Assignment – 2

Mathematical Methods

1. Find the angle between the vectors $|\alpha\rangle = (1+i)\hat{\imath} + 1\hat{\jmath} + i\hat{k}$ and $|\beta\rangle = (4-i)\hat{\imath} + 0\hat{\jmath} + (2-2i)\hat{k}$.
2. Prove that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Prove same for inverse.
3. Using the standard basis $\hat{\imath}, \hat{\jmath}, \hat{k}$ in 3D, (a)construct a matrix representing rotation through angle θ counterclockwise about the origin about the z-axis. (b) Construct a matrix representing a rotation by 120° about an axis through the point $(1,1,1)$ (counterclockwise, looking down the axis). (c) Construct a matrix representing reflection through x-y plane.
4. Show that a matrix S carries an orthonormal basis into another orthonormal basis iff it is unitary.
5. Prove that $\det(T) = \lambda_1\lambda_2 \dots \lambda_n\,; Trace\ (T) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ where $\lambda's$ are the n-solutions of the characteristic equation. What happens if some eigenvalues are degenerate.
6. Show that eigenvectors of a hermitian matrix belonging to distinct eigenvalues are orthogonal.
7. Let $T = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$. Verify that it is hermitian. Find its eigenvalues and normalized eigenvectors. Construct the unitary diagonalizing matrix S and check explicitly that it diagonalizes T. Check that $\det(T)$ and $Trace\ (T)$ are same before and after diagonalization.
8. Consider the hermitian matrix $T = \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$. Find eigenvalues and using those write T in diagonal form. Find eigenvectors (in the degenerate sector, construct two linearly independent vectors, which is always possible for hermitian matrix but not in general). Orthogonalize them and show that both are orthogonal to the third eigenvector. Construct the unitary matrix S that diagonalizes T and show that S diagonalizes T explicitly.
9. Show that trace and determinant of an operator is unaffected by a unitary change of basis.
10. What is the condition for two matrices two be simultaneously diagonalizable.
11. Show that if two matrices commute in one basis, they commute in any basis.
12. Show that trace of sum of matrices equals sum of traces of the matrices.
13. Show that every even power of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ equals unit matrix and every odd power equals itself.
14. If A is a nonsingular matrix, show that $\det(A^{-1}) = \frac{1}{\det(A)}$.
15. If $AB = 0$, show that wither A and B are both singular or one of them is a zero matrix.
16. If U is a non-singular matrix and $U^{-1}AU$ and $U^{-1}BU$ are diagonal, prove that A and B commute.
17. If a matrix $\boldsymbol{P}$ diagonalizes $\boldsymbol{A}$, prove that the columns of $\boldsymbol{P}$ are eigenvectors of $\boldsymbol{A}$.
18. Find common set of eigenvectors and simultaneously diagonalize the following matrices: $\begin{bmatrix} 1 & -3 & 6 \\ 1 & 5 & -2 \\ 2 & 2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 8 & -4 & -4 \\ -2 & -2 & 4 \\ -3 & -9 & 12 \end{bmatrix}$.

19. Solve eigenvalue problem and diagonalize whenever possible. Find a set of linearly independent eigenvectors in each case. Reduce each matrix to diagonal or triangular form:
(a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & -1 \\ 2 & 4 & 1 \\ 0 & -2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 4 & 0 \\ 2 & 2 & 1 \\ -10 & -12 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & 1 \\ 5 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$.
20. Diagonalize the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find the similarity matrix P that such that $P^{-1}AP$ is diagonal.
21. Let $|e_1\rangle = (1,0,0)^T, |e_2\rangle = (0,1,0)^T, |e_3\rangle = (0,0,1)^T$. Define a new basis $|f_1\rangle = (1,1,0)^T, |f_2\rangle = (0,1,1)^T, |f_3\rangle = (1,0,1)^T$. Find the change of basis matrix. Express the $(2,1,)^T$ vector in the new basis.
22. Simultaneously diagonalize the matrices $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.
23. Show that if λ is an eigenvalue of an orthogonal matrix, then $1/\lambda$ is also an eigenvalue.
24. Prove that a real matrix is unitary if it is orthogonal. Also prove that if H is a hermitian matrix, then $U = e^{iH}$ is a unitary matrix.
25. Diagonalize $\begin{pmatrix} 2 & 3-i \\ 3+i & -1 \end{pmatrix}$ by suitable similarity transformation.
26. If D is a diagonal matrix, then show that D^n is the diagonal matrix with elements equal to the n^{th} power of D. Also show that if $D = C^{-1}MC$, then $D^n = C^{-1}M^nC$.
27. State and verify "Caley-Hamilton theorem".
28. Show that $\delta(ax) = \frac{\delta(x)}{a}$. Also show that $\delta\big(f(x)\big) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx_i}\right|}$.
29. Let $\widehat{D}$ be the differentiation operator for functions: $\widehat{D}|F\rangle = \left|\frac{df}{dx}\right\rangle$. Show that in the position basis $\langle x|D|x'\rangle = D_{xx'} = \delta'(x-x') = \delta(x-x')\frac{d}{dx'}$. Also show that $\widehat{D}$ is not hermitian but $i\widehat{D}$ is hermitian.
30. Set up the eigenvalue problem and find characteristic frequencies for the system of masses and springs shown below.