Assignment – 1

Mathematical Methods

1. Show that the null vector of a vector space is unique.
2. Show that the following row vectors are linearly independent: (1, 1, 0), (1,0,1), (3,2,1). Show the opposite for (1,1,0), (1,0,1) and (0,1,1).
3. Determine whether the following sets of vectors are linearly dependent or independent: (a) $(1, 2, -1), (0,5, -3)$ and $(-4,1,3)$. (b) $(1,2,0, -2, -1), (3, -1, 4, 1, -2), (0,2, -3,4,1), (3,3,3,4,1)$ and $(-1,0,2,1,3)$.
4. Show that the three vectors $(1,1,1), (1,0, -1)$ and $(1, -2,1)$ are a set of linearly independent vectors. Verify that they are also orthogonal. Express $(1,0,0)$ and $(2, -1,3)$ as a linear combination of the three L.I. vectors.
5. Find the value of x for which the given set of vectors are linealry independent: (a) $(1,2,3), (4,5,6), (x, 8,9)$. (b) $(0, -1,2), (0,1,6), (1,2, x)$.
6. Form an orthonormal basis in two dimensions starting with $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}; \vec{B} = 2\hat{\imath} - 6\hat{\jmath}$. Can you generate another orthonormal basis starting with these two vectors? If so, produce another.
7. If $|u_i\rangle, 1 \leq i \leq n$, is a set of n vectors, Their Gram-Schmidt determinant (or simply Gram determinant) is defined as an $n \times n$ determinant whose elements are the scalar products of the vectors with each other
$$D = \begin{bmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_1, u_n \rangle \\ \vdots & \ddots & \vdots \\ \langle u_n, u_1 \rangle & \cdots & \langle u_n, u_n \rangle \end{bmatrix}$$
Prove that the set of vectors is linearly independent if and only if $D = 0$.
8. Prove Schwarz and Triangle inequalities.
9. Consider the collection of all polynomials (with complex coefficients) of degree less than N in x. Does this set constitute a vector space (with thew polynomials as vectors)? If so, suggest a convenient basis, and give the dimension. If not, which defining property does it lack? What if we require the polynomials to be even functions?
10. Consider the ordinary vectors in three dimensions: $a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$. with complex components. Does the subset of all vectors with $a_z = 0$ constitute a vector space? If so, what is its dimension?
What about the vectors whose $z -$ component is 1? What about the subset of vectors whose components are all equal?
11. Prove that components of a given vector with respect to a given basis are unique.
12. Show that product of two unitary matrices are unitary. Under what condition the product of two hermitian matrices hermitian? Is the sum of two unitary? Is the sum of two hermitian matrices hermitian?
13. Show that rows and columns of unitary matrix constitute orthnormal sets.
14. An operator A is given by the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. What is its action?
15. Prove that determinant of a unitary matrix is a complex number of unit modulus.
16. A transformation $\widehat{\boldsymbol{A}}$ on vectors $\vec{r}$ has the effect $\widehat{\boldsymbol{A}}\vec{r} = \vec{a} \times \vec{r}$, where $\vec{a}$ is a given vector. Show that $\widehat{\boldsymbol{A}}^3 + a^2\widehat{\boldsymbol{A}} = \mathbf{0},$ the null operator where $a = |\vec{a}|$.
17. If the vectors of a set are orthogonal to each other, prove that it is a set of linearly independent vectors.

18. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ in 3-dimensional space are linearly independent iff $\vec{a}.(\vec{b} \times \vec{c}) \neq 0$.
19. Suppose the rows of a real 3×3 matrix A are interpreted as the components in a given basis of three (three-component) vectors$\vec{a}, \vec{b}, \vec{c}$. Show that one can write the determinant of A as $|A| = \vec{a}.(\vec{b} \times \vec{c})$.
20. Show that eigenvalues of hermitian matrices are real.
21. The basis vectors of the unit cell of a crystal are denoted by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ (which are not necessarily mutually orthogonal). The matrix G has elements $G_{ij} = \vec{e}_i \cdot \vec{e}_j$ and H_{ij} are elements of the matrix $H = G^{-1}$. Show that the vectors $\vec{f}_i = \sum_j H_{ij} \vec{e}_j$ are reciprocal vectors and that $H_{ij} = \vec{f}_i \cdot \vec{f}_j$.
22. If A is a hermitian matrix and U is unitary, prove that $U^{-1}AU$ is hermitian.
23. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Is the matrix hermitian? Are the eigenvectors orthogonal?
24. Consider the matrix $\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Is it hermitian? Find eigenvalues and eigenvectors . Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω.
25. Consider the rotation matrix $\Omega = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. Show that it is unitary. Show that its eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$. Find the corresponding eigenvectors and show that they are orthogonal.

Note: Assignments to be submitted by 31 August.