UNIT-II

Fluid Dynamics

For Graduate & Post Graduate Students

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2025

DIGIMATHS

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Fluid Dynamics

UNIT-II: Pressure at a point of moving fluid, Euler equation of motion, Equations of motion in cylindrical and shperical polar co-ordinates, Bernoulli equations, Impulsive motion, Kelvin circulation theorem, Vorticity equation, Engery equation for incompressible flow, Kinetic energy of irrotational flow, Kelvin minimum energy theorem, Kinetic energy of infinite fluid, Uniqueness theorem.

2.1:. Pressure at a point of a Moving Fluid

Let P be a point in a ideal (inviscid) fluid moving with velocity $\vec{q}$. We insert an elementary rigid plane area δA into this fluid at point P. This plane area also moves with the velocity $\vec{q}$ of the local fluid at P.

If $\delta\vec{F}$ denotes the force exerted on one side of δA by the fluid particles on the other side,

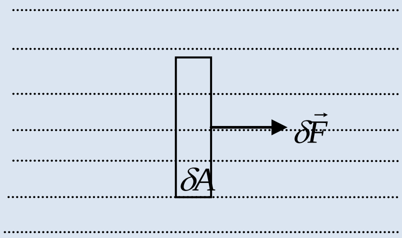


figure 2.1

Then this force will act normal to δA.

Further, if we assume that $\lim_{\delta A \to 0} \frac{\delta \vec{F}}{\delta A}$ exists uniquely, then this limit is called the (hydrodynamic) fluid pressure at point P and is denoted by p.

Theorem:-Prove that the pressure p at a point P in a moving inviscid fluid is same in all direction.

Proof:- Let $\vec{q}$ be the velocity of the fluid. We consider an elementary tetrahedron $PQRS$ of the fluid at a point P of the moving fluid. Let the edges of the tetrahedron be $PQ = \delta x,\ PR = \delta y,\ PS = \delta z$ at time t, where $\delta x, \delta y, \delta z$ are taken along the co-ordinate axes OX, OY, OZ respectively. This tetrahedron is also moving with the velocity $\vec{q}$ of the local fluid at P.

Let p be the pressure on the face QRS where area is δs. Suppose that $< l, m, n >$ are the d.c.'s of the normal to δs drawn outward from the tetrahedron.

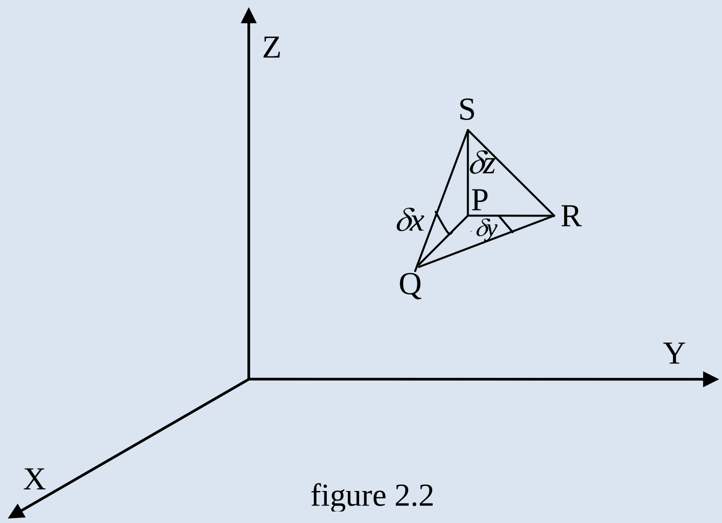


figure 2.2

Then,

$$l\delta s = \text{projection of the area } \delta s \text{ on } yz - \text{plane.}$$

$$= \text{area of face } PRS \text{ (triangle)}$$

$$= \frac{1}{2}\delta y \delta z = \frac{\delta y \delta z}{2}$$

Similarly, $$m\delta s = \text{area of face } PQS = \frac{1}{2}\delta z \delta x = \frac{\delta z \delta x}{2}$$

and, $$n\delta s = \text{area of face } PQR = \frac{1}{2}\delta x \delta y = \frac{\delta x \delta y}{2}$$

The total force exerted by the fluid, outside the tetrahedron, on the face QRS is

$$= -p\delta s(l\hat{i} + m\hat{j} + n\hat{k})$$

$$= -p(l\delta s\hat{i} + m\delta s\hat{j} + n\delta s\hat{k})$$

$$= -\frac{p}{2}(\delta y\delta z\hat{i} + \delta z\delta x\hat{j} + \delta x\delta y\hat{k})$$

Let p_x, p_y, p_z be the pressures on the faces PRS, PQS, PRQ. The force exerted on these faces by the exterior fuid are $\frac{1}{2}p_x\delta y\delta z\hat{i}, \frac{1}{2}p_y\delta z\delta x\hat{j}, \frac{1}{2}p_z\delta x\delta y\hat{k}$ respectively.

Thus, the total surface force on the tetrahedron is

$$= -\frac{p}{2}(\delta y\delta z\hat{i} + \delta z\delta x\hat{j} + \delta x\delta y\hat{k}) + \frac{1}{2}p_x\delta y\delta z\hat{i} + \frac{1}{2}p_y\delta z\delta x\hat{j} + \frac{1}{2}p_z\delta x\delta y\hat{k}$$

$$= \frac{1}{2}\left[(p_x - p)\delta y\delta z\hat{i} + \frac{1}{2}(p_y - p)\delta z\delta x\hat{j} + \frac{1}{2}(p_z - p)\delta x\delta y\hat{k}\right] \qquad (1)$$

In addition to surface force (fluid forces), the fluid may be subjected to body forces which are due to external causes such as gravity. Let $\vec{F}$ be the mean body force per unit mass within the tetrahedron.

Volume of the tetrahedron $PQRS$ is $\frac{1}{3}h\delta s$ i.e. $\frac{1}{6}\delta x\delta y\delta z,$ where h is the perpendicular from P on the face QRS.

Thus, the total force acting on the tetrahedron $PQRS$ is $+\frac{1}{6}\rho\vec{F}\delta x\delta y\delta z$ (2)

Where ρ is the mean density of the fluid.

From $(1)\,\&\,(2),$ the net force acting in the tetrahedron is

$$= \frac{1}{2}\left[(p_x - p)\delta y\delta z\hat{i} + \frac{1}{2}(p_y - p)\delta z\delta x\hat{j} + \frac{1}{2}(p_z - p)\delta x\delta y\hat{k}\right] + \frac{1}{6}\rho\vec{F}\delta x\delta y\delta z$$

Now, the acceleration of the tetrahedron is $\frac{D\vec{q}}{Dt}$ and the mass $\frac{1}{6}\rho\delta x\delta y\delta z$ of fluid inside it is constant.

Thus, the equation of motion of the fluid contained in the tetrahedron is

$$\Rightarrow \frac{1}{2}\left[(p_x - p)\delta y\delta z\hat{i} + \frac{1}{2}(p_y - p)\delta z\delta x\hat{j} + \frac{1}{2}(p_z - p)\delta x\delta y\hat{k}\right] + \frac{1}{6}\rho\vec{F}\delta x\delta y\delta z$$

$$= \frac{1}{6}\rho\vec{F}\delta x\delta y\delta z\left(\frac{D\vec{q}}{Dt}\right) \qquad (\vec{f} = m\vec{a})$$

i.e. $$= \left[(p_x - p)l\delta s\hat{i} + (p_y - p)m\delta s\hat{j} + (p_z - p)n\delta s\hat{k}\right] + \frac{1}{3}\rho\vec{F}\delta hs = \frac{1}{3}\rho h\delta s\left(\frac{D\vec{q}}{Dt}\right)$$

on dividing by δs and letting the tetrahedron shrink to zero about P, in which case $h \to 0$, it follows that

$$p_x - p = 0,\ p_y - p = 0,\ p_z - p = 0$$

i.e. $$p_x = p_y = p_z = p \qquad (3)$$

Since the choice of axes is arbitrary, the relation (3) establishes that at any point P

Of a moving ideal fluid, the pressure p is same in all direction.

Equation of Motion

Euler's Equation of Motion of an Ideal Fluid (Equation of Conservation of Momentum).

To obtain Euler's dynamical equation, we shall make use of Newton's second law of motion.Consider a region τ of fluid bounded by a closed surface S which consists of the same fluidparticle at all times. Let $\vec{q}$ be the velocity and ρ be the density of the fluid.

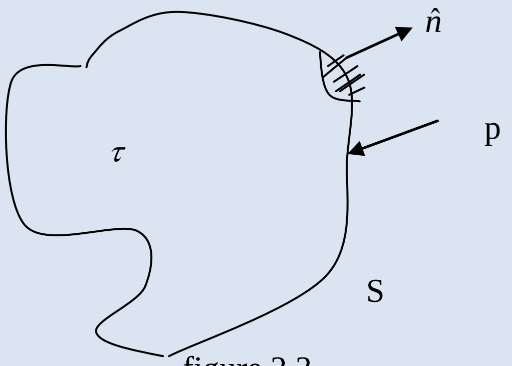


figure 2.3

Then $\rho d\tau$ is an element of mass within S and it remains constant

The linear momentum of volume τ is

$$\vec{M} = \int_{\tau} \vec{q}\rho d\tau \mid \ mass \times velocity = momentum$$

Rate of change of momentum is

$$\frac{d\vec{M}}{dt} = \frac{d}{dt}\int_{\tau} \vec{q}\rho d\tau = \int_{\tau} \frac{d\vec{q}}{dt}\rho d\tau \qquad (1)$$

The fluid within τ is acted upon by two types of forces.

The first type of forces are the surface forces which are due to the fluid exterior to τ .

Since the fluid is ideal, the surface force is simply the pressure p directed along the inward normal at all point of S.

The total surface force on S is

$$\int_{S} p(-\hat{n})dS = -\int_{S} p\hat{n}dS = \int_{\tau} -\nabla p d\tau \qquad (By\, Gauss\, div. Theorem) \qquad (2)$$

The second type of forces are the body forces which are due to some external agent. Let $\vec{F}$ be the body force per unit mass acting on the fluid. Then $\vec{F}\rho d\tau$ is the body force on the element of mass $\rho d\tau$ and the total body force on the mass within τ is

$$\int_{\tau} \vec{F}\rho d\tau \qquad (3)$$

By Newton's second law of motion, we have

Rate of change of momentum $= total\ force$

$$\Rightarrow \int_{\tau} \frac{d\vec{q}}{dt} \rho d\tau = \int_{\tau} \vec{F} \rho d\tau - \int_{\tau} \nabla p d\tau$$

$$\Rightarrow \int_{\tau} \left(\frac{d\vec{q}}{dt} \rho - \vec{F}\rho + \nabla \rho \right) d\tau = 0$$

Since $d\tau$ is arbitrary, we get $\frac{d\vec{q}}{dt}\rho - \vec{F}\rho + \nabla\rho = 0$

i.e. $$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla\rho \qquad (4)$$

which holds at every point of the fluid and is known as Euler's dynamical equation for an ideal fluid.

Remark.: The above method for obtaining the Euler's equation of motion, is also known as flux method.

Other Forms of Euler's Equation of Motion.

(i) We know that

$$\frac{d}{dt} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q}.\nabla$$

Therefore equation (4) becomes

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p \qquad (5)$$

But $$(\vec{q}.\nabla)\vec{q} = \nabla\left(\frac{1}{2}\vec{q}^{\,2}\right) + \vec{\xi} \times \vec{q},\ \vec{\xi} = curl\,\vec{q}$$

Therefore, Euler's equation becomes

$$\frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^{\,2}\right) + \vec{\xi} \times \vec{q} = \vec{F} - \frac{1}{\rho}\nabla p \qquad (6)$$

Equation (6) is called Lamb's hydrodynamical equation.

(ii) **Cartesian Form**

Let $\vec{q}=(u,v,w), \vec{F}=(X,Y,Z) \& \nabla p=\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}+\frac{\partial p}{\partial z}\right),$

Then equation (5) gives

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \quad (7)$$

Equation (7) are the required equations in Cartesian form of Euler's equations.

(iii) **Equation of Motion in Cylindrical Co-ordinates.** (r,θ,z).

Here, $\quad \vec{q}=(u,v,w), d\vec{r}=(dr, rd\theta, dz)$

$$\nabla p=\left(\frac{\partial p}{\partial r},\frac{1}{r}\frac{\partial p}{\partial \theta},\frac{\partial p}{\partial z}\right)$$

Let $\quad \vec{F}=(F_r, F_\theta, F_z)$

Also, the acceleration components in cylindrical co-ordinates are

$$\frac{d\vec{q}}{dt}=\left(\frac{du}{dt}-\frac{v^2}{r},\frac{dv}{dt}+\frac{uv}{r},\frac{dw}{dt}\right)$$

Thus, the equation of motion

$$\frac{d\vec{q}}{dt}=\vec{F}-\frac{1}{\rho}\nabla\rho \quad \text{becomes}$$

$$\left.\begin{aligned}
&\frac{du}{dt}-\frac{v^2}{r}=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\
&\frac{dv}{dt}+\frac{uv}{r}=F_\theta-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}
\end{aligned}\right\} \quad (8)$$

$$\frac{dw}{dt} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

(iv) **Equations of Motion in Spherical co-ordinates** (r, θ, ψ).

Here, $\vec{q} = (u, v, w),\ d\vec{r} = (dr, rd\theta, r\sin\theta d\psi)$

$$\nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \psi}\right)$$

Let $\vec{F} = (F_r, F_\theta, F_\psi)$

Also, the acceleration components in spherical co-ordinates are

$$\frac{d\vec{q}}{dt} = \left(\frac{du}{dt} - \frac{v^2 + w^2}{r}, \frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r}, \frac{dw}{dt} + \frac{vw\cot\theta}{r}\right)$$

Thus, the equation of motion take the form

$$\left.\begin{aligned}
&\frac{du}{dt} - \frac{v^2 + w^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\
&\frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r} = F_\theta - \frac{1}{r\rho}\frac{\partial p}{\partial \theta} \\
&\frac{dw}{dt} + \frac{vw\cot\theta}{r} = F_\psi - \frac{1}{\rho}\frac{\partial p}{\partial \psi}
\end{aligned}\right\} \qquad (9)$$

Remark:- The two equations, the equation of continuity and the Euler's equation of motion, comprise the equations of motion of an ideal fluid. Thus the equations

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{q}) = 0$$

And
$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

Are fundamental to any theoretical study of ideal fluid flow. These equations are solved subject to the appropriate boundary and initial conditions dictated by the physical characteristics of the flow.

Lagrange's Equation of Motion: Let initially a fluid element be at (a,b,c) at time $t = t_0$ when its volume is dV_0 and density is ρ_0. After time t, let the same fluid element be at (x, y, z) when its volume is dV and density is ρ. The equation of continuity is

$$\rho J = \rho_0 \tag{1}$$

Where $$J = \frac{\partial(x, y, z)}{\partial(a,b,c)}$$

The components of acceleration are

$$\ddot{x} = \frac{\partial^2 x}{\partial t^2},\ \ddot{y} = \frac{\partial^2 y}{\partial t^2},\ \ddot{z} = \frac{\partial^2 z}{\partial t^2}$$

Let the body force $\vec{F}$ be conservative so that we can express it in terms of a body force potential function Ω as

$$\vec{F} = -\nabla\Omega \tag{2}$$

By Euler's equation of motion,

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla\rho \tag{3}$$

Its Cartesian equivalent is

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= -\frac{\partial\Omega}{\partial x} - \frac{1}{\rho}\frac{\partial\rho}{\partial x} \\ \frac{\partial^2 y}{\partial t^2} &= -\frac{\partial\Omega}{\partial y} - \frac{1}{\rho}\frac{\partial\rho}{\partial y} \\ \frac{\partial^2 z}{\partial t^2} &= -\frac{\partial\Omega}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z} \end{aligned}\right\} \tag{4}$$

We note that a,b,c,t are the independent variables and our object is to determine x, y, z in terms of a,b,c,t and so investigate completely the motion.

To deduce equations containing only differentiations w.r.t. the independent variables a,b,c,t we multiply the equations in (4) by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$ and add to get

$$\left.\begin{aligned}
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a}+\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a}&=-\frac{\partial \Omega}{\partial a}-\frac{1}{\rho}\frac{\partial \rho}{\partial a}\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b}+\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b}&=-\frac{\partial \Omega}{\partial b}-\frac{1}{\rho}\frac{\partial \rho}{\partial b}\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c}+\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c}+\frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c}&=-\frac{\partial \Omega}{\partial c}-\frac{1}{\rho}\frac{\partial \rho}{\partial c}
\end{aligned}\right\}\qquad(5)$$

This equation (5) together with equation (1) constitute Lagrange's Hydrodynamical Equations.

Example: A homogeneous incompressible liquid occupies a length $2l$ of a straight tube of uniform small bore and is acted upon by a body force which is such that the fluid is attracted to a fixed point of the tube, with a force varying as the distance from the point. Discuss the motion and determine the velocity and pressure within the liquid.

Solution. We note that the small bore of the tube permits us to ignore any variation of velocity across any cross-section of the tube and to suppose that the flow is unidirectional.

Let u be the velocity along the tube and p be the pressure at a general point P at distance x from the centre of force O. Also, let h be the distance of the centre of mass G of the fluid, as shown in the figure.

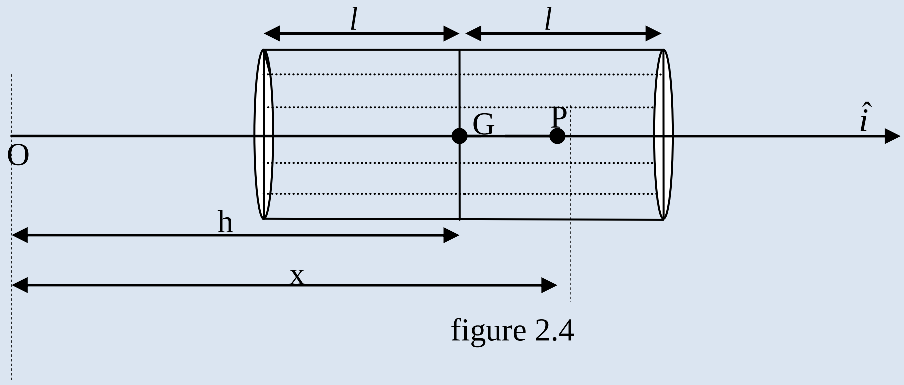


figure 2.4

Equations of motion of the fluid are :

(i) Equation of Continuity

Here, $\vec{q} = (u,0,0)$

Therefore, equation of continuity becomes

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(t) \qquad (1)$$

(ii) Euler's Equation

In this case, it becomes $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x}$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad |\text{using } (1) \qquad (2)$$

Where $-\mu x\hat{i}$ is the body force per unit mass, μ being a positive constant.

We observe that equation (2) can be written as

$$\frac{du}{dt} = -\mu x - \frac{1}{\rho}\frac{dp}{dx} \qquad (3)$$

Integrating w.r.t. x, we get

$$x\frac{du}{dt} = -\mu\frac{x^2}{2} - \frac{p}{\rho} + C \qquad (4)$$

Where C is a constant and at most can be a function of t only w.r.t. (x, y, z)

Let Π be the pressure at the free surfaces $x = h - l$, and $x = h + l$ of the liquid.

Then using these boundary conditions, equation (4) becomes

$$(h-l)\frac{du}{dt} = -\frac{1}{2}\mu(h-l)^2 - \frac{\Pi}{\rho} + C$$

$$(h+l)\frac{du}{dt} = -\frac{1}{2}\mu(h+l)^2 - \frac{\Pi}{\rho} + C$$

Which on subtraction give

$$\frac{du}{dt} = -\mu h \qquad (5)$$

But in the fluid motion all fluid particles move with the same velocity u and $u = \frac{dh}{dt}$

$\therefore$ Equation (5) becomes

$$\frac{d^2h}{dt} = -\mu h \qquad (6)$$

Now, we solve the different equation (6), which can be written as

$$(D^2 + \mu)h = 0$$

Here auxiliary equation is

$$D^2 + \mu = 0 \Rightarrow D = \pm(\mu)^{1/2} i$$

Therefore, the solution of (6) is

$$h = A\cos\left((\mu)^{1/2} t + \in\right)$$

Where $A \,\&\in$ are constants which can be determined from initial conditions.

To Calculate Pressure:- We have from $(3) \& (5)$

$$-\mu x - \frac{1}{\rho}\frac{dp}{dx} = -\mu h$$

$$\Rightarrow \frac{1}{\rho}\frac{dp}{dx} = \mu(h-x)$$

Integrating w.r.t. x, we get

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{2(-1)} + D \qquad (7)$$

The boundary condition $x = h - l, p = \Pi$ gives

$$\frac{\Pi}{\rho} = \mu\frac{l^2}{-2} + D$$

i.e. $$D = \frac{\Pi}{\rho} + \mu\frac{l^2}{2}$$

Therefore, equation (7) becomes

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{-2} + \Pi/\rho + \frac{\mu l^2}{2}$$

$$= \frac{\Pi}{\rho} - \frac{\mu}{2}\left[(h-x)^2 - l^2\right]$$

Example. Homogeneous liquid is in motion in a vertical plane, within a curved tube of uniform small bore, under the action of gravity. Calculate the period of oscillation.

Solution. Let O be the lowest point of the tube, AB the equilibrium level of the liquid and h the height of AB above O. Let $\alpha \& \beta$ be respectively the inclinations of the tube to the horizontal at $A \& B \& \theta$ be the inclination of the tube at a distance s along the tube form O. Let $a \& b$ denote the arc length of $OA \& OB$ respectively and suppose that at time t, the liquid is displaced through a small distance z along the tube form its equilibrium position.

Due to the assumption of uniform small bore the flow is unidirectional along the tube.

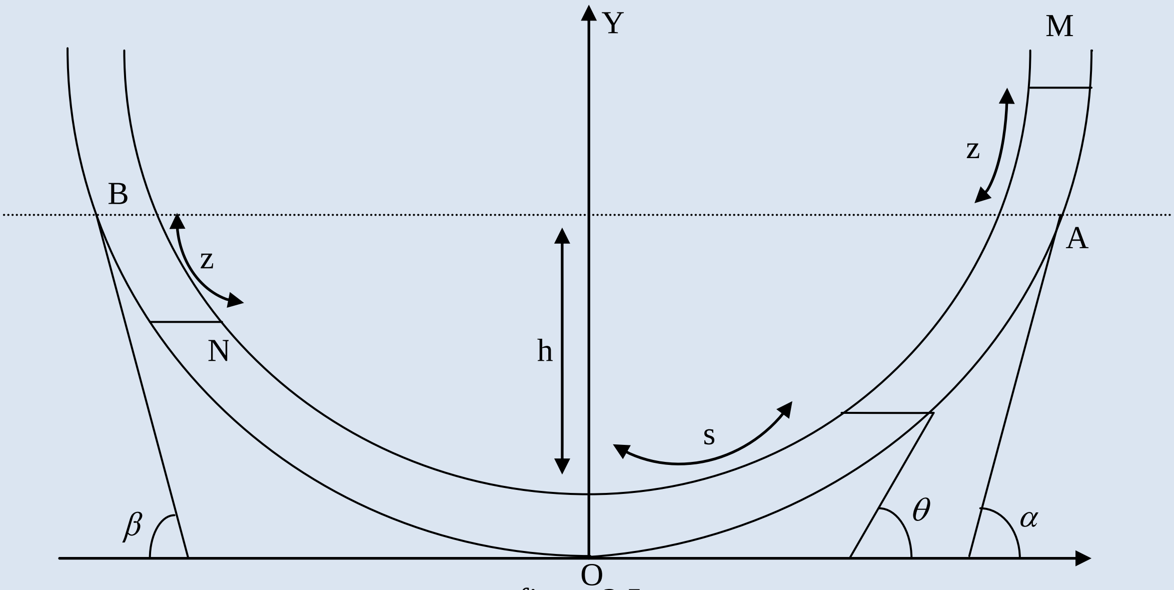


figure 2.5

Let the velocity be $u(s,t)$

The equation of continuity gives $\frac{\partial u}{\partial s} = 0$ (1)

$\Rightarrow$ u is independent of s

Euler's equation of motion becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} = -g\sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

Using equation (1), this gives

$$\frac{du}{dt} \equiv \frac{\partial u}{\partial t} = -g\sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

i.e. $$\frac{du}{dt} = -g\sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial s} \quad (2)$$

integrating it w.r.t. s, we find

$$s\frac{du}{dt} = -gy - \frac{p}{\rho} + C \quad (3)$$

Where C may be a function of time t at the most

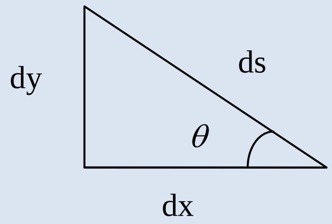


figure 2.6

The boundary conditions at free surface are

$(i) \quad p = \Pi \; for \; y = h + z\sin\alpha, s = OM = a + z \; at \; M$

$(ii) \quad p = \Pi \; for \; y = h - z\sin\beta, s = OM = -(b - z) \; at \; N$

Using these boundary conditions in (3), we get

$$(a+z)\frac{du}{dt} = -g(h + z\sin\alpha) - \frac{\Pi}{\rho} + C$$

$$-(b-z)\frac{du}{dt} = -g(h - z\sin\beta) - \frac{\Pi}{\rho} + C$$

Subtracting these we get

$$(a+b)\frac{du}{dt} = -gz(\sin\alpha + \sin\beta) \qquad (4)$$

Since

$$u = \frac{dz}{dt} \Rightarrow \frac{du}{dt} = \frac{d^2z}{dt^2}$$

Equation (4) becomes

$$(a+b)\frac{d^2z}{dt^2} = -gz(\sin\alpha + \sin\beta)$$

$$\Rightarrow \qquad \frac{d^2z}{dt^2} = -\mu z \qquad (5)$$

where $$\mu = \frac{-gz(\sin\alpha + \sin\beta)}{a+b}$$

We observe that equation (5) represents the simple harmonic motion. It's period T is given by

$$T = \frac{2\pi}{\mu} = 2\pi\left[\frac{a+b}{g(\sin\alpha + \sin\beta)}\right]^{\frac{1}{2}}$$

Example: A sphere of radius $R,$ whose centre is at rest, vibrates radically in an infinite incompressible fluid of density $\rho,$ which is at rest infinity. If the pressure at infinity is $\Pi,$ show that the pressure at the surface of the sphere at time t is

$$\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$$

If $R = a(2+\cos nt),$ show that, to prevent cavitation in the Π must not be less than $3\rho a^2 n^2.$

Solution. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface should be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the center of the sphere which is taken as origin), and time t only. Let p be pressure distance r'. Let P be the pressure on the surface of the sphere of radius and V be the velocity there, Then the equation of continuity is

$$r'^2 v' = R^2 V = F(t) \tag{1}$$

From $(1),$ $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}$$

Again equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \tag{2}$$

Or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad [\text{using } (2)] \tag{3}$$

Integrating with respect to r', (3) reduces to

$$-\frac{F'(t)}{r'^2}+\frac{1}{2}v'^2=-\frac{p}{\rho}+C$$

When $r'=\infty$, then $v'=0\ \&\ p=\Pi$ so that $C=\Pi/\rho$

$$\therefore \qquad -\frac{F'(t)}{r'^2}+\frac{1}{2}v'^2=-\frac{\Pi-p}{\rho}$$

Or $$p=\Pi+\frac{1}{2}\rho\left[2\frac{F'(t)}{r'^2}+\frac{1}{2}v'^2\right] \tag{4}$$

But $p=P\ \&\ v'=V$ when $r'=R$. Hence (4) gives

$$P=\Pi+\frac{1}{2}\rho\left[\frac{2}{R}\{F'(t)\}_{r'=R}-V^2\right] \tag{5}$$

Also $V=dR/dt$. Hence using (1), we have

$$\{F'(t)\}_{r'=R}=\frac{d}{dt}(R^2V)=\frac{d}{dt}\left(R^2\frac{dR}{dt}\right)=\frac{d}{dt}\left(\frac{R}{2}.\frac{dR^2}{dt}\right)$$

$$=\frac{R}{2}\frac{d^2R^2}{dt^2}+\frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt}=\frac{R}{2}\frac{d^2R^2}{dt^2}+R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V and $\{F'(t)\}_{r'=R}$, (5) reduces to

$$P=\Pi+\frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2}+R\left(\frac{dR}{dt}\right)^2\right\}-\left(\frac{dR}{dt}\right)^2\right]$$

Or $$P=\Pi+\frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2}+\left(\frac{dR}{dt}\right)^2\right] \tag{6}$$

Second Part: From $r'^2v'=const.$, we conclude that v' is maximum when r' is minimum i.e. $r'=R$. Hence pressure is minimum on $r'=R$ by using Bernoulli's theorem

Given $R = a(2 + \cos nt)$

(7)

$\therefore$ $dR/dt = -an \sin nt$

and $$\frac{dR^2}{dt} = 2a^2(2 + \cos nt)(-n \sin nt)$$

$\therefore$ $$\frac{dR^2}{dt} = -2a^2n^2(2 + \cos nt)\cos nt + 2a^2n^2 \sin^2 nt$$

with the above values, (6) reduces to

$$P = \Pi + (3/2)\rho a^2 n^2 \sin^2 nt - a^2 n^2 \rho(2\cos nt + \cos^2 nt) \qquad (8)$$

From (7), R varies from $3a$ to a. Thus the sphere has the greatest radius $3a$, there is a possibility of a cavitation there because pressure would be minimum there. Hence the minimum value of pressure $P'(say)$ on the surface of the sphere is given by replacing $t = 0$ or $nt = 2m\pi$ in (8). We thus obtain

$$P' = \Pi - 3\rho a^2 n^2 \qquad (9)$$

To prevent cavitation in the fluid. P' givn by (9) must be positive i.e. Π must not be less than $3\rho a^2 n^2$.

Example: Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated ; prove that if Π be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is $\Pi \dfrac{\log r - \log b}{\log a - \log b}$.

Solution. Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder v' will be radial and v' will function of r' The radial distance from the cylinder $|z| = b$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad (1)$$

From (1), $$\frac{\partial v'}{\partial t'} = \frac{F'(t)}{r'} \qquad (2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{[using (2)]}$$

Integrating,

$$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C \qquad (3)$$

Initially when $r' = 0$. then $v' = 0$ & $p = P$

$\therefore$ $$F'(0)\log r' = -\frac{p}{\rho} + C \qquad (4)$$

again , $P = \Pi$ when $r' = a$ and $P = 0$ when $r' = b$

$\therefore$ $$F'(0)\log a = -\frac{\Pi}{\rho} + C \text{ and } F'(0)\log b = C \qquad (5)$$

Solving (5) for $F'(0)$ & C, we have

$$C = -\log b\frac{\Pi}{\rho\log(a/b)}, \quad F'(0) = -\frac{\Pi}{\rho\log(a/b)},$$

Putting these values in (4), we get

$$\frac{P}{\rho} = \frac{\Pi}{\rho\log(a/b)}\log r' - \frac{\Pi\log b}{\rho\log(a/b)}$$

$$P = \Pi \frac{\log r' - \log b}{\log(a/b)} = \Pi \frac{\log r' - \log b}{\log a - \log b} \qquad (6)$$

For the required result, replace r' by r in (6).

Example: A spherical hollow of radius a initially exists in an infinite fluid, subject to constant pressure at infinity. Show that the pressure that distance r' from the centre when the radius of the cavity is r is to the pressure at infinity as

$$3r^2 r'^4 + (a^3 - 4r^3)r'^3 - (a^3 - r^3)r^3 : 3r^2 r'^4$$

Solution: Let v' be the velocity at a distance r' at any time t and p be the pressure there. Again, let v be the velocity of the inner surface of radius r. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \qquad (1)$$

From (1),

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad (2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho} \frac{\partial p}{\partial r'}$$

or

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2} v'^2\right) = -\frac{1}{\rho} \frac{\partial p}{\partial r'} \qquad \text{Using } (2)$$

Integrating,

$$-\frac{F'(t)}{r'} + = -\frac{1}{\rho} \frac{\partial p}{\partial r'}$$

Bernoulli's Equation (Theorem)

For Steady Flow. We shall obtain a special form of Euler's dynamical equation in terms of pressure. The Euler's dynamical equation is

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho} \nabla p \qquad (1)$$

Where $\vec{q}$ is velocity, $\vec{F}$ is the body force, p and ρ are pressure and density respectively.

$\vec{F}$ be conservative so that it can be expressed in terms of a body force potential function Ω as

$$\vec{F} = -\nabla\Omega \qquad (2)$$

When the flow is steady, then $\dfrac{\partial \vec{q}}{\partial t} = 0$ $\qquad (3)$

Therefore, in case of steady motion with a conservative body force equation (1), on using (2) & (3), gives

$$\nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\vec{\xi} = -\nabla\Omega - \frac{1}{\rho}\nabla p$$

$$\because \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q}$$

$$or\ \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\vec{\xi}\ \&\ \frac{\partial q}{\partial t} = 0$$

$$\Rightarrow \nabla\left(\frac{1}{2}\vec{q}^2 + \Omega\right) + \frac{1}{\rho}\nabla p = \vec{q}\times\vec{\xi} \qquad (4)$$

Further, if we suppose that the liquid is barotropic i.e. (density is a function of pressure p only), then we can write

$$\frac{1}{\rho}\nabla p = \nabla\int\frac{dp}{\rho}$$

Using this in (4), we get

$$\nabla\left[\frac{1}{2}\vec{q}^2 + \Omega + \int\frac{dp}{\rho}\right] = \vec{q}\times\vec{\xi} \qquad (5)$$

Multiplying (5) scalarly by $\vec{q}$ and nothing that

$$\vec{q}.(\vec{q}\times\vec{\xi}) = (\vec{q}\times\vec{q}).\vec{\xi} = 0, \text{ we get}$$

$$\vec{q}.\nabla\left[\frac{1}{2}\vec{q}^2+\Omega+\int\frac{dp}{\rho}\right]=0 \qquad (6)$$

If $\hat{s}$ is a unit vector along the streamline through general point of the fluid and s measures distance along this stream line, then since $\hat{s}$ is parallel to $\vec{q}$, therefore equation (6) gives

$\because \hat{s}$ is parallel to $\vec{q}$

$$\frac{\partial}{\partial s}\left[\frac{1}{2}\vec{q}^2+\Omega+\int\frac{dp}{\rho}\right]=0 \quad \vec{q}=k\hat{s}$$

$$\hat{s}\nabla\equiv\frac{\partial}{\partial s}$$

Hence along any particular streamline, we have

$$\frac{1}{2}\vec{q}^2+\Omega+\int\frac{dp}{\rho}=C \qquad (7)$$

Where C is constant which takes different values for different stremlines. Equation (7) is known as Bernoulli's equation. This result applies to steady flow of ideal barotropic fluids in which the body forces are conservative.

Now, if $\hat{s}$ is a unit vector taken along a vortexline, then, similarly, we get

$\frac{1}{2}\vec{q}^2+\Omega+\int\frac{dp}{\rho}=C$ along any particular vortexline. (Here,we multiply scalarly by $\vec{\xi}$)

Remark.

(i) If $\vec{q}\times\vec{\xi}=\vec{0}$ i.e. if $\vec{q}\,\&\,\vec{\xi}$ are parallel, then streamlines and vortex lines coincide and $\vec{q}$ is said to be Beltrami vector.

If $\vec{\xi}=\vec{0}$, the flow is irrotational

For both of these flow patterns,

$$\frac{1}{2}\vec{q}^2+\Omega+\int\frac{dp}{\rho}=C$$

where C is same at all points of the fluid.

(ii) For homogeneous incompressible fluids, ρ is constant and

$$\int \frac{dp}{\rho} = \frac{p}{\rho}$$

The Bernoulli's equation becomes

$$\frac{p}{\rho} + \frac{1}{2}\vec{q}^2 + \Omega = C$$

So that if $\vec{q}$ is known, the pressure can be calculated.

For Unsteady Irrotational Flow.

Here also, we suppose that the body forces are conservative i.e. $\vec{F} = -\nabla\Omega$

For irrotational flow, $\vec{q} = -\nabla\phi \Rightarrow \nabla\times\vec{q} = 0$

The equation of motion

$$\frac{\partial\vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times(\nabla\times\vec{q}) = \vec{F} - \frac{1}{\rho}\nabla p$$

In the present case becomes,

$$-\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\vec{q}^2\right) = -\nabla\Omega - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \qquad \nabla\left(\frac{1}{2}\vec{q}^2 + \Omega + \int\frac{dp}{\rho} - \frac{\partial\phi}{\partial t}\right) = 0 \qquad | \text{ Barotropic fluid.}$$

Integrating, we get

$$\frac{1}{2}\vec{q}^2 + \Omega + \int\frac{dp}{\rho} - \frac{\partial\phi}{\partial t} = f(t) \qquad (2)$$

Which is the required equation.

If the liquid is homogeneous, then $\int\frac{dp}{\rho} = \frac{p}{\rho}$ and the equation (2) become

$$\frac{1}{2}\vec{q}^{\,2} + \Omega + \frac{p}{\rho} - \frac{\partial \phi}{\partial t} = f(t)$$

Further, for study case,

$$\frac{\partial \phi}{\partial t} = 0, \quad f(t) = const.$$

$$\frac{1}{2}\vec{q}^{\,2} + \Omega + \frac{p}{\rho} = const.$$

Example: A long straight pipe of length L has a slowly tapering circular cross section. It is inclined so that its axis makes and angle α to the horizontal with its smaller cross-section downwards. The radius of the pipe at its upper end is twice that of at its lower end and water is pumped at a steady rate through the pipe to emerge at atmospheric pressure. It the pumping pressure is twice the atmospheric pressure, show that the fluid leaves the pipe with a speed U give by

$$U^2 = \frac{32}{25}\left[gL\sin\alpha + \frac{\Pi}{\rho}\right], \text{ where } \Pi \text{ is atmospheric pressure.}$$

Solution. The assumption that the pipe is slowly tapering means that any variation in the velocity over any cross-section can be ignored. Let the velocity at the wider end of the pipe be V and the emerging velocity be U (velocity at the lower end). The only body force is that of gravity, so $\vec{F} = -g\hat{j}$ and consequently $\Omega = gy$

$$\because \vec{F} = -\nabla\Omega \Rightarrow -g\hat{j} = -\nabla\Omega = -\frac{\partial \Omega}{\partial x}\hat{i} - \frac{\partial \Omega}{\partial y}\hat{j} - \frac{\partial \Omega}{\partial z}\hat{k}$$

$$\Rightarrow -g = \frac{\partial \Omega}{\partial y} \Rightarrow \Omega = gy$$

Bernoulli's equation, $\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = C$ | $\because$ For water ρ is const.

Becomes $\frac{p}{\rho} + \frac{1}{2}q^2 + gy = C$

Applying this equation of the two ends of the pipe, we get

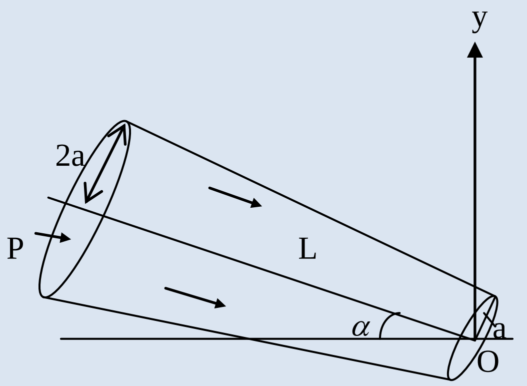
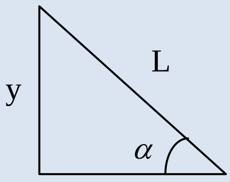


figure 2.6



$y = L\sin\alpha$

$$\frac{2\Pi}{\rho} + \frac{1}{2}V^2 + gL\sin\alpha = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \ (2) \ | \text{ for lower end } \ y = 0$$

Let a and $2a$ be the radii of the lower and upper ends respectively, then by the principle of conservation of mass

$$\pi(2a)^2 V = \pi a^2 U$$

$$\Rightarrow V = \frac{U}{4}$$

From $(2)\,\&\,(3)$, we obtain

$$\Pi + \frac{1}{2}\rho\left(\frac{U^2}{16}\right) + g\rho L\sin\alpha = \frac{1}{2}\rho U^2$$

$$\Rightarrow \quad \frac{1}{2}\rho\left(U^2 - \frac{U^2}{16}\right) = \Pi + g\rho L\sin\alpha$$

$$\Rightarrow \quad \frac{15}{32}\rho U^2 = \Pi + g\rho L\sin\alpha$$

$$\Rightarrow \quad U^2 = \frac{32}{15}\left[gL\sin\alpha + \frac{\Pi}{\rho}\right]$$

Hence the result.

Example: A straight tube ABC, of small bore, is bent so as to make the angle ABC a right angle and AB equal to BC. The end C is closed and the tube is placed with end A upwards and AB vertical, and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half. Also find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmospheric being neglected.

Solution. Let $AB = BC = a$

When the liquid in AB has fallen through a distance z at time t, then let P be any point in the vertical column such that

$AM = z,\ BP = x,\ BM = a - z$ If u & p be the velocity and pressure at P, then equation of motion is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad | \ u \equiv u(x,t) \qquad (1)$$

and equation of continuity is

$$\frac{\partial u}{\partial x} = 0 \quad \text{i.e. } u = u(t)$$

Therefore, equation (1) becomes

$$\frac{\partial u}{\partial t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating w.r.t. x, we get

$$x\frac{\partial u}{\partial t} = -gx - \frac{1}{\rho}p + C \qquad (2)$$

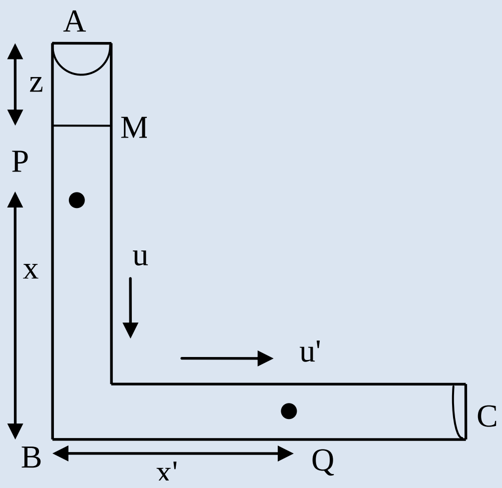


figure 2.7

Using the boundary condition $p=0$ at $x=a-z,$ we get

$$C=(a-z)\frac{\partial u}{\partial t}+g(a-z)$$

Therefore, equation (2) becomes

$$x\frac{\partial u}{\partial t}=-gx-\frac{1}{\rho}p+(a-z)\frac{\partial u}{\partial t}+g(a-z)$$

i.e. $$\frac{p}{\rho}=-(x-a+z)\left(\frac{\partial u}{\partial t}+g\right) \qquad (3)$$

Now, we take a point Q in $BC,$ where $BQ=x'$ and let u', p' be the velocity and pressure at $Q,$ then

$$\frac{p'}{\rho}=-(x'-a)\left(\frac{\partial u'}{\partial t}\right) \qquad | \ z=0 \ \& \ g \text{ is not effecting} \qquad (4)$$

equating the pressure at B, when $x=0, x'=0$, we get

$$(a-z)\left(\frac{\partial u}{\partial t}+g\right)=a\frac{\partial u'}{\partial t} \qquad | \text{From } (3) \ \& \ (4)$$

$$=-a\frac{\partial u}{\partial t} \qquad | \because u'=-u$$

Initially, when C is just opened, then $z = 0, t = 0$ and we have

$$a\left[\left(\frac{\partial u}{\partial t}\right)_{t=0} + g\right] = -a\left(\frac{\partial u}{\partial t}\right)_{t=0}$$

$$\Rightarrow \left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{-g}{2} \text{ i.e. } \left(\frac{\partial u}{\partial t}\right)_{0} = -\frac{g}{2} \qquad (5)$$

Therefore, from equation (3), initially, the pressure at P is given by

$$\frac{p_0}{\rho} = -(x-a)\left[\left(\frac{\partial u}{\partial t}\right)_0 + g\right] \qquad | \; p_0 \equiv (p)_{t=0}$$

$$= \frac{-g}{2}(x-a)$$

$$\Rightarrow p_0 = \frac{1}{2}\rho g(a-x)$$

(6)

But when the end C is closed, the liquid is at rest and the hydrostatic pressure at P

$$p_1 = \rho g h = \rho g(a-x) \qquad | \; h = AP = a - x \qquad (7)$$

From (6) & (7), we get

$$p_0 = \frac{1}{2}\rho_1$$

Thus, the pressure is diminished to one-half.

Now, form (4), initial pressure at Q is given by

$$\frac{p'_0}{\rho} = -(x'-a)\left(\frac{\partial u'}{\partial t}\right)_{t=0} = (x'-a)\left(\frac{\partial u}{\partial t}\right)_{t=0} = (a-x')\frac{g}{2}$$

$$\Rightarrow \qquad p'_0 = \frac{1}{2}\rho g(a-x')$$

When the end C is closed, the initial pressure (hydrostatic) p_2 at Q (or B or C) is $\rho g a$

Therefore, instantaneous change in pressure

$$= p_2 - p'_0 = \rho g a - \frac{1}{2}\rho g(a - x') = \frac{1}{2}\rho g(a + x')$$

Example: A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π. Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is

$$\Pi + \frac{\rho}{2}\left[\frac{d^2(R^2)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Solution. In the incompressible liquid, outside the sphere, the fluid velocity q will be radial and thus will be a function of r, the radial distance from the centre of the sphere (the origin), and time t only.

The equation of continuity in spherical polar co-ordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 u) = 0 \quad (1)$$

$$\because \bar{q} = (u,0,0),\ u = u(r,t),\ \nabla \equiv \left(\frac{\partial}{\partial r},0,0\right)$$

$$\nabla.q = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)$$

i.e. spherical symmetry

$$\Rightarrow \quad r^2 u = const. = f(t)$$

On the surface of the sphere,

$$r = R\ ,\ u = \dot{R}$$

Therefore,

$$f(t) = R^2\dot{R}$$

and thus

$$r^2 u = R^2 \dot{R} \qquad (2)$$

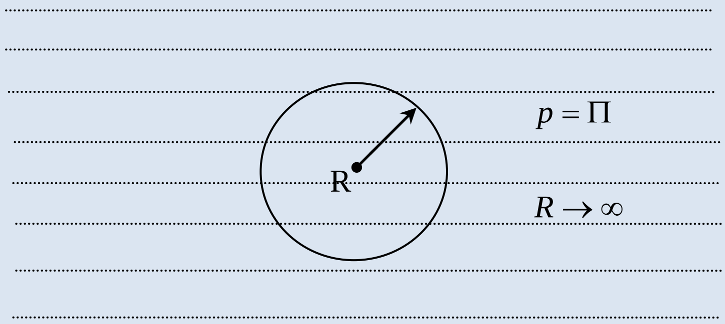


figure 2.8

We observe that $u \to 0$ *as* $r \to \infty$, as required

From (1), it is clear that $curl\,\vec{q} = \vec{0}$

$\Rightarrow$ the motion is irrotational and $\vec{q} = -\nabla \phi$

$$\Rightarrow \; u = -\frac{\partial \phi}{\partial r} \Rightarrow \; -\frac{\partial \phi}{\partial r} = \frac{f}{r^2} \qquad \text{| From (2)}$$

$$\Rightarrow \; \phi = \frac{f}{r} \qquad (3)$$

The pressure equation for irrotational non-steady fluid motion in the absence of body forces is

$$\frac{p}{\rho} + \frac{1}{2}\vec{q}^{\,2} - \frac{\partial \phi}{\partial t} = C(t)$$

i.e.

$$\frac{p}{\rho} + \frac{1}{2}u^2 - \frac{\partial \phi}{\partial t} = C(t)$$

where $C(t)$ is a function of time t.

As $r \to \infty,\; p \to \Pi, u = f/r^2 \to 0, \phi \to 0$

So that $$C(t) = \Pi/\rho \text{ for all } t \qquad (5)$$

Therefore, from (2),(3),(4)&(5), we get

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\partial}{\partial t}(f/r) - \frac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2 \tag{6}$$

But $$\frac{\partial f}{\partial t} = \frac{d}{dt}\left(R^2\dot{R}\right) = \ddot{R}R^2 + 2R\dot{R}^2$$

At the surface of the sphere, we have $r = R$ and equation (6) gives

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{R}\left(2R\dot{R}^2 + \ddot{R}R^2\right) - \frac{1}{2}\dot{R}^2$$

$$\Rightarrow \qquad \frac{p}{\rho} = \frac{\Pi}{\rho} + 2\dot{R}^2 + R\ddot{R} - \frac{1}{2}\dot{R}^2$$

$$= \frac{\Pi}{\rho} + \frac{1}{2}\left(3\dot{R}^2 + 2\ddot{R}R\right)$$

(7)

Now,

$$\frac{d^2\left(R^2\right)}{dt^2} + \left(\dot{R}\right)^2 = \frac{d}{dt}\left(2R\dot{R}\right) + \left(\dot{R}\right)^2$$

$$= \left(2R\ddot{R} + 2\dot{R}^2\right) + \dot{R}^2$$

$$= 2R\ddot{R} + 3\dot{R}^2$$

Therefore, from (7), we obtain

$$p = \Pi + \frac{1}{2}\rho\left[\frac{d^2\left(R^2\right)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Hence the result.

Example: An infinite mass of ideal incompressible fluid is subjected to a force $\mu r^{-7/3}$ per unit mass directed towards the origin. If initially the fluid is at rest and there is cavity in the form of the sphere $r = a$ in it, show that the cavity will be completely filled after an interval of time $\pi a^{5/3}(10\mu)^{-1/2}$.

Solution. The motion is entirely radial and consequently irrotational and the present case in the case of spherical symmetry. The equation of continuity is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2u\right)=0 \Rightarrow r^2u = const. = f\left(t\right) \tag{1}$$

On the surface of the sphere, $r = R, \quad \dot{R} = v\,(say)$

Therefore,

$$r^2\dot{r} = f(t) = R^2\dot{R}$$

$$\Rightarrow \qquad \dot{f}(t) = R^2\dot{R} + \dot{R}2R\dot{R} = R^2\frac{dv}{dt} + 2Rv^2$$

$$\Rightarrow \qquad \frac{\dot{f}(t)}{R} = 2v^2 + R\frac{dv}{dt} = 2v^2 + R\frac{dv}{dt}\frac{dR}{dt}$$

$$= 2v^2 + Rv\frac{dv}{dt} \tag{2}$$

The Euler's equation of motion, in radial direction, using $\dot{r} = u$, is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

But $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{f(t)}{r^2}\right) = \frac{\dot{f}(t)}{r^2}, \; F_r = -\mu r^{-7/3}$$

So, we need to integrate the Euler's equation

$$\frac{\dot{f}(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}u^2\right) = \frac{-\mu}{r^{7/3}} - \frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) \tag{3}$$

Let us assume that the cavity has cavity radius R at time t and its velocity then is $\dot{R} = v$. Integrating (3) over the whole liquid $(r = R \, to \, r = \infty)$ at time t, we obtain

$$\left[\frac{-f(t)}{r}\right]_R^{\infty} + \left[\frac{1}{2}u^2\right]_v^0 = \frac{3\mu}{4}\left[\frac{1}{r^{4/3}}\right]_R^{\infty} - \left[\frac{p}{\rho}\right]_R^{\infty}$$

Since the fluid is at rest at infinity, $u_\infty = 0$. Also $p_\infty = 0$, $p_R = 0$ (cavity), thus we get

$$\frac{\dot{f}(r)}{R} - \frac{1}{2}v^2 = \frac{3\mu}{4}\frac{1}{R^{4/3}}$$

$$\Rightarrow \qquad 2R^3 v\frac{dv}{dR} + 3v^2 = -\frac{3\mu}{2}\frac{1}{R^{4/3}} \qquad | \text{ using } (2)$$

To make it exact, we multiply by R^2 so that

$$2R^3 v\frac{dv}{dR} + 3v^2 = -\frac{3\mu}{2}\frac{1}{R^{4/3}}$$

$$\Rightarrow \qquad \frac{d(R^3v^2)}{dR} = -\frac{3\mu}{2}R^{2/3}$$

Integrating, we get

$$R^3v^2 = A - \frac{9\mu}{10}R^{5/3} \qquad (4)$$

When $R = a$, $\dot{R} \equiv v = 0$, which gives $A = \frac{9\mu}{10}a^{5/3}$

Now, we take $v = R < 0$ because as the cavity fills, R decrease with time.

Thus (4) gives

$$\frac{dR}{dt} = -\left(\frac{9\mu}{10}\right)^{1/2}\left(\frac{a^{5/3} - R^{5/3}}{R^3}\right)^{1/2}$$

Therefore,

$$\left(\frac{9\mu}{10}\right)^{1/2} t = -\int_a^0 \frac{R^{3/2}}{\left(a^{5/3} - R^{5/3}\right)^{1/2}}$$

$$= \frac{6a^{5/3}}{5}\int_0^{\pi/2} \sin^2\theta d\theta \qquad | \; R^{5/3} = a^{5/3}\sin^2\theta \text{ i.e. } R = a(\sin\theta)^{6/5}$$

$$= \frac{3\pi a^{5/3}}{10}$$

Thus,

$t = \pi a^{5/3}(10\mu)^{-1/2}$ Hence the result.

Impulsive Motion

Impulsive motion occurs in a fluid when there is rapid but finite charge in the fluid velocity $\vec{q}$ over a short interval δt of time t, or a high pressure on a boundary acting over time δt, or the rapid variation in the velocity of a rigid body immersed in the fluid. Such type of actions are termed as impulsive actions.

The situation of impulsive action is effectively modeled mathematically by letting the body force or pressure approach to infinity while $\delta t \to 0$ in such a way that the integral of body force or pressure over the time interval δt remains finite in this limit.

It the flow is incompressible, infinitely rapid propagation of the effect of the impulsive action takes place, so that an impulsive pressure is produced instantaneously throughout the fluid. Here, we consider only the incompressible fluid with constant density ρ. The impulsive body force $\vec{I}$ and impulsive pressure P are defined as

$$\vec{I} = \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} \vec{F} dt$$

$$P = \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} p dt \qquad (1)$$

We note that finite body forces such as gravity do not contribute to the impulsive body force $\vec{I}$.

To determine the equation of impulsive motion, we consider the Euler's equation

$$\frac{D\vec{q}}{Dt} \equiv \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = \vec{F} - \frac{1}{\rho}\nabla p$$

Integrating w.r.t. time t from to $t+\delta t$ and taking limit as $\delta t \to 0$, we get

$$\underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} \frac{D\vec{q}}{Dt} dt \equiv \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} \frac{\partial \vec{q}}{\partial t} dt + \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} (\vec{q}.\nabla) dt$$

$$= \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} \vec{F} dt - \frac{1}{\rho} \underset{\delta t \to 0}{Lt} \int_t^{t+\delta t} \nabla p dt \qquad (2)$$

Assuming that fluid is accelerated impulsively at $t=0$ and since we expert a finite change in $\vec{q}$ as a result of the impulsive, we get form $(1)\&(2)$

$$\vec{q}'-\vec{q}=\vec{I}-\frac{1}{\rho}\nabla P \tag{3}$$

where $\vec{q}'\&\vec{q}$ denote respectively the fluid velocity before and after the impulsive action.

Thus, the equation of impulsive motion is

$$\rho(\vec{q}'-\vec{q})=\rho\vec{I}-\nabla P \tag{4}$$

Which holds at each point of the fluid.

In Cartesian co-ordinates, (4) can be expressed as

$$\rho(u'-u)=\rho X'-\frac{\partial p}{\partial x}$$

$$\rho(v'-v)=\rho Y'-\frac{\partial p}{\partial y}$$

$$\rho(w'-w)=\rho Z'-\frac{\partial p}{\partial z}$$

where

$$\vec{q}=(u,v,w),\ \vec{q}'=(u',v',w'),\vec{I}=(X',Y',Z')$$

when there is no externally applied impulse, then $\vec{I}=\vec{0}$ and equation (4) becomes

$$-\nabla P=\rho(\vec{q}'-\vec{q}) \tag{5}$$

Further, if the motion is irrotational, then $\vec{q}=-\nabla\phi,\vec{q}'=-\nabla\phi'$, where $\phi\&\phi'$ denote the velocity potential just before and just after the impulsive action, then (5) becomes

$$P=\rho(\phi'-\phi) \tag{6}$$

Where we have ignored the constant of integration since an extra pressure, constant throughout the liquid, would not effect the impulsive motion.

Corollary. If the fluid is at rest prior to the impulsive action, then the velocity $\vec{q}$ generated in the fluid by the impulse is given by

$$\vec{q} = \vec{I} - \frac{1}{\rho}\nabla P \qquad (7)$$

$$| \; In(3),\, put\, \vec{q} = 0 \,\&\, \vec{q}' \equiv \vec{q}$$

For this case, equation (5) can be put as

$$-\nabla P = \rho\vec{q} \qquad (8)$$

And equation (6) becomes

$$P = \rho\phi \qquad (9)$$

Equations $(6)\,\&\,(9)$ give the relation between impulsive pressure P and the velocity potential $\phi.$

Remark. From the above discussion, we observe that, likewise, an irrotational motion can be brought to rest by applying an impulsive pressure $-\rho\phi$ throughout the fluid.

Example. A sphere of radius a is surrounded by an infinite liquid of density $\rho,$ the pressure at infinity being $\Pi.$ The sphere is suddenly annihilated. Show that the pressure at distance r from the centre immediately falls to $\pi\left(1-\frac{a}{r}\right).$ Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{a}{2},$ the impulsive pressure sustained by the surface of the sphere is $\sqrt{7\Pi\rho a^2/6}.$

Solution. Let v' be the velocity at a distance r' from the centre of the sphere at any time $t\,\&\,p$ be the pressure. The equation of continuity (case of spherical symmetry) is

$$\frac{1}{r'^2}\frac{d}{dr}(r'^2 v') = 0 \qquad \Rightarrow r'^2 v^2 = f(t) \qquad (1)$$

Equation of motion is

$$\frac{\partial v'}{\partial t}+y'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad | \text{ No body force}$$

or $$\frac{f(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

integrating w.r.t. r', we get

$$-\frac{f(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{\rho}+C$$

Since $r'\to\infty \Rightarrow p=\Pi, v'=0$ so that $C=\Pi/\rho$.

Thus $$\frac{-f(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p}{\rho} \qquad (2)$$

When, sphere is suddenly annihilated i.e. $r'=a, v'=0, p=0,$ then

$$-\frac{f(t)}{a}=\Pi/\rho \quad \text{i.e.} \quad f(t)=-\frac{\Pi a}{\rho} \qquad (3)$$

The velocity v' vanishes just after annihilation, so from $(2)\&(3),$ we get

$$\frac{\Pi a}{\rho r'}=\frac{\Pi-p}{\rho}\Rightarrow\frac{a\Pi}{r'}=\Pi-p$$

Thus, the pressure at the time of annihilation $(r'=r)$ is

$$\frac{a\Pi}{r'}=\Pi-p\Rightarrow p=\Pi\left(1-\frac{a}{r}\right)$$

Which proves the first result.

Now, let P be the impulsive pressure at a distance r', then from the relation $-\nabla P=\rho\vec{q},$ we get

$$-\frac{dP}{dr'}=\rho v'\Rightarrow dP=-\rho v'\,dr'$$

From the equation of continuity, we have

$$r^2 v = r'^2 v' = f(t) \tag{4}$$

So $$dP = -\rho v (r^2/r'^2) dr' \tag{5}$$

Where r is the radius of the inner surface and v is the velocity there, Integrating (5), we get

$$P = \rho v (r^2/r'^2) + C_1$$

When $r' \to \infty, P = 0$ so that $C_1 = 0$

Thus $$P = \rho v (r^2/r'^2) \tag{6}$$

Equation (6) determines the impulsive pressure P at a distance r'. The velocity v at the inner surface of the sphere $(p = 0)$ is obtained from (2) as

$$-\frac{f(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \tag{7}$$

From (4), $f(t) = \frac{d(r^2 v)}{dt} = r^2 \frac{dv}{dt} + v.2r\frac{dr}{dt} = r^2 \frac{dv}{dr}\frac{dr}{dt} + 2rv^2$

$$\Rightarrow \qquad f(t) = rv\frac{dv}{dr} + 2v^2$$

Thus (7) becomes

$$rv\frac{dv}{dr} + 2v^2 - \frac{1}{2}v^2 = -\frac{\Pi}{\rho}$$

Or $$rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{\Pi}{\rho}$$

$$\Rightarrow \qquad 2r^3 v\frac{dv}{dr} + 3v^2 r^2 = -\frac{2\Pi}{\rho}r^2$$ | Multiplying by r^2

$$\Rightarrow \qquad \frac{d(r^3 v^2)}{dr} = -\frac{2\Pi}{\rho}r^2$$

Integrating, we get

$$r^3v^2 = -\frac{2\Pi}{3\rho}r^3 + C_2$$

Since $r = a, v = 0$ so we find $C_2 = \frac{2\Pi a^3}{3\rho}$

Therefore, $r^3v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3)$

The velocity v at the surface of the sphere $r = a/2,$ on which the liquid strikes,

is $v^2 = \frac{2\Pi}{3\rho}\frac{a^3 - (a/2)^3}{(a/2)^3} = \frac{14}{3}\frac{\Pi}{\rho}$

From relation $(6),$ using $r = a/2,$ we get $P = \frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}\frac{a^2}{r'}}$ (8)

Which determines the impulsive pressure at a distance r' from the centre of the sphere.

Thus, the impulsive pressure at the surface of the radius $a/2$ is given by

$$P = \frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}\frac{a^2}{a/2}} = \sqrt{7\Pi\rho a^2/6}$$. Hence the result

Stream Function

When motion is the same in all planes parallel to xy plane (say) and there is no velocity parallel to the $z-$axis i.e. when u, v are functions of x, y, t only and $w = 0,$ we may regard the motion as two-dimensional and consider only the cases confined to the xy plane. When we speak of the flow across a curve in this plane, we shall mean the flow across unit length of a cylinder whose trace on the xy plane is the question, the generators of the cylinder being parallel to the $z-$axis.

For a two-dimensional motion in xy plane, $\vec{q}$ is a function of x, y, t only and the differential equation of the streamlines (lines of flow) are

$$\frac{dx}{u}=\frac{dy}{v} \text{ i.e. } vdx-udy=0 \qquad (1)$$

and the corresponding equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2)$$

We note that equation (2) is the condition of exactness of (1), it follows that (1) must be an exact differential, $d\psi$ (say). Thus

$$vdx-udy=d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

so that

$$v=-\frac{\partial \psi}{\partial y}, v=\frac{\partial \psi}{\partial x}$$

This function ψ is called the stream function or the current function or Lagrange's stream function.

Obviously, the streamlines are given by the solution of (1) i.e. $d\psi=0$ i.e. $\psi=const.$ (For unsteady flow, streamlines are given by $\psi=f(t)$)

Thus, the stream function is constant along a streamline.

From the above discussion, it is clear that the existence of stream function is merely a consequence of the continuity and incompressibility of the fluid. The stream function always exists in all types of two dimensional motion whether rotational or irrotational. However, it should be noted again that velocity potential exists only for irrotational motion whether two dimensional or three dimensional.

Physical Interpretation of Stream Function:-

Let P be a point on a curve C in $xy-$plane. Let an element ds of the curve makes an angle θ with $x-$axis.

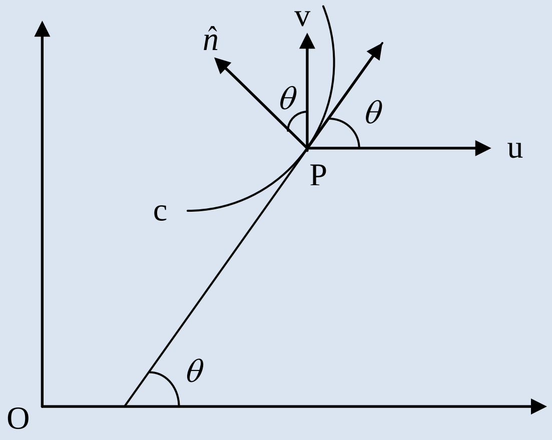


figure 2.9

The direction cosines of the normal at P are $(\cos(90+\theta), \cos\theta, 0)$ i.e. $(-\sin\theta, \cos\theta, 0)$

The flow across the curve C from right to left is

$$= \int_C \vec{q}.\hat{n}ds, \text{ where } \hat{n} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}, \quad \vec{q} = u\,\hat{i} + v\,\hat{j}$$

$$= \int_C (-u\sin\theta + v\cos\theta)ds$$

$$= \int_C \left(\frac{\partial\psi}{\partial y}\sin\theta + \frac{\partial\psi}{\partial x}\cos\theta \right) ds \quad u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}$$

$$= \int_C \left(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial x}\frac{dy}{ds} \right) ds \quad \cos\theta = \frac{dx}{ds}, \sin\theta = \frac{dy}{ds}$$

$$= \int_C \left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial x}dy \right)$$

$$= \int_C d\psi = (\psi_B - \psi_A)$$

where ψ_A & ψ_B are the values of ψ at the initial and final points of the curve. Thus, the difference of the values of a stream function at any two points represents the flow across that curve, joining the two points.

Corollary. If we suppose that the curve C be the streamline, then no fluid crosses its boundary, then

$$(\psi_B - \psi_A) = 0 \Rightarrow \psi_B = \psi_A$$

i.e. ψ is constant along c.

Relation Between (i.e. C-R equations): $\phi \,\&\, \psi$ -

We know that the velocity potential ϕ is given by

$$\vec{q} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

i.e. $$u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y} \qquad (1)$$

Also, the stream function ψ is given by

$$u = -\frac{\partial\psi}{\partial x},\ v = \frac{\partial\psi}{\partial y} \qquad (2)$$

From $(1)\,\&\,(2)$, we get

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad (3)$$

Equation in (3) imply

$$\nabla^2\phi = 0 \,\&\, \nabla^2\psi = 0$$

i.e. $\phi \,\&\, \psi$ are harmonic functions.

again, from (3), we get

$$\nabla\phi = grad\,\phi = -\vec{q} = -(u\,\hat{i} + v\,\hat{j})$$

$$= -\left(-\frac{\partial\psi}{\partial y}\,\hat{i} + \frac{\partial\psi}{\partial x}\,\hat{j}\right)$$

$$= \frac{\partial \psi}{\partial y} \hat{i} - \frac{\partial \psi}{\partial x} \hat{j}$$

$$= \frac{\partial \psi}{\partial y} (\hat{j} \times \hat{k}) + \frac{\partial \psi}{\partial x} (\hat{i} \times \hat{k})$$

$$= \left(\frac{\partial \psi}{\partial y} \hat{i} - \frac{\partial \psi}{\partial x} \hat{j} \right) \times \hat{k}$$

$$= \nabla \psi \times \hat{k} = grad \, \psi \times \hat{k}$$

i.e. $$grad \, \phi = (grad \, \psi) \times \hat{k} = -\hat{k} \times grad \, \psi$$

i.e. $$\nabla \phi = \nabla \psi \times \hat{k} \qquad (4)$$

Again, from (3), we note that

$$\frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} \left(- \frac{\partial \phi}{\partial y} \right)$$

$$\Rightarrow \qquad \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} = 0$$

i.e. $$\nabla \phi . \nabla \psi = 0 \qquad (5)$$

Thus, for irrotational incompressible two-dimensional flow (steady or unsteady), $\phi(x, y), \psi(x, y)$ are harmonic functions and the family of curves $\phi = const.$ (equipotentials) and $\psi = const.$ (streamlines) intersect orthogonally.

Exercise. Show that $u = 2cxy, v = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function and the streamlines.

Remark. We shall consider the study of two dimensional motion later on. At present we continue discussing three dimensional irrotational flow of incompressible fluids.

Three Dimensional Irrotational Flow

Acyclic and Cyclic Irrotational Motion: An irrotational motion is called acyclic if the velocity potential ϕ is a single valued function i.e. when at every field point, a unique velocity potential exists, otherwise the irrotational motion is said to be cyclic. Clearly, only acyclic irrotational motion is possible in a simply connected region.

For a possible fluid motion, even if ϕ is multivalued at a particular point, the velocity at that point must be single-valued. Hence if we obtain two different values of ϕ, these values can only differ by a constant.

At present, we restrict ourself to acylic irrotational motion for which we prove a number of results related to ϕ.

Mean Value of Velocity Potential Over Spherical Surfaces:

Theorem: The mean value of a ϕ over any spherical surface S drawn in the fluid throughout whose interior $\nabla^2\phi = 0$, is equal to the value of ϕ at the centre of the sphere.

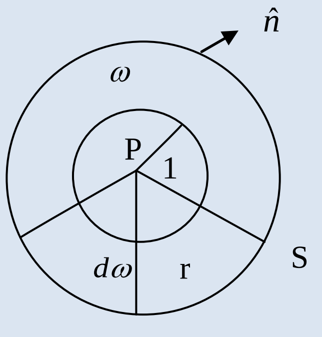


figure 2.10

Proof. Let $\phi(P)$ be the value of ϕ at the centre P of a spherical surface S of radius r, wholly lying in the liquid and let $\bar{\phi}$ denotes the mean value of ϕ over S. Let us draw another concentric sphere ω of unit radius. Then a cone with vertex P which intercepts area ds from the sphere S, intercepts an area $d\omega$ from the sphere ω and we have

$$\frac{dS}{d\psi} = \frac{r^2}{1^2} \Rightarrow dS = r^2 d\omega \tag{1}$$

Now, by definition

$$\bar{\phi} = \frac{\int_S \phi dS}{\int_S dS} = \frac{1}{4\pi r^2} \int_S \phi dS$$

$$= \frac{1}{4\pi r^2} \int_S \phi r^2 d\omega = \frac{1}{4\pi} \int_S \phi d\omega$$

$$\Rightarrow \qquad \frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial r} d\omega = \frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial r} \frac{dS}{r^2}$$

$$= \frac{1}{4\pi r^2} \int_S \frac{\partial \phi}{\partial r} dS \Rightarrow \qquad \frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial r} d\omega = \frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial r} \frac{dS}{r^2} \tag{2}$$

Since the normal $\hat{n}$ to the surface is along the radius r, therefore on S, we have

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial n} = \nabla \phi . \hat{n} \tag{3}$$

From $(2) \& (3)$, we find

$$\frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi r^2} \int_S \nabla \phi . \hat{n} dS$$

$$= \frac{1}{4\pi r^2} \int_\tau div(\nabla \phi) d\tau \qquad \text{| Gauss theorem}$$

$$= \frac{1}{4\pi r^2} \int_\tau \nabla^2 \phi d\tau = 0, \qquad | \ \nabla^2 \phi = 0$$

where τ is the volume enclosed by the surface S.

Thus $\qquad \frac{\partial \bar{\phi}}{\partial r} = 0 \ \Rightarrow \bar{\phi} = const.$

This shows that $\bar{\phi}$ is independent of choice of r and hence mean value of ϕ is same over all spherical surfaces having the same centre P. When S shrinks to point P, then $\bar{\phi} = \phi(P)$

Corollary. The velocity potential ϕ cannot have a maximum or minimum value in the interior of any region throughout which $\nabla^2\phi = 0$.

Proof. If possible suppose that ϕ has a maximum value $\phi(P)$ at a point P. We draw a sphere with centre P and radius $\in$, where $\in$ is small. Then the mean value $\bar{\phi}$ of ϕ must be less than $\phi(P)$ i.e. $\bar{\phi} < \phi(P)$ as $\phi(P)$ is maximum. This is a contradiction to the mean potential theorem in which $\bar{\phi} = \phi(P)$. Thus ϕ cannot have a maximum value. Similarly ϕ cannot have a minimum value.

Theorem: In an irrotational motion the maximum value of the fluid velocity occurs at the boundary.

Proof. Let P be any interior point of the fluid and Q be a neighbouring point also lying in the fluid. Let us take the direction of $x-$axis along the direction of $\vec{q}$ at P. Let q_P & q_Q denote the speed of particles at P & Q respectively.

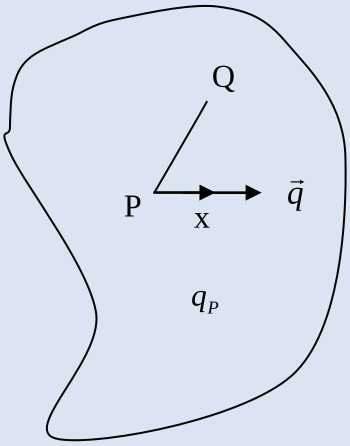


figure 2.10

Then $$q_P^{\,2} = \left(\frac{\partial\phi}{\partial x}\right)_P^2$$

and $$q_Q^{\,2} = \left(\frac{\partial \phi}{\partial x}\right)_Q^2 + \left(\frac{\partial \phi}{\partial y}\right)_Q^2 + \left(\frac{\partial \phi}{\partial z}\right)_Q^2$$

Since $$\nabla^2 \phi = 0 \Rightarrow \frac{\partial}{\partial x}(\nabla^2 \phi) = 0 \Rightarrow \nabla^2 \left(\frac{\partial \phi}{\partial x}\right) = 0$$

$\Rightarrow \dfrac{\partial \phi}{\partial x}$ satisfies Laplace equation. Therefore, by mean value theorem (corollary), $\dfrac{\partial \phi}{\partial x}$ cannot be maximum or minimum at P. Thus, there are points such as Q in the neighbourhood of P such that

$$\left(\frac{\partial \phi}{\partial x}\right)_Q^2 > \left(\frac{\partial \phi}{\partial x}\right)_P^2 \Rightarrow q_Q^{\,2} > q_P^{\,2}$$

$\Rightarrow q_Q^{\,2}$ cannot be maximum in the interior of fluid and its maximum value $|\vec{q}|$, if any, must therefore occur on the boundary.

Note. $q = |\vec{q}|$ may be minimum in the interior of the fluid as $\vec{q} = \vec{0}$ at the stagnation point. i.e. q in minimum at stagnation points.

Corollary. In steady irrotational flow, the pressure has its minimum value on the boundary.

Proof. From Bernoulli's equation, we have

$$\frac{p}{\rho} + \frac{1}{2} q^2 = const. \qquad (1)$$

Equation (1) shows that p is least when q^2 is greatest and by above theorem, q^2 is greatest at the boundary . Thus, the minimum value of p must occur only on the boundary.

Note. The maximum value of p occurs at the stagnation points, where $\vec{q} = \vec{0}$.

Theorem. If liquid of infinite extent is in irrotational motion and is bounded internally by one or more closed surfaces S, the mean value of ϕ over a large sphere Σ, of radius R, which enclosed S, is of the form

$$\bar{\phi} = \frac{M}{R} + C$$

where M & C are constant, provided that the liquid is at rest at infinity.

Proof. Suppose that the volume of fluid acrossing each of internal surfaces contained within Σ, per unit, is a finite quantity say $-4\pi M$ (i.e. $-4\pi M$ represent the flux of fluid across Σ is $\frac{\partial \phi}{\partial R}$ radially outwards, the equation of continuity gives

$$\int_{\Sigma} \frac{\partial \phi}{\partial R} d\Sigma = -4\pi M \qquad (1)$$

But $\qquad d\Sigma = R^2 d\omega$

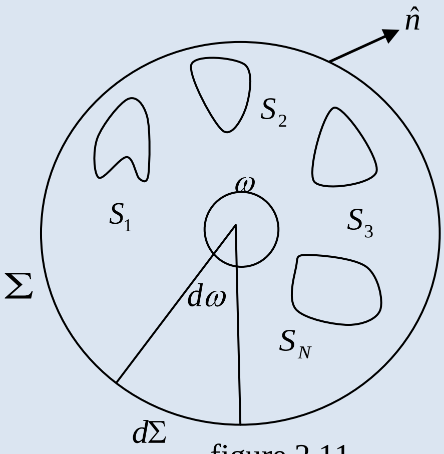


figure 2.11

Therefore,

$$\frac{1}{4\pi} \int_{\Sigma} \frac{\partial \phi}{\partial R} R^2 d\omega = -M$$

$$\Rightarrow \frac{1}{4\pi} \int_{\Sigma} \frac{\partial \phi}{\partial R} d\omega = \frac{-M}{R^2}$$

$$\Rightarrow \frac{1}{4\pi} \frac{\partial}{\partial R} \int_{\Sigma} \phi d\omega = \frac{-M}{R^2}$$

Integrating w.r.t. R, we get

$$\frac{1}{4\pi}\int_{\Sigma}\phi d\omega = \frac{M}{R} + C$$

where C is independent of R

$$\Rightarrow \frac{1}{4\pi}\int_{\Sigma}\phi\left(\frac{d\Sigma}{R^2}\right)d\omega = \frac{M}{R} + C$$

$$\Rightarrow \frac{\int_{\Sigma}\phi d\Sigma}{4\pi R^2} = \frac{M}{R} + C$$

$$\Rightarrow \bar{\phi} = \frac{M}{R} + C \qquad (2)$$

To show that C is an absolute constant, we have to prove that it is independent of co-ordinates of centre of sphere Σ. Let the centre of the sphere Σ be displaced by distance δx in an arbitrary direction while keeping R constant, then from (2),

$$\frac{\partial\bar{\phi}}{\partial x} = \frac{\partial C}{\partial x} \qquad (3)$$

| $\because R$ is constant

Also, $$\frac{\partial\bar{\phi}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{4\pi}\int_{\Sigma}\phi dw\right] = \frac{1}{4\pi}\int_{\Sigma}\frac{\partial\phi}{\partial x}dw = 0,$$

since $\frac{\partial\phi}{\partial x} = 0$ on Σ when $R \to \infty$ as the liquid is at rest at infinity.

$\therefore$ From (3), we get

$$\frac{\partial C}{\partial x} = 0 \Rightarrow C \text{ is an absolute constant.}$$

Hence, $\bar{\phi} = \frac{M}{R} + C,$ where