Derivatives :

Derivability of a function : A function f defined on $[a, b]$ is said to be derivable or differentiable at $c \in (a, b)$ if $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists. This limit is called derivative of f at c and is denoted by $f'(c)$. The process of obtaining $f'(c)$ is called differentiation.

Left hand derivative : If $\lim\limits_{h \to 0-} \frac{f(c+h) - f(c)}{h}$ exists, then it is called the left hand derivative of f at c and is denoted by $Lf'(c)$.

Right hand derivative : If $\lim\limits_{h \to 0+} \frac{f(c+h) - f(c)}{h}$ exists, then it is called the right hand derivative of f at c and is denoted by $Rf'(c)$.

Hence f is derivable at $x = c$ if $Lf'(c) = Rf'(c) =$ finite value.

Example 1. *Differentiate $(x^2 + x)^4$ with respect to x.*

Solution. Let $y = (x^2 + x)^4$

Putting $u = x^2 + x$, we have $\quad y = u^4$

Here y is a differentiable function of u and u is a differentiable function of x.

$$\therefore \quad \frac{dy}{du} = 4u^3 \quad \text{and} \quad \frac{du}{dx} = 2x + 1$$

By chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (4u^3) \cdot (2x + 1) = 4(x^2 + x)^3 (2x + 1).$$

Example 2. *Differentiate $\left(\frac{3 + 4x}{2 - x}\right)^2$ w.r.t. x.*

Example 3. *Differentiate $\sqrt{1 + x^2}$ w.r.t. x.*

Solution. Let $\quad y = \sqrt{1 + x^2}$

Putting $u = 1 + x^2$, we have $\quad y = u^{1/2}$

$$\therefore \quad \frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

and
$$\frac{du}{dx} = 2x$$

By chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{2} u^{-1/2} \cdot 2x = \frac{1}{2} (1 + x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}.$$

Example 4. *If $y = v^3 + 2v^2 + 5$, $v = 3u + 1$ and $u = 9x + 1$, find $\frac{dy}{dx}$.*

Solution. Here
$$y = v^3 + 2v^2 + 5 \quad \Rightarrow \quad \frac{dy}{dv} = 3v^2 + 4v$$
$$v = 3u + 1 \quad \Rightarrow \quad \frac{dv}{du} = 3$$
$$u = 9x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 9$$

By chain rule,
$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$
$$= (3v^2 + 4v) \cdot 3 \cdot 9$$
$$= 27 [3 (3u + 1)^2 + 4 (3u + 1)]$$
$$= 27 [3 (9u^2 + 1 + 6u) + 12u + 4]$$
$$= 27 [27 u^2 + 30u + 7]$$
$$= 27 [27 (9x + 1)^2 + 30 (9x + 1) + 7]$$
$$= 27 [27 (81x^2 + 1 + 18x) + 270x + 30 + 7]$$
$$= 27 [2187 x^2 + 756x + 64].$$

Example 5. *If $y = \sqrt{\frac{1 - x}{1 + x}}$; find $\frac{dy}{dx}$.*

Example 6. *Differentiate $(x + \sqrt{x^2 + a^2})^n$ w.r.t. x*

Example 7. *Find $\frac{dy}{dx}$ where $y = [(2x + 3)^2 + 8]^3$.*

$\boxed{\text{Sign of the Derivative}}$

Let f be a function defined on $[a,b]$ & $f'(c)$ exists when $c \in [a,b]$

Case I : If $f'(x) > 0$:

Choose $\varepsilon > 0$ s.t. $f'(c) - \varepsilon > 0$

Since $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$.

For a given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$

whenever $|x - c| < \delta$

$$f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon \quad \text{whenever} \quad x \in (c - \delta, c + \delta)$$

$$0 < \frac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon \quad \text{whenever} \quad x \in (c - \delta, c + \delta)$$

i.e
$$\frac{f(x) - f(c)}{x - c} > 0 \quad \text{whenever} \quad x \in (c - \delta, c + \delta)$$

$$f(x) - f(c) > 0 \quad \text{if} \quad x \in (c, c + \delta)$$
$$f(x) - f(c) < 0 \quad \text{if} \quad x \in (c - \delta, c)$$

Thus, $f'(c) > 0 \Rightarrow f$ is increasing at c.

lly one can prove that if $f'(c) < 0 \Rightarrow f$ is decreasing at c.

2.4. Darboux's Theorem

If a function f defined on $[a, b]$ is such that
(i) f is derivable on $[a, b]$
(ii) $f'(a)$ and $f'(b)$ are of opposite signs,
then there exists some point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let us suppose

Differentiate the following functions w.r.t. x using chain rule of differentiability :

1. (i) $(4x^3 - 5x^2 + 1)^4$ (ii) $\sqrt{15x^2 - x + 1}$
(iii) $\sqrt{2x^2 - 3x + 5}$ (iv) $\frac{1}{\sqrt{x + 1}}$
(v) $\sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$ (vi) $\sqrt{\frac{1 - x^2}{1 + x^2}}$
(vii) $\sqrt{\frac{x^2 - 2ax}{a^2 - 2ab}}$ (viii) $\left(\frac{5x}{3x + 2}\right)^3 - 2\left(\frac{5x}{3x + 2}\right)^2 + 10$
(ix) $2^{\cos x}$ (x) $\left(\frac{x - \sqrt{x}}{1 - 2x}\right)^2$
(xi) $\log \sin x$ (xii) $\tan x^2$

2. (i) If $y = \frac{u}{u - 1}$ and $u = x - \frac{1}{x}$; find $\frac{dy}{dx}$.
(ii) If $y = \frac{3 - u}{2 + u}$ and $u = \frac{4x}{1 - x^2}$; find $\frac{dy}{dx}$.

3. If $y = \sqrt{x^2 + a^2}$, prove that $y \cdot \frac{dy}{dx} - x = 0$.

4. If $y = \sqrt{1 + x^5}$, prove that $\frac{dy}{dx} = \frac{5x^4}{2y}$

5. If f be defined and derivable on $[a, b]$ such that $f(a) = f(b) = 0$ and $f'(a), f'(b)$ are of same sign, then show that f must vanish atleast once in (a, b).

6. If f is derivable on $[a, b]$ such that $f(a) = f(b) = 0$ and $f(x) \neq 0$ for any $x \in (a, b)$, then prove that $f'(a)$ and $f'(b)$ must be of opposite signs.

7. If f is defined and derivable on an interval, then the range of f' is either an interval or a singleton.

8. Let $f : [-1, 1] \to R$ such that $f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$

Does there exist a function g such that $g'(x) = f(x)$ for all $x \in [-1, 1]$.