Definitions of Probability

Mathematical or Classical or a Priori Probability

Let n be the number of mutually exclusive, equally likely and exhaustive cases in a trial out of which m are favourable to the event A. Then the probability of happening of A, usually denoted by $P(A)$, is given by

$$P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{m}{n}$$

Remarks

1. The non-happening of the event A is called the complementary event of A and is denoted by $\bar{A}$. The number of cases favourable to $\bar{A}$ is $n-m$. So probability of non-happening of A is given by

$$P(\bar{A}) = \frac{n-m}{n}$$

$$= 1 - \frac{m}{n}$$

$$= 1 - P(A).$$

2. Odds in favour of an event A is defined as the ratio of number of favourable cases to the number of unfavourable cases.

$\therefore$ Odds in favour of an event A $= \frac{m}{n-m}$

Similarly, odds against an event A is defined as the ratio of no. of unfavourable cases to the number of favourable cases.

$\therefore$ Odds against the event A $= \frac{n-m}{m}$

Limitations of Classical Definition

The classical definition of probability fails in the following situations:

(i) If the various outcomes of the random experiment are not equally likely, for example the probability that a ceiling fan in a room will fall is not $\frac{1}{2}$ since the events of the fan 'falling' and 'not falling' though mutually exclusive and exhaustive, are not equally likely.

(ii) If the exhaustive number of outcomes of the random experiment is infinite or unknown

Statistical or Empirical Definition of Probability

If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.

Suppose an event A occurs m times in n repetitions of a random experiment. Then the ratio $\frac{m}{n}$ is called the relative frequency of the event A. The limiting value of the relative frequency when the number of trials becomes indefinitely large, is called the probability of happening of the event A. Thus

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Limitations of Empirical Probability

(i) If an experiment is repeated a large number of times, the experimental conditions may not remain identical and homogeneous.

(ii) The relative frequency $\frac{m}{n}$, may not attain a unique value, no matter however large n may be.

Example What is the chance that a non-leap year selected at random will contain 53 Sundays?

Soln. A non-leap year consists of 365 days, that is, 52 complete weeks and one day over. The day over may be one of seven days Monday, Tuesday, - - , Sunday

$\therefore$ No. of exhaustive cases = 7

No. of favourable cases = 1.

So, required probability = $\frac{1}{7}$.

Example What is the chance that a leap year selected at random will contain 53 Sundays?

Soln. A leap year consists of 366 days, that is, 52 complete weeks and two days over. The following are the possible combinations for these two 'over' days:

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday

$\therefore$ No. of exhaustive cases $= 7$

No. of favourable cases $= 2$

So required probability $= \frac{2}{7}$

<u>Example</u> Out of $2n+1$ tickets consecutively numbered, three are drawn at random. Find the probability that the numbers on them are in A.P.

<u>Soln</u> 3 tickets out of $(2n+1)$ tickets can be drawn in ${}^{2n+1}C_3$ ways.

$\therefore$ No. of exhaustive cases $= {}^{2n+1}C_3 = \frac{(2n+1)\,2n\,(2n-1)}{3!}$

To find the number of favourable cases, we enumerate cases starting with $1, 2, 3, \ldots$ These cases are given as follows:

Favourable cases	No. of favourable cases
(i) $(1,2,3), (1,3,5), (1,4,7), \ldots, (1, n+1, 2n+1)$	n
(ii) $(2,3,4), (2,4,6), (2,5,8), \ldots, (2, n+1, 2n)$	$n-1$
(iii) $(3,4,5), (3,5,7), (3,6,9), \ldots, (3, n+2, 2n+1)$	$n-1$
$\ldots$	$\ldots$
Last but one $(2n-2, 2n-1, 2n)$	1
Last $(2n-1, 2n, 2n+1)$	1

Total no. of favourable cases

$$= n + 2\,[(n-1) + (n-2) + \ldots + 2 + 1]$$

$$= n + 2 \cdot \frac{(n-1)\,n}{2}$$

$$= n + n^2 - n = n^2$$

So required probability $= \frac{n^2\,(3!)}{(2n+1)\,2n\,(2n-1)} = \frac{3n}{4n^2-1}$

Axiomatic Probability

Given a sample space S of a random experiment, the probability of the occurence of any event A is defined as a set function P(A) satisfying the following axioms.

Axiom 1 $\quad$ P(A) is defined, is real and $P(A) \geq 0$

(Axiom of non-negativity)

Axiom 2 $\quad P(S) = 1$ $\quad$ (Axiom of certainity)

Axiom 3 $\quad$ If $\{A_n\}$ is any finite or infinite sequence of disjoint events of S, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad \text{or} \quad P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(Axiom of additivity)

Some theorems on Probability

Theorem $\quad$ Probability of the impossible event is zero, i.e.,

$$P(\phi) = 0$$

Proof. The impossible event ϕ contains no sample point. So the certain event S and the impossible event ϕ are mutually exclusive.

$$\text{Now} \quad S \cup \phi = S$$

$$\Rightarrow \quad P(S \cup \phi) = P(S)$$

Using axiom of additivity, we get

$$P(S) + P(\phi) = P(S)$$

$$\text{Hence} \quad P(\phi) = 0$$

<u>Theorem</u> $P(\bar{A}) = 1 - P(A)$

Proof. A and $\bar{A}$ are mutually disjoint events such that

$$A \cup \bar{A} = S$$

$$\therefore \quad P(A \cup \bar{A}) = P(S)$$

$$\Rightarrow \quad P(A) + P(\bar{A}) = 1$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

<u>Cor.</u> $0 \leqslant P(A) \leqslant 1$ for every event A.

Proof. By axiom 1, $P(\bar{A}) \geqslant 0$

$$\Rightarrow \quad 1 - P(A) \geqslant 0$$

$$\Rightarrow \quad P(A) \leqslant 1.$$

Also, by axiom 1, $P(A) \geqslant 0$

Hence $0 \leqslant P(A) \leqslant 1$

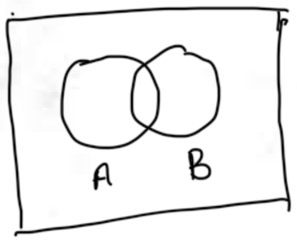
<u>Cor.</u> $P(\phi) = 0$.

Proof. $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$

<u>Theorem</u> For any two events A and B, we have

(i) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(ii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$



Proof. (i) We have $B = (A \cap B) \cup (\bar{A} \cap B)$

where $A \cap B$ and $\bar{A} \cap B$ are disjoint events.

$$\therefore \quad P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$= P(A \cap B) + P(\bar{A} \cap B)$$

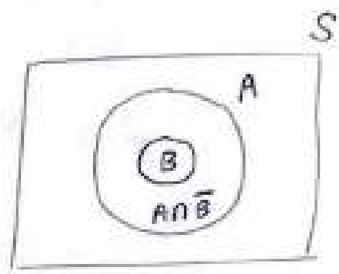
Hence $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(ii) We have $A = (A\cap B)\cup(A\cap\bar{B})$

where $A\cap B$ and $A\cap\bar{B}$ are disjoint.

$$\therefore \quad P(A) = P(A\cap B) + P(A\cap\bar{B})$$

Hence $\quad P(A\cap\bar{B}) = P(A) - P(A\cap B)$



Theorem If $B\subset A$, then

(i) $\quad P(A\cap\bar{B}) = P(A) - P(B)$

(ii) $\quad P(B) \leq P(A)$

Proof (i) Since $B\subset A$, B and $A\cap\bar{B}$ are mutually exclusive events such that

$$A = B\cup(A\cap\bar{B})$$

$$\therefore \quad P(A) = P(B) + P(A\cap\bar{B})$$

$$\Rightarrow \quad P(A\cap\bar{B}) = P(A) - P(B)$$

(ii) Since $P(A\cap\bar{B}) \geq 0$, we have

$$P(A) - P(B) \geq 0$$

$$\Rightarrow \quad P(A) \geq P(B)$$

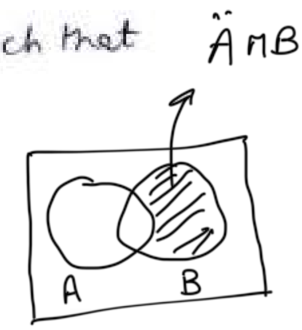
Hence $\quad P(B) \leq P(A)$.

Addition theorem of Probability

Theorem If A and B are any two events, then

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

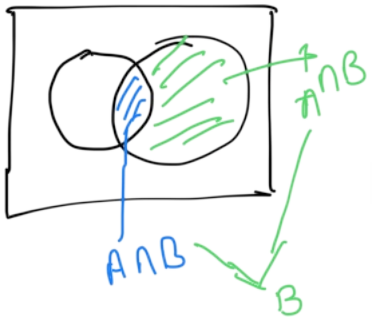
Proof. The events A and $\bar{A}\cap B$ are disjoint such that



$$A\cup B = A\cup(\bar{A}\cap B)$$

$$\therefore \quad P(A\cup B) = P(A) + P(\bar{A}\cap B) \qquad (1)$$

Also $B = (A \cap B) \cup (\bar{A} \cap B)$



where $A \cap B$ and $\bar{A} \cap B$ are disjoint events.

$$\therefore \; P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow \quad P(\bar{A} \cap B) = P(B) - P(A \cap B) \qquad (2)$$

From (1) and (2), we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Cor. For any three events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Proof.
$$\begin{aligned} P(A \cup B \cup C) &= P[(A \cup B) \cup C] \\ &= P(A \cup B) + P(C) - P[(A \cup B) \cap C] \\ &= P(A) + P(B) - P(A \cap B) + P(C) \\ &\qquad - P[(A \cap C) \cup (B \cap C)] \\ &= P(A) + P(B) - P(A \cap B) + P(C) \\ &\qquad - P(A \cap C) - P(B \cap C) + P[(A \cap C) \cap (B \cap C)] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) \\ &\qquad - P(C \cap A) + P(A \cap B \cap C). \end{aligned}$$

Example If $P(\bar{A}) = a$, $P(\bar{B}) = b$, then prove that

$$P(A \cap B) \geq 1 - a - b$$

Soln We know $P(A \cup B) \leq 1$

$$\Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$$

$$\Rightarrow 1 - a + 1 - b - P(A \cap B) \leq 1$$

$$\Rightarrow 1 - a - b \leq P(A \cap B)$$

Hence $P(A \cap B) \geq 1 - a - b$

Boole's Inequality

For any n events $A_1, A_2, \ldots, A_n$; we have

(a) $$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1) \qquad (1)$$

(b) $$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \qquad (2)$$

Proof. (a) We know

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Since $P(A_1 \cup A_2) \leq 1$, it follows that

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$\Rightarrow \quad P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 \qquad (3)$$

which shows that (1) is true for $n = 2$.

Let us suppose that (1) is true for $n = r$ so that

$$P\left(\bigcap_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - (r-1)$$

Then
$$\begin{aligned} P\left(\bigcap_{i=1}^{r+1} A_i\right) &= P\left[\left(\bigcap_{i=1}^{r} A_i\right) \cap A_{r+1}\right] \\ &\geq P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1 \\ &\geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1 \\ &= \sum_{i=1}^{r+1} P(A_i) - r \end{aligned}$$

So (1) is true for $n = r+1$. Hence by principle of mathematical induction, (1) is true for all naturals n.

(b) Applying the inequality (1) to the events $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n$;

we get

$$P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n) \geq P(\overline{A}_1) + P(\overline{A}_2) + \ldots + P(\overline{A}_n) - (n-1)$$

$$= [1-P(A_1)] + [1-P(A_2)] + \ldots + [1-P(A_n)] - (n-1)$$

$$= 1 - P(A_1) - P(A_2) - \ldots - P(A_n)$$

This implies

$$P(A_1) + P(A_2) + \ldots + P(A_n) \geq 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n)$$

$$= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n})$$

$$= P(A_1 \cup A_2 \cup \ldots \cup A_n)$$

Hence

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n)$$

<u>Theorem</u> For n events $A_1, A_2, \ldots, A_n$;

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

<u>Proof.</u> We shall prove the theorem by the principle of mathematical induction.

Clearly result is true for $n = 2$

We know that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3)$$
$$- P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

$$\geq P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3)$$

$$[\because P(A_1 \cap A_2 \cap A_3) \geq 0]$$

which can be written as

$$P\left(\bigcup_{i=1}^{3} A_i\right) \geq \sum_{i=1}^{3} P(A_i) - \sum_{1 \leq i < j \leq 3} P(A_i \cap A_j)$$

Thus the result is true for $n = 3$.

Let us suppose that the result is true for $n = r$ so that

$$P\left(\bigcup_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - \sum_{1 \leq i < j \leq r} P(A_i \cap A_j)$$

Now

$$\begin{aligned}
P\left(\bigcup_{i=1}^{r+1} A_i\right) &= P\left[\left(\bigcup_{i=1}^{r} A_i\right) \cup A_{r+1}\right] \\
&= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right] \\
&= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \\
&\geq \sum_{i=1}^{r} P(A_i) - \sum_{1 \leq i < j \leq r} P(A_i \cap A_j) + P(A_{r+1}) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \\
&= \sum_{i=1}^{r+1} P(A_i) - \sum_{1 \leq i < j \leq r} P(A_i \cap A_j) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \qquad (1)
\end{aligned}$$

Using Boole's inequality,

$$P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \leq \sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

So (1) implies

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum_{1 \leq i < j \leq r} P(A_i \cap A_j) - \sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

i.e. $$P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum_{1 \leq i < j \leq r+1} P(A_i \cap A_j)$$

which shows that the result is true for $n = r+1$. Hence by the principle of mathematical induction, result is true for all positive integers n.

Example For the events $A_1, A_2, \ldots, A_n$; assuming

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i),$$

prove that

(i) $$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$$

(ii) $$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Soln Given

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Using this inequality for the events $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$; we get

$$P\left(\bigcup_{i=1}^{n} \bar{A}_i\right) \leq \sum_{i=1}^{n} P(\bar{A}_i)$$

$$\Rightarrow \quad P\left(\overline{\bigcap_{i=1}^{n} A_i}\right) \leq \sum_{i=1}^{n} P(\bar{A}_i)$$

$$\Rightarrow \quad 1 - P\left(\bigcap_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(\bar{A}_i)$$

$$\Rightarrow \quad P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$$

which proves (i)

Also
$$\begin{aligned} P\left(\bigcap_{i=1}^{n} A_i\right) &\geq 1 - \sum_{i=1}^{n} P(\bar{A}_i) \\ &= 1 - \sum_{i=1}^{n} [1 - P(A_i)] \\ &= 1 - n + \sum_{i=1}^{n} P(A_i) \\ &= \sum_{i=1}^{n} P(A_i) - (n-1) \end{aligned}$$

which proves (ii).

Example A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :

$$P(B) = \frac{3}{2} P(A) \quad \text{and} \quad P(C) = \frac{1}{2} P(B)$$

Soln Let $P(A) = p$. Then $P(B) = \frac{3}{2} p$ and $P(C) = \frac{3}{4} p$

Since A, B, C are mutually exclusive and exhaustive events, we have $A \cup B \cup C = S$

$$\therefore \quad P(A) + P(B) + P(C) = 1$$

$$\Rightarrow \quad p + \frac{3}{2} p + \frac{3}{4} p = 1$$

$$\Rightarrow \quad \frac{13}{4} p = 1$$

$$\Rightarrow \quad p = \frac{4}{13}$$

Example Let A and B be two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$. Show that

(a) $P(A \cup B) \geq \frac{3}{4}$ (b) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$

Soln (a) We have $A \subseteq A \cup B$

$$\Rightarrow \quad P(A) \leq P(A \cup B)$$

$$\Rightarrow \quad \frac{3}{4} \leq P(A \cup B)$$

So $P(A \cup B) \geq \frac{3}{4}$

(b) Since $A \cap B \subseteq B$, we have $P(A \cap B) \leq P(B)$

$$\therefore \quad P(A \cap B) \leq \frac{5}{8}$$

Also $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$

So $\frac{3}{4} + \frac{5}{8} - P(A \cap B) \leq 1$.

$$\Rightarrow \quad \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$$

$$\Rightarrow \quad \frac{3}{8} \leq P(A \cap B)$$

Hence $\quad \frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$

Example For any two events A and B,

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Proof. We have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

where $A \cap \bar{B}$ and $A \cap B$ are disjoint. So

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

$$\Rightarrow P(A) \geq P(A \cap B) \quad \text{since } P(A \cap \bar{B}) \geq 0.$$

Similarly $\quad P(B) \geq P(A \cap B)$

$$\Rightarrow P(B) - P(A \cap B) \geq 0$$

$$\Rightarrow P(A) + P(B) - P(A \cap B) \geq P(A)$$

$$\Rightarrow P(A \cup B) \geq P(A)$$

Also $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Since $P(A \cap B) \geq 0$, we have

$$P(A) + P(B) - P(A \cup B) \geq 0$$

$$\Rightarrow P(A) + P(B) \geq P(A \cup B)$$

$$\Rightarrow P(A \cup B) \leq P(A) + P(B).$$

Hence $\quad P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

Conditional Probability

Let A and B be two events. Then the probability of occurence of the event A given that B has occured, is called the conditional probability of A given that B has occured and is denoted by $P(A|B)$.

Consider a random experiment of tossing three fair coins. Then the sample space is

$$S = \{HHH, HHT, HTH, THH, THT, HTT, TTH, TTT\}$$

Let A denote the event of getting three heads and B denote the event of getting at least two heads.

Then $\quad P(A|B) = \frac{1}{4}$

Multiplication theorem of Probability

For any two events A and B,

$$P(A \cap B) = P(A) \cdot P(B/A) \quad , \quad P(A) > 0$$

$$= P(B) \cdot P(A/B) \quad , \quad P(B) > 0.$$

Proof. We have

$$P(A) = \frac{n(A)}{n(S)} \; , \quad P(B) = \frac{n(B)}{n(S)} \quad \text{and} \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

For the event $A|B$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurence of the event A. Hence

$$P(A/B) = \frac{n(A \cap B)}{n(B)}$$

Now $\quad P(A \cap B) = \frac{n(A \cap B)}{n(B)} \cdot \frac{n(B)}{n(S)} = P(A|B)\, P(B)$

Similarly, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)}$

$$= \frac{n(A)}{n(S)} \cdot \frac{n(A \cap B)}{n(A)}$$

$$= P(A)\, P(B/A)$$

Hence the probability of simultaneous occurence of two events is equal to the product of the probability of one of these events and the conditional probability of the other, given that the first one has occured.

Remarks

1. We have $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

and $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

Thus the conditional probabilities $P(B/A)$ and $P(A/B)$ are defined if and only if $P(A) \neq 0$, $P(B) \neq 0$, respectively.

2. $P(B/B) = 1$.

Independent Events

Two or more events are said to be independent if the occurence of one of them does not affect the occurence of any other.

Defn An event A is said to be independent or statistically independent of another event B if $P(A/B) = P(A)$.

Similarly, an event B is said to be independent of event A if $P(B/A) = P(B)$.

Theorem If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.

Proof. Since the event A is independent of B, we have

$$P(A|B) = P(A)$$

$$\therefore \quad \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} = P(B)$$

$$\Rightarrow \quad P(B/A) = P(B).$$

Hence B is independent of A.

Remark. We have seen that if A is independent of B, then B is independent of A. Hence, instead of saying that 'A is independent of B' or 'B is independent of A', we may say that A and B are independent events.

Multiplication theorem of Probability for independent events.

Theorem Let A and B be two events with $P(A) \neq 0$, $P(B) \neq 0$.

Then A and B are independent if and only if

$$P(A \cap B) = P(A)\, P(B)$$

Proof. Suppose A and B are independent.

Then A is independent of B

So $P(A/B) = P(A)$.

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

Hence $P(A \cap B) = P(A)\, P(B)$.

Conversely suppose that $P(A \cap B) = P(A)\, P(B)$.

Then $\dfrac{P(A \cap B)}{P(B)} = P(A)$ and $\dfrac{P(A \cap B)}{P(A)} = P(B)$

$\Rightarrow P(A/B) = P(A)$ and $P(B/A) = P(B)$

Hence A and B are independent events.

Remark. For any event A in S

(a) A and ϕ are independent

(b) A and S are independent.

Proof (a) $P(A \cap \phi) = P(\phi) = 0$

$P(A) \cdot P(\phi) = P(A) \cdot 0 = 0$

$\therefore P(A \cap \phi) = P(A)\, P(\phi)$

So A and ϕ are independent.

(b) $P(A \cap S) = P(A) = P(A)\, P(S)$

So A and S are independent.

Remark

1. Two mutually disjoint events with positive probabilities cannot be independent.

Let A and B be two mutually disjoint events s.t. $P(A) > 0$ and $P(B) > 0$. Then $A \cap B = \phi$. So $P(A \cap B) = 0$. But $P(A)P(B) > 0$. Therefore $P(A \cap B) \neq P(A)P(B)$. Hence A and B are not independent.

2. Two independent events with positive probabilities cannot be disjoint.

Let A and B be two independent events s.t. $P(A) > 0$ and $P(B) > 0$. Then $P(A \cap B) = P(A)P(B) > 0$. Hence A and B are not disjoint.

3. $P(\phi | B) = 0$

4. $P(\overline{A} | B) = 1 - P(A/B)$.

Proof. $$P(\overline{A} | B) = \frac{P(\overline{A} \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= 1 - \frac{P(A \cap B)}{P(B)}$$

$$= 1 - P(A|B)$$

5. If A and B are two independent and mutually exclusive events then at least one of them must have zero probability.

Proof Since A and B are independent, $P(A \cap B) = P(A)P(B)$.

Further A and B are mutually exclusive

So $\quad P(A\cap B) = P(\phi) = 0$

$\therefore P(A)P(B) = 0$

Hence $\quad P(A) = 0$ or $P(B) = 0$

Example If $P(A) = p_1$, $P(B) = p_2$, $P(A\cap B) = p_3$, express the following in terms of p_1, p_2, p_3.

(a) $P(\overline{A\cup B})$

(b) $P(\bar{A}\cup\bar{B})$

(c) $P(\bar{A}\cap B)$

(d) $P(\bar{A}\cup B)$

(e) $P(\bar{A}\cap\bar{B})$

(f) $P(A\cap\bar{B})$

(g) $P(A|B)$

(h) $P(B|\bar{A})$

(i) $P[\bar{A}\cap(A\cup B)]$

Soln (a)

$$P(\overline{A\cup B}) = 1 - P(A\cup B)$$

$$= 1 - [P(A) + P(B) - P(A\cap B)]$$

$$= 1 - p_1 - p_2 + p_3.$$

(b) $P(\bar{A}\cup\bar{B}) = P(\overline{A\cap B}) = 1 - P(A\cap B) = 1 - p_3$.

(c) $P(\bar{A}\cap B) = P(B) - P(A\cap B) = p_2 - p_3$

(d) $P(\bar{A}\cup B) = P(\bar{A}) + P(B) - P(\bar{A}\cap B)$

$$= 1 - p_1 + p_2 - (p_2 - p_3)$$

$$= 1 - p_1 + p_3.$$

(e) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3$

(f) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = p_1 - p_3$

(g) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{p_3}{p_2}$

(h) $P(B/\overline{A}) = \dfrac{P(B \cap \overline{A})}{P(\overline{A})} = \dfrac{p_2 - p_3}{1 - p_1}$

(i) $P[\overline{A} \cap (A \cup B)] = P[(\overline{A} \cap A) \cup (\overline{A} \cap B)]$

$= P(\phi \cup (\overline{A} \cap B))$

$= P(\overline{A} \cap B) = p_2 - p_3.$

<u>Example</u> Let $P(A) = p$, $P(A|B) = q$, $P(B/A) = r$.

Find relations between p, q, r for the following cases:

(a) A and B are mutually exclusive

(b) A and B are mutually exclusive and collectively exhaustive

(c) A is subevent of B

(d) B is subevent of A

(e) $\overline{A}$ and $\overline{B}$ are mutually exclusive

<u>Soln</u> Given $P(A) = p$, $P(B/A) = r$, $P(A|B) = q$.

So $P(A \cap B) = P(A)\, P(B/A) = pr$.

Also $P(B) = \dfrac{P(A \cap B)}{P(A/B)} = \dfrac{pr}{q}$

(a) Since A and B are mutually exclusive, $A \cap B = \phi$

$\therefore P(A \cap B) = 0 \Rightarrow pr = 0$.

(b) Since A and B are mutually exclusive and collectively exhaustive, we have

$$A \cap B = \phi \quad \text{and} \quad A \cup B = S$$

$$\therefore \quad P(A \cap B) = 0 \quad \text{and} \quad P(A) + P(B) = 1$$

$$\Rightarrow \quad pr = 0 \quad \text{and} \quad p + \frac{pr}{q} = 1$$

$$\Rightarrow \quad p = 1$$

(c) Since A is a subevent of B, $A \subseteq B$

$$\therefore \quad A \cap B = A$$

$$\Rightarrow P(A \cap B) = P(A)$$

$$\Rightarrow \quad pr = p$$

$$\Rightarrow \quad p(r-1) = 0$$

$$\Rightarrow \quad p = 0 \quad \text{or} \quad r = 1$$

(d) Since B is a subevent of A, $B \subseteq A$.

$$\therefore \quad B \cap A = B$$

$$\Rightarrow P(B \cap A) = P(B)$$

$$\Rightarrow \quad rp = \frac{rp}{q}$$

$$\Rightarrow \quad rp(q-1) = 0$$

$$\Rightarrow \quad r = 0 \text{ or } p = 0 \text{ or } q = 1$$

(e) Since $\bar{A}$ and $\bar{B}$ are mutually exclusive, $\bar{A} \cap \bar{B} = \phi$

$$\therefore \quad P(\bar{A} \cap \bar{B}) = 0$$

$$\Rightarrow P(\overline{A \cup B}) = 0$$

$$\Rightarrow 1 - P(A \cup B) = 0$$

$$\Rightarrow \quad P(A \cup B) = 1$$

$$\Rightarrow \quad P(A) + P(B) - P(A \cap B) = 1.$$

$$\Rightarrow \quad p + \frac{rp}{q} - rp = 1$$

$$\Rightarrow \quad p(q+r) = q(1+pr)$$

Example Let A and B be the two possible outcomes of an experiment and suppose

$$P(A) = 0.4, \quad P(A\cup B) = 0.7 \text{ and } P(B) = p$$

(i) For what choice of p are A and B mutually exclusive?
(ii) For what choice of p are A and B independent?

Soln (i) We have

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\therefore \quad 0.7 = 0.4 + p - P(A\cap B)$$

$$\Rightarrow \quad P(A\cap B) = p - 0.3$$

If A and B are mutually exclusive, then $P(A\cap B) = 0$

$$\therefore \quad p = 0.3$$

(ii) When A and B are independent, $P(A\cap B) = P(A)\,P(B)$

$$\therefore \quad p - 0.3 = 0.4p$$

$$\Rightarrow \quad 0.6p = 0.3$$

$$\Rightarrow \quad p = \frac{1}{2}$$

Example The odds that a book on Statistics will be favourably reviewed by 3 independent critics are 3 to 2, 4 to 3 and 2 to 3 respectively. What is the probability that of the three reviews:

(i) All will be favourable
(ii) Majority of the reviews will be favourable.
(iii) Exactly one review will be favourable.
(IV) At least one of the reviews will be favourable.

Soln. Let A, B and C denote respectively the events that the book is favourably reviewed by first, second and third critic respectively. Then

$$P(A) = \frac{3}{5}, \quad P(B) = \frac{4}{7} \text{ and } P(C) = \frac{2}{5}$$

$$\therefore P(\bar{A}) = \frac{2}{5}, \quad P(\bar{B}) = \frac{3}{7} \text{ and } P(\bar{C}) = \frac{3}{5}$$

(i) Probability that all the three reviews will be favourable is $P(A \cap B \cap C) = P(A)\,P(B)\,P(C) = \frac{3}{5} \cdot \frac{4}{7} \cdot \frac{2}{5} = \frac{24}{175}$

(ii) Probability that majority of the reviews will be favourable is $P(A \cap B \cap C) + P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$

$$= P(A)\,P(B)\,P(C) + P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(\bar{B})\,P(C) + P(\bar{A})\,P(B)\,P(C)$$

$$= \frac{3}{5} \cdot \frac{4}{7} \cdot \frac{2}{5} + \frac{3}{5} \cdot \frac{4}{7} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{3}{7} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{4}{7} \cdot \frac{2}{5}$$

$$= \frac{94}{175}$$

(iii) Probability that exactly one review will be favourable

$$= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$$

$$= P(A)\,P(\bar{B})\,P(\bar{C}) + P(\bar{A})\,P(B)\,P(\bar{C}) + P(\bar{A})\,P(\bar{B})\,P(C)$$

$$= \frac{3}{5} \cdot \frac{3}{7} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{4}{7} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{3}{7} \cdot \frac{2}{5} = \frac{63}{175}$$

(IV) Probability that at least one of the reviews will be favourable $= P(A \cup B \cup C)$

$$= 1 - P(\overline{A \cup B \cup C})$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = 1 - \frac{2}{5} \cdot \frac{3}{7} \cdot \frac{3}{5} = \frac{157}{175}$$

Example Three groups of children contains respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Find the chance that the three selected consist of 2 girls and 1 boy.

Soln let E_1, E_2, E_3 denote the events of selecting a boy from group I, group II and group III respectively.

Then reqd probability $= P(E_1 \bar{E_2} \bar{E_3}) + P(\bar{E_1} E_2 \bar{E_3}) + P(\bar{E_1} \bar{E_2} E_3)$

$$= P(E_1) P(\bar{E_2}) P(\bar{E_3}) + P(\bar{E_1}) P(E_2) P(\bar{E_3}) + P(\bar{E_1}) P(\bar{E_2}) P(E_3)$$

$$= \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4}$$

$$= \frac{2}{64} + \frac{6}{64} + \frac{18}{64} = \frac{26}{64} = \frac{13}{32}$$

Example An urn contains 10 white and 3 black balls while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball?

Soln. Two balls drawn from the first urn may be either

(i) both white

(ii) one white and one black

or (iii) both black.

Let the happening of these events be denoted by A_1, A_2, A_3 respectively. Then

$$P(A_1) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{10 \times 9}{2} \times \frac{2}{13 \times 12} = \frac{15}{26}$$

$$P(A_2) = \frac{10 \times 3}{{}^{13}C_2} = \frac{30}{78} = \frac{5}{13}$$

$$P(A_3) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{3.2}{13.12} = \frac{1}{26}$$

Let W denote the event of drawing a white ball from second urn. Then

$$P(W/A_1) = \frac{5}{10} = \frac{1}{2}$$

$$P(W/A_2) = \frac{4}{10} = \frac{2}{5}$$

$$P(W/A_3) = \frac{3}{10}$$

$$\therefore \quad P(W) = P(A_1)P(W/A_1) + P(A_2)P(W/A_2) + P(A_3)P(W/A_3)$$

$$= \frac{15}{26} \times \frac{1}{2} + \frac{5}{13} \times \frac{2}{5} + \frac{1}{26} \times \frac{3}{10}$$

$$= \frac{15}{52} + \frac{2}{13} + \frac{3}{260}$$

$$= \frac{75 + 40 + 3}{260}$$

$$= \frac{118}{260}$$

$$= \frac{59}{130}$$

Theorem For a fixed B with $P(B) > 0$, $P(A/B)$ is probability function.

Proof.

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

(ii) $$P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(iii) If $\{A_n\}$ is any finite or infinite sequence of disjoint events, then

$$P\left(\bigcup_n A_n / B\right) = \frac{P[(\bigcup_n A_n) \cap B]}{P(B)}$$

$$= \frac{P[\bigcup_n (A_n \cap B)]}{P(B)}$$

$$= \frac{\sum_n P(A_n \cap B)}{P(B)}$$

$$= \sum_n \frac{P(A_n \cap B)}{P(B)}$$

$$= \sum_n P(A_n / B)$$

Theorem For any three events A, B and C,

$$P(A \cup B / C) = P(A/C) + P(B/C) - P(A \cap B / C)$$

Proof. We have

$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C).$$

Dividing bothsides by $P(C)$, we get

$$\frac{P[(A\cup B)\cap C]}{P(C)} = \frac{P(A\cap C)}{P(C)} + \frac{P(B\cap C)}{P(C)} - \frac{P(A\cap B\cap C)}{P(C)}$$

Hence

$$P(A\cup B|C) = P(A|C) + P(B|C) - P(A\cap B|C)$$

<u>Theorem</u> For any three events A, B and C,

$$P(A\cap \overline{B}|C) + P(A\cap B|C) = P(A|C)$$

<u>Proof.</u>

The events $A\cap\overline{B}\cap C$ and $A\cap B\cap C$ are disjoint such that their union is $A\cap C$.

So $\quad P(A\cap\overline{B}\cap C) + P(A\cap B\cap C) = P(A\cap C)$

Dividing by $P(C)$, we get

$$\frac{P(A\cap\overline{B}\cap C)}{P(C)} + \frac{P(A\cap B\cap C)}{P(C)} = \frac{P(A\cap C)}{P(C)}$$

Hence $\quad P(A\cap\overline{B}|C) + P(A\cap B|C) = P(A|C)$.

<u>Theorem</u> If A and B are independent events, then

(i) A and $\overline{B}$

(ii) $\overline{A}$ and B

(iii) $\overline{A}$ and $\overline{B}$, are also independent.

Proof. Since A and B are independent, we have

$$P(A\cap B) = P(A)\,P(B)$$

(i)

$$\begin{aligned} P(A\cap\overline{B}) &= P(A) - P(A\cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1-P(B)) \\ &= P(A)\,P(\overline{B}) \end{aligned}$$

Hence A and $\overline{B}$ are independent events.

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$$= P(B) - P(A)\,P(B)$$

$$= (1 - P(A))\,P(B)$$

$$= P(\bar{A})\,P(B)$$

$\therefore$ $\bar{A}$ and B are independent events

(iii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)\,P(B)$$

$$= [1 - P(A)]\,[1 - P(B)]$$

$$= P(\bar{A})\,P(\bar{B})$$

Hence $\bar{A}$ and $\bar{B}$ are independent events.

<u>Example</u> It is given that $P(A_1 \cup A_2) = \frac{5}{6}$, $P(A_1 \cap A_2) = \frac{1}{3}$ and $P(\bar{A}_2) = \frac{1}{2}$. Determine $P(A_1)$ and $P(A_2)$. Hence show that A_1 and A_2 are independent events.

<u>Soln</u> Given $P(\bar{A}_2) = \frac{1}{2}$

$\therefore$ $P(A_2) = 1 - P(\bar{A}_2) = 1 - \frac{1}{2} = \frac{1}{2}$

Also $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$\therefore$ $\frac{5}{6} = P(A_1) + \frac{1}{2} - \frac{1}{3}$

$\Rightarrow$ $P(A_1) = \frac{5}{6} - \frac{1}{2} + \frac{1}{3} = \frac{5 - 3 + 2}{6} = \frac{2}{3}$

Since $P(A_1)\,P(A_2) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} = P(A_1 \cap A_2)$,

it follows that A_1 and A_2 are independent.

Example If $P(A) > 0$, $P(B) > 0$ and $P(A|B) = P(B|A)$

then $P(A) = P(B)$

Soln Given $P(A|B) = P(B/A)$

$$\therefore \quad \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(A)}$$

Since $A \cap B = B \cap A$, we have $P(A) = P(B)$

Example If A and B are two events and $P(B) \neq 1$,

prove that

$$P(A|\overline{B}) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

where $\overline{B}$ denotes the event complementary to B and

hence deduce that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Also show that

$P(A) >$ or $< P(A|B)$ according as $P(A|\overline{B}) >$ or $< P(A)$.

Soln

(i) $$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

(ii) Since $P(A|\overline{B}) \leq 1$, we have

$$\frac{P(A) - P(A \cap B)}{1 - P(B)} \leq 1$$

$$\Rightarrow \quad P(A) - P(A \cap B) \leq 1 - P(B)$$

$$\Rightarrow \quad P(A) + P(B) - 1 \leq P(A \cap B)$$

Hence $P(A \cap B) \geq P(A) + P(B) - 1$

(iii)
$$\frac{P(A/\bar{B})}{P(A)} = \frac{P(A\cap\bar{B})}{P(\bar{B})\,P(A)}$$
$$= \frac{P(\bar{B}\cap A)}{P(A)\,P(\bar{B})}$$
$$= \frac{P(\bar{B}/A)}{P(\bar{B})}$$
$$= \frac{1-P(B/A)}{1-P(B)}$$

Suppose $P(A/\bar{B}) > P(A)$.

Then $\dfrac{P(A/\bar{B})}{P(A)} > 1$

$$\Rightarrow \frac{1-P(B/A)}{1-P(B)} > 1$$
$$\Rightarrow 1-P(B/A) > 1-P(B)$$
$$\Rightarrow P(B) > P(B/A)$$
$$\Rightarrow P(B) > \frac{P(B\cap A)}{P(A)}$$
$$\Rightarrow P(A) > \frac{P(A\cap B)}{P(B)}$$
$$\Rightarrow P(A) > P(A/B)$$

Similarly, we can show that

$P(A/\bar{B}) < P(A)$ implies $P(A) < P(A/B)$.

Example If A and B are two mutually exclusive events, show that

$$P(A/\bar{B}) = \frac{P(A)}{1-P(B)}$$

Soln Since A and B are mutually exclusive,

$$A \cap B = \phi .$$

Now $$P(A/\bar{B}) = \frac{P(A\cap\bar{B})}{P(\bar{B})}$$

$$= \frac{P(A)-P(A\cap B)}{1-P(B)}$$

$$= \frac{P(A)}{1-P(B)}$$

Example If A and B are two mutually exclusive events and $P(A\cup B) \neq 0$, then

$$P(A/A\cup B) = \frac{P(A)}{P(A)+P(B)}$$

Soln Since A and B are mutually exclusive,

$$P(A\cap B) = P(\phi) = 0. \qquad (1)$$

Now $$P(A/A\cup B) = \frac{P[A\cap(A\cup B)]}{P(A\cup B)}$$

$$= \frac{P(A)}{P(A)+P(B)-P(A\cap B)}$$

$$= \frac{P(A)}{P(A)+P(B)} \qquad [\text{using } (1)]$$

Example If A, B and C are three events such that

$$P(A|C) \geq P(B|C) \text{ and } P(A|\bar{C}) \geq P(B|\bar{C}),$$

then show that $P(A) \geq P(B)$

Soln Since $P(A|C) \geq P(B|C)$, we have

$$\frac{P(A\cap C)}{P(C)} \geq \frac{P(B\cap C)}{P(C)}$$

$$\Rightarrow \quad P(A\cap C) \geq P(B\cap C) \qquad (1)$$

Also $P(A|\bar{C}) \geq P(B|\bar{C})$

So $$\frac{P(A\cap\bar{C})}{P(\bar{C})} \geq \frac{P(B\cap\bar{C})}{P(\bar{C})}$$

$$\Rightarrow \quad P(A\cap\bar{C}) \geq P(B\cap\bar{C}) \qquad (2)$$

Adding (1) and (2),

$$P(A\cap C) + P(A\cap\bar{C}) \geq P(B\cap C) + P(B\cap\bar{C})$$

i.e. $P(A) \geq P(B)$ Proved.

Example An event A is known to be independent of the events B, $B\cup C$ and $B\cap C$. Show that it is also independent of C.

Soln Given A is independent of B, $B\cup C$ and $B\cap C$.

So $$P(A\cap B) = P(A)\,P(B). \qquad (1)$$

$$P[A\cap(B\cup C)] = P(A)\,P(B\cup C) \qquad (2)$$

$$P[A\cap(B\cap C)] = P(A)\,P(B\cap C) \qquad (3)$$

From (2), $P[(A\cap B)\cup(A\cap C)] = P(A)\,[P(B) + P(C) - P(B\cap C)]$

i.e. $P(A\cap B)+P(A\cap C)-P(A\cap B\cap C) = P(A)P(B)+P(A)P(C)-P(A)P(B\cap C)$

Using (1) and (3), this gives

$$P(A\cap C) = P(A)P(C)$$

Hence A is independent of C.

Example Show that if an event C is independent of two mutually exclusive events A and B, then C is also independent of $A\cup B$.

Soln. Since A and B are mutually exclusive events,

$$P(A\cap B) = P(\phi) = 0.$$

Also C is independent of A and B.

So $$P(C\cap A) = P(C)P(A)$$
$$P(C\cap B) = P(C)P(B).$$

Now

$$P[(A\cup B)\cap C] = P[(A\cap C)\cup(B\cap C)]$$
$$= P(A\cap C)+P(B\cap C)-P(A\cap B\cap C)$$
$$= P(A)P(C)+P(B)P(C)$$
$$[\because P(A\cap B\cap C)=0]$$
$$= [P(A)+P(B)]P(C)$$
$$= P(A\cup B)P(C).$$

Hence C is independent of $A\cup B$.

Example If $A\cap B=\phi$ then show that $P(A)\leq P(\overline{B})$.

Soln Since $A\cap B=\phi$, we have $A\subseteq \overline{B}$

$$\therefore P(A)\leq P(\overline{B}).$$

Pairwise Independent Events

Let $A_1, A_2, \ldots, A_n$ be n events associated with sample space S. Then $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if and only if

$$P(A_i \cap A_j) = P(A_i)\,P(A_j), \quad i \neq j = 1,2,3,\ldots,n.$$

Mutually Independent Events

Let S denote the sample space for a number of events. The events in S are said to be mutually independent if the probability of simultaneous occurence of any finite number of them is equal to the product of their probabilities.

Conditions for mutual independence of n events.

The events $A_1, A_2, \ldots, A_n$ are mutually independent if and only if the following conditions hold.

(i) $$P(A_i \cap A_j) = P(A_i)P(A_j), \quad (i \neq j;\ i,j = 1,2,3,\ldots,n)$$

(ii) $$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$$
$$(i \neq j \neq k;\ i,j,k = 1,2,3,\ldots,n)$$

$\ldots\ldots\ldots\ldots$

(n-1) $$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

Thus, for mutual independence of $A_1, A_2, \ldots, A_n$, we get ${}^nC_2$ conditions from (i), ${}^nC_3$ conditions from (ii), $\ldots\ldots$, ${}^nC_n$ conditions from last equation. Hence total number of conditions for mutual independence of $A_1, A_2, \ldots, A_n$ is

$${}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n - {}^nC_0 - {}^nC_1 = 2^n - 1 - n$$

In particular, for three events A_1, A_2 and A_3, we have the following $2^3-1-3 = 4$ conditions for their mutual independence

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$
$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$

Remark It is obvious that mutual independence of events implies pairwise independence. However, the converse is not true i.e. the events may be pairwise independent but not mutually independent.

Theorem If A, B, C are mutually independent events then $A \cup B$ and C are also independent.

Proof. Since A, B, C are mutually independent events, we have

$$P(A \cap B) = P(A) P(B)$$
$$P(B \cap C) = P(B) P(C)$$
$$P(C \cap A) = P(C) P(A)$$
$$P(A \cap B \cap C) = P(A) P(B) P(C)$$

Now
$$\begin{aligned} P[(A \cup B) \cap C] &= P[(A \cap C) \cup (B \cap C)] \\ &= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A) P(C) + P(B) P(C) - P(A) P(B) P(C) \\ &= [P(A) + P(B) - P(A) P(B)] P(C) \\ &= P(A \cup B) P(C) \end{aligned}$$

Hence $A \cup B$ and C are independent.

Theorem If A, B and C are random events in a sample space such that A, B, C are pairwise independent and A is independent of $B \cup C$, then A, B and C are mutually independent.

Proof. Since A, B, C are pairwise independent, we have

$$P(A \cap B) = P(A)\,P(B) \qquad (1)$$
$$P(B \cap C) = P(B)\,P(C) \qquad (2)$$
$$P(C \cap A) = P(C)\,P(A) \qquad (3)$$

Also A is independent of $B \cup C$.

So $\quad P[A \cap (B \cup C)] = P(A)\,P(B \cup C) \qquad (4)$

Now $\quad P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \qquad (5)$$

and $\quad P(A)\,P(B \cup C) = P(A)\,[P(B) + P(C) - P(B \cap C)]$

$$= P(A)P(B) + P(A)\,P(C) - P(A)\,P(B \cap C) \qquad (6)$$

From (5) & (6), using (1), (3), (4); we get

$$P(A \cap B \cap C) = P(A)\,P(B \cap C)$$

$$\Rightarrow \quad P(A \cap B \cap C) = P(A)\,P(B)\,P(C) \qquad [\text{using } (2)]$$

Hence A, B, C are mutually independent.

Example An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i $(i = 1, 2, 3)$ be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.

Soln No. of exhaustive cases $= 4$

A_1 is the event that the first digit of the number of the ticket drawn is 1 and the cases favourable to A_1 are 112, 121

$$\therefore P(A_1) = \frac{2}{4} = \frac{1}{2}$$

Cases favourable to event A_2 are 112 and 211.

$$\therefore P(A_2) = \frac{2}{4} = \frac{1}{2}$$

Cases favourable to event A_3 are 121 and 211.

$$\therefore P(A_3) = \frac{2}{4} = \frac{1}{2}$$

No. of cases favourable to $A_1 \cap A_2 = 1$

No. of cases favourable to $A_2 \cap A_3 = 1$

No. of cases favourable to $A_1 \cap A_3 = 1$

$$\therefore P(A_1 \cap A_2) = \frac{1}{4}$$

$$P(A_2 \cap A_3) = \frac{1}{4}$$

$$P(A_1 \cap A_3) = \frac{1}{4}$$

Since $P(A_1 \cap A_2) = P(A_1) P(A_2)$, $P(A_2 \cap A_3) = P(A_2) P(A_3)$

and $P(A_1 \cap A_3) = P(A_1) P(A_3)$, it follows that

A_1, A_2, A_3 are pairwise independent.

Also there is no case favourable to the event $A_1 \cap A_2 \cap A_3$.

So $P(A_1 \cap A_2 \cap A_3) = P(\phi) = 0 \neq P(A_1) P(A_2) P(A_3)$

Hence A_1, A_2 and A_3, though pairwise independent, are not mutually independent.

Example: Two fair dice are thrown independently. Three events A, B, C are defined as follows:

A: Odd face with first die

B: Odd face with second die

C: Sum of points on two dice is odd.

Are the events A, B and C,

(i) pairwise independent

(ii) mutually independent.

Soln. In a random toss of two dice, sample space is

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

Cases favourable to A are

(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)

Cases favourable to B are

(1,1), (1,3), (1,5), (2,1), (2,3), (2,5)

(3,1), (3,3), (3,5), (4,1), (4,3), (4,5)

(5,1), (5,3), (5,5), (6,1), (6,3), (6,5)

Cases favourable to C are

(1,2), (2,1), (3,2), (2,3), (5,2), (2,5), (1,4), (4,1), (3,4), (4,3), (5,4), (4,5)

(1,6), (6,1), (3,6), (6,3), (5,6), (6,5)

Cases favourable to $A \cap B$ are

$(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)$

Cases favourable to $B \cap C$ are

$(2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (6,1), (6,3), (6,5)$

Cases favourable to $A \cap C$ are

$(1,2), (3,2), (5,2), (1,4), (3,4), (5,4), (1,6), (3,6), (5,6)$

There is no case favourable to $A \cap B \cap C$.

Now $P(A) = \frac{18}{36} = \frac{1}{2}$

$P(B) = \frac{18}{36} = \frac{1}{2}$

$P(C) = \frac{18}{36} = \frac{1}{2}$

$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$

$P(B \cap C) = \frac{9}{36} = \frac{1}{4}$

$P(C \cap A) = \frac{9}{36} = \frac{1}{4}$

$P(A \cap B \cap C) = 0$

Since $P(A \cap B) = P(A)\,P(B)$,

$P(B \cap C) = P(B)\,P(C)$,

$P(A \cap C) = P(A)\,P(C)$,

it follows that A, B, C are pairwise independent.

Also $P(A \cap B \cap C) \neq P(A)\,P(B)\,P(C)$

Hence A, B, C are not mutually independent.

Bayes theorem

Let $E_1, E_2, \ldots, E_n$ be n mutually disjoint events such that $P(E_i) \neq 0$ for $i = 1, 2, 3, \ldots, n$. Then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)} \qquad (i = 1, 2, \ldots, n)$$

Proof:- Since $A \subset \bigcup_{i=1}^{n} E_i$, we have

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i)$$

Now $\{A \cap E_i\}_{i=1}^{n}$ are mutually disjoint events. So by addition law of probability,

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

But $P(A \cap E_i) = P(E_i)\,P(A/E_i)$ for $i = 1, 2, \ldots, n$. (1)

So $$P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i) \qquad (2)$$

Also we have

$$P(A \cap E_i) = P(A)\,P(E_i/A) \quad \text{for } i = 1, 2, \ldots, n.$$

Hence

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)} \qquad [\text{using (1) \& (2)}]$$

Example A doctor is to visit a patient and from past experience it is known that the probabilities that he will come by train, bus or scooter are respectively $\frac{3}{10}$, $\frac{1}{5}$ and $\frac{1}{10}$; the probability that he will use some other means of transport being, therefore, $\frac{2}{5}$. If he comes by train, the probability that he will be late is $\frac{1}{4}$; if by bus $\frac{1}{3}$; if by scooter $\frac{1}{12}$; if he uses some other means of transport, it can be assumed that he will not be late. When he arrives, he is late. What is the probability that he comes by train.

Soln Let E_1, E_2, E_3, E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively. Let A be the event that the doctor visits the patient late. Then

$$P(E_1) = \frac{3}{10}, \quad P(E_2) = \frac{1}{5}, \quad P(E_3) = \frac{1}{10}, \quad P(E_4) = \frac{2}{5}$$

Also $P(A/E_1) = \frac{1}{4}$, $P(A/E_2) = \frac{1}{3}$, $P(A/E_3) = \frac{1}{12}$

$P(A/E_4) = 0$

By Bayes theorem,

$$\text{reqd probability} = P(E_1/A)$$

$$= \frac{P(E_1)P(A/E_1)}{\sum_{i=1}^{4} P(E_i)P(A/E_i)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{1}{2}$$

Example In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{5}$. The probability that his answer is correct, given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it

Soln let E_1, E_2, E_3 denote the events that the examinee guesses, copies and knows the answer, respectively

let A denote the event that the answer is correct.

Then $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$, $P(E_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$

$P(A/E_1) = \frac{1}{4}$, $P(A/E_2) = \frac{1}{8}$, $P(A/E_3) = 1$

So by Bayes theorem,

$$\text{reqd probability} = P(E_3/A)$$

$$= \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{8} + \frac{1}{2} \times 1}$$

$$= \frac{\frac{1}{2}}{\frac{1}{12} + \frac{1}{48} + \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{4+1+24}{48}} = \frac{24}{29}$$

Example There are two coins one unbiased and the other two headed. One of the coin is choosen and tossed. If head turns up, what is the probability that biased coin was selected.

Soln let E_1 denote the event that unbiased coin was choosen
and E_2 " " that biased coin was choosen.

let A be the event of getting head. Then

$$P(E_1) = \frac{1}{2}, \quad P(E_2) = \frac{1}{2}$$

$$P(A/E_1) = \frac{1}{2}, \quad P(A/E_2) = 1$$

By Bayes theorem,

$$\text{reqd probability} = P(E_2/A)$$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1} = \frac{1/2}{\frac{3}{4}} = \frac{2}{3}$$

Example There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads. what is the probability that it was the two headed coin?

Soln Let E_1, E_2, E_3 denote the events that Ist, 2nd and 3rd coin is selected respectively and A be the event that head turns up. Then to find $P(E_1/A)$.

Now $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{3}$, $P(E_3) = \frac{1}{3}$

$P(A/E_1) = 1$, $P(A/E_2) = \frac{3}{4}$, $P(A/E_3) = \frac{1}{2}$

So by Bayes theorem,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{3}{4} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2}}$$

$$= \frac{1}{1 + \frac{3}{4} + \frac{1}{2}} = \frac{4}{9}$$ Ans

Example A bag contains three coins, one of which is a coin with two heads while the other two coins are normal and not biased. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two-headed coin?

Soln Let E_1, E_2, E_3 denote the events that Ist, 2nd and 3rd coin is selected respectively and A be the event that head turns up each time. We have to find $P(E_1/A)$.

Now $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{3}$, $P(E_3) = \frac{1}{3}$

$P(A/E_1) = 1$, $P(A/E_2) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $P(A/E_3) = \frac{1}{16}$

By Bayes theorem,

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{16} + \frac{1}{3} \times \frac{1}{16}} = \frac{16}{18} = \frac{8}{9}$$ Ans

Example The probabilities of X, Y and Z becoming managers are $\frac{4}{9}$, $\frac{2}{9}$ and $\frac{1}{3}$ respectively. The probabilities that the Bonus Scheme will be introduced if X, Y and Z become managers are $\frac{3}{10}$, $\frac{1}{2}$ and $\frac{4}{5}$ respectively.

(i) What is the probability that Bonus Scheme will be introduced ?

(ii) If the Bonus scheme has been introduced, what is the probability that the manager appointed was X ?

Soln. Let E_1 denote the event that X is appointed as manager,

E_2 " " " Y is " as manager

and E_3 " " " Z is " as manager.

Let A denote the event that Bonus scheme is Introduced.

Then $P(E_1) = \frac{4}{9}$, $P(E_2) = \frac{2}{9}$, $P(E_3) = \frac{1}{3}$

$P(A/E_1) = \frac{3}{10}$, $P(A/E_2) = \frac{1}{2}$, $P(A/E_3) = \frac{4}{5}$

(i) Probability that Bonus scheme will be introduced,

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= \frac{4}{9}\times\frac{3}{10} + \frac{2}{9}\times\frac{1}{2} + \frac{1}{3}\times\frac{4}{5}$$

$$= \frac{12}{90} + \frac{1}{9} + \frac{4}{15} = \frac{12+10+24}{90} = \frac{46}{90} = \frac{23}{45}$$

(ii) Using Bayes theorem,

$$\text{reqd prob.} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(A)}$$

$$= \frac{\frac{4}{9}\times\frac{3}{10}}{\frac{23}{45}} = \frac{12/90}{23/45} = \frac{6}{23}$$

Example There are two bags A and B. Bag A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $\frac{6}{7}$, find the value of n.

Soln. Let E_1 denote the event that bag A is selected and

E_2 " " " bag B is selected.

Let E denote the event that two balls drawn are white.

Then

$$P(E_1) = \frac{1}{2}, \quad P(E_2) = \frac{1}{2}$$

$$P(E/E_1) = \frac{{}^{n}C_2}{{}^{n+2}C_2} = \frac{n(n-1)}{2!} \cdot \frac{2!}{(n+2)(n+1)} = \frac{n(n-1)}{(n+2)(n+1)}$$

$$P(E/E_2) = \frac{{}^{2}C_2}{{}^{n+2}C_2} = \frac{1}{\frac{(n+2)(n+1)}{2!}} = \frac{2}{(n+2)(n+1)}$$

Prob that two white balls drawn are from bag A, is given by

$$P(E_1/E) = \frac{P(E_1)\,P(E/E_1)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2} \cdot \frac{2}{(n+2)(n+1)}}$$

$$= \frac{n(n-1)}{n(n-1)+2}$$

But $P(E_1|E) = \frac{6}{7}$ (given)

So

$$\frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$$\Rightarrow \quad 7n(n-1) = 6n(n-1)+12$$

$$\Rightarrow \quad n^2-n-12 = 0$$

$$\Rightarrow \quad n = 4, -3$$

Since n cannot be negative, we have $n=4$ Ans.

Example A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA?

Soln Let E_1 and E_2 denote the events that the letter came from TATANAGAR and CALCUTTA respectively. Let A denote the event that two consecutive visible letters on the envelope are TA. Then

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(A/E_1) = \frac{2}{8}, \quad P(A/E_2) = \frac{1}{7}$$

Using Bayes theorem, reqd probability is

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{2}{8}+\frac{1}{2}\times\frac{1}{7}} = \frac{\frac{1}{7}}{\frac{2}{8}+\frac{1}{7}} = \frac{\frac{1}{7}}{\frac{22}{56}} = \frac{4}{11}$$

Example A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.

Soln. Let E_1 denote the event that both A & B solve the problem correctly, E_2 denote the event that exactly one of them solves the problem correctly and E_3 denote the event that neither of them solves the problem correctly.

Let E denote the event that A and B get the same answer. Then

$$P(E_1) = \frac{1}{6} \times \frac{1}{8} = \frac{1}{48}$$

$$P(E_2) = \frac{1}{6} \times \frac{7}{8} + \frac{5}{6} \times \frac{1}{8} = \frac{12}{48} = \frac{1}{4}$$

$$P(E_3) = \frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$$

$$P(E/E_1) = 1, \quad P(E/E_2) = 0, \quad P(E/E_3) = \frac{1}{525}$$

Reqd Prob. $= P(E_1/E)$

$$= \frac{P(E_1)\,P(E/E_1)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2) + P(E_3)\,P(E/E_3)}$$

$$= \frac{\frac{1}{48} \times 1}{\frac{1}{48} \times 1 + \frac{12}{48} \times 0 + \frac{35}{48} \times \frac{1}{525}} = \frac{1}{1 + \frac{1}{15}} = \frac{15}{16}$$

Example A speaks truth 4 out of 5 times. A die is tossed. He reports that it is a six. What is the chance that actually there was six?

Soln let E_1 denote the event that A speaks truth

and E_2 " " " A tells a lie.

let A denote the event that 'A' reports a six.

Then $P(E_1) = \frac{4}{5}$, $P(E_2) = \frac{1}{5}$

$$P(A/E_1) = \frac{1}{6}, \quad P(A/E_2) = \frac{5}{6}$$

By Bayes theorem, reqd prob. is

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{4}{5} \times \frac{1}{6}}{\frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{5}{6}} = \frac{4}{4+5} = \frac{4}{9}$$

Example Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person choses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Soln. Let E_1, E_2, E_3 be the events that boxes I, II and III are chousen respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Let A denote the event that the coin drawn is of gold.

Then

$$P(A/E_1) = \frac{2}{2} = 1$$

$$P(A/E_2) = 0$$

$$P(A/E_3) = \frac{1}{2}$$

Prob. that other coin in the box is also of gold.

= Prob that gold coin is drawn from box I

$= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \quad \underline{\text{Ans}}$$

<u>Example</u> From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel, a ball is drawn and is found to be white. What is the probability that out of four balls transferred, 3 are white and 1 is black?

<u>Soln</u> Let E_1 denote the event of transfer of 4 black balls,

E_2 denote the event of transfer of one white and 3 black balls.

E_3 denote the event of transfer of 2 white and 2 black balls.

E_4 denote the event of transfer of 3 white and 1 black balls.

Let A denote the event that the ball drawn from second vessel is white. We have to find $P(E_4/A)$.

Now $$P(E_1) = \frac{{}^5C_4}{{}^8C_4} = \frac{5.4.3.2}{4!} \cdot \frac{4!}{8.7.6.5} = \frac{1}{14}$$

$$P(E_2) = \frac{{}^3C_1 \times {}^5C_3}{{}^8C_4} = \frac{3 \times 10}{70} = \frac{3}{7}$$

$$P(E_3) = \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = \frac{3 \times 10}{70} = \frac{3}{7}$$

$$P(E_4) = \frac{{}^3C_3 \times {}^5C_1}{{}^8C_4} = \frac{5}{70} = \frac{1}{14}$$

$P(A/E_1) = 0$, $P(A/E_2) = \frac{1}{4}$

$P(A/E_3) = \frac{2}{4}$, $P(A/E_4) = \frac{3}{4}$

By Bayes theorem,

$$P(E_4/A) = \frac{P(E_4)P(A/E_4)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + P(E_4)P(A/E_4)}$$

$$= \frac{\frac{1}{14} \times \frac{3}{4}}{\frac{1}{14} \times 0 + \frac{3}{7} \times \frac{1}{4} + \frac{3}{7} \times \frac{2}{4} + \frac{1}{14} \times \frac{3}{4}}$$

$$= \frac{3}{6+12+3}$$

$$= \frac{1}{7}$$ Ans

Example There are three urns having the following compositions of black and white balls :

Urn 1 : 7 white and 3 black balls

Urn 2 : 4 white and 6 black balls

Urn 3 : 2 white and 8 black balls

One of these urns is chosen at random with probabilities 0.20, 0.60 and 0.20 respectively and two balls are drawn from the chosen urn without replacement. What is the probability that both these balls are white ? If both these balls are found to be white, what is the probability that these are from Urn I ?

Soln Let E_1, E_2 and E_3 denote the events of choosing Ist, 2nd and 3rd urn respectively. Let A be the event that two balls drawn from the selected urn are white.

Then $P(E_1) = 0.20$, $P(E_2) = 0.60$, $P(E_3) = 0.20$

$$P(A/E_1) = \frac{7}{10} \times \frac{6}{9} = \frac{42}{90}$$

$$P(A/E_2) = \frac{4}{10} \times \frac{3}{9} = \frac{12}{90}$$

$$P(A/E_3) = \frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$$

(i) Probability that both balls drawn are white $= P(A)$

$$= P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)$$

$$= 0.20 \times \frac{42}{90} + 0.60 \times \frac{12}{90} + 0.20 \times \frac{2}{90} = \frac{16}{90} = \frac{8}{45}$$

(ii) By Bayes theorem,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{0.20 \times \frac{42}{90}}{\frac{8}{45}}$$

$$= \frac{1}{5} \times \frac{42}{90} \times \frac{45}{8} = \frac{21}{40}$$ Ans

Example Urn A contains 2 white, 1 black and 3 red balls; Urn B contains 3 white, 2 black and 4 red balls and Urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn. They happen to be red and black. What is the probability that both balls came from urn B ?

Soln Let E_1, E_2, E_3 denote the events of choosing Urn A, Urn B and Urn C respectively. Let A be the event that two balls drawn from the selected urn are red and black.

Then $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{3}$, $P(E_3) = \frac{1}{3}$

$$P(A/E_1) = \frac{1 \times 3}{^6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/E_2) = \frac{4 \times 2}{^9C_2} = \frac{8}{36} = \frac{2}{9}$$

$$P(A/E_3) = \frac{3 \times 2}{^9C_2} = \frac{6}{36} = \frac{1}{6}$$

By Bayes theorem,

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{9} + \frac{1}{3} \times \frac{1}{6}}$$

$$= \frac{2/9}{\frac{1}{5} + \frac{2}{9} + \frac{1}{6}}$$

$$= \frac{2/9}{\frac{18+20+15}{90}} = \frac{2}{9} \times \frac{90}{53} = \frac{20}{53}$$ Ans.