right)^{-1}$$

$$= \left(\frac{z-a}{t-a}\right)^n \left(\frac{t-\cancel{a}-z+\cancel{a}}{t-a}\right)^{-1} = \boxed{\frac{(z-a)^n}{(t-a)^{n+1}} \left(\frac{1}{t-z}\right)}$$

$$U_{n+1} = \left(\frac{z-a}{t-a}\right)^n \left\{1 + \left(\frac{z-a}{t-a}\right) + \left(\frac{z-a}{t-a}\right)^2 + \cdots\right\}$$

$$U_{n+1} = \frac{(z-a)^{n+1}}{2\pi i} \int_C \frac{f(t)\,dt}{(t-a)^{n+1}(t-z)}$$

$$|U_{n+1}| \le \frac{|z-a|^{n+1}}{2\pi} \int_C \frac{|f(t)|\,|dt|}{|t-a|^{n+1}(|t|-|z|)} \qquad \because |a-b| \ge |a|-|b|$$

$\Rightarrow f(z)$ is analytic, so continuous on the closed boundary C. Let $M = \max. |f(z)|$ on C.

$$|U_{n+1}| \le \frac{M}{\cancel{2\pi}} \left(\frac{r}{R}\right)^{n+1} \cdot \left(\frac{1}{R-r}\right) \cancel{2\pi} R$$

$$\le \frac{M}{1} \left(\frac{r}{R}\right)^{n+1} \cdot \left(\frac{1}{1-\left(\frac{r}{R}\right)}\right) \to 0$$

$$\boxed{\begin{aligned} \because\ & t-z \\ &= \underbrace{t-a}-\underbrace{z+a} \\ &= R-r \end{aligned}}$$

$$\boxed{\because S_\infty = \frac{1}{1-x}}$$

@ $n \to \infty$

For $\lim_{n\to\infty} \left(\frac{r}{R}\right)^{n+1} = 0$ as $\frac{r}{R} < 1$

$$\boxed{\therefore \lim_{n\to\infty} U_{n+1} = 0}$$

Good 13/09/24

Put in (1),

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)\,(z-a)^n}{n!}} \quad \text{OR}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n} \quad \text{where } a_n = \frac{f^n(a)}{n!}$$