Uniform Continuous Function

Definition: A function is said to be uniformly continuous on the interval $[a,b]$ if for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta \ \forall\ x_1, x_2 \in [a,b].$$

Note: The continuity of a function is defined on a point but uniform continuity of a function is defined in an interval.

Also, A f^n is not uniformly continuous in an interval I, if $\exists$ some $\varepsilon > 0$ s.t. for any $\delta > 0$, $\exists$ a pair of points $x, y \in I$ s.t.

$$|f(x) - f(y)| \geq \varepsilon \quad \text{whenever} \quad |x - y| < \delta.$$

Theorem: If a function f is uniformly continuous on $[a,b]$, then it is continuous on $[a,b]$. But converse is not true.

Proof: Let f is uniformly cont. on $[a,b]$

Then for given $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\forall\ x_1, x_2 \in [a,b]$

$$|f(x_1) - f(x_2)| < \varepsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta$$

Let $c \in [a,b]$ then put $x_1 = x$ & $x_2 = c$, we get

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta$$

$\Rightarrow$ f is continuous at any $c \in [a,b]$

$\Rightarrow$ f is cont. on $[a,b]$

Converse

Let $f(x) = \frac{1}{x}$; $x \in (0,1)$

x is cont. on $(0,1)$, therefor $\frac{1}{x}$ is cont. on $(0,1)$

Let $\delta > 0$ be any number. Choose +ve integer n s.t. $\frac{1}{n} < \delta$

Let $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{2n}$; $x_1, x_2 \in (0,1)$

$$|x_1 - x_2| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \left|\frac{1}{2n}\right| < \delta$$

$$\Rightarrow |f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = n > \varepsilon \text{ if } \varepsilon = \frac{1}{2}$$

Thus, we have for any $\delta > 0$, $\exists$ $\varepsilon = \frac{1}{2}$ s.t.

$$|f(x_1) - f(x_2)| > \varepsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta$$

So, f is not uniformly continuous on $(0,1)$.

Theorem: If a function f is continuous on a closed interval $[a,b]$ then it is uniformly continuous on $[a,b]$.

Proof: Here, the function f is continuous on the closed interval $[a,b]$

For a given $\varepsilon > 0$, the interval $[a,b]$ can be divided into a finite number of sub intervals $[a, x_1], [x_1, x_2], \dots [x_{n-1}, b]$ s.t. for any two points α & β in the same sub-interval, we have

$$|f(\alpha) - f(\beta)| < \frac{\varepsilon}{2} \quad \rightarrow (1)$$

Let $\delta = \min\{(x_1 - a), (x_2 - x_1), \dots (b - x_{n-1})\}$

Let α, β be any two points in $[a,b]$ s.t. $|\alpha - \beta| < \delta$

Case-I α, β lie in the same subinterval.

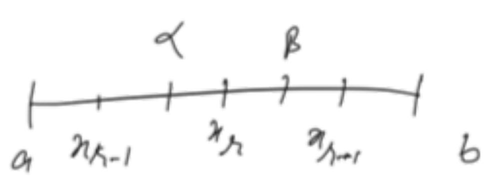
From (1), we have $\forall\ \alpha, \beta \in [a,b]$, $|f(\alpha) - f(\beta)| < \frac{\varepsilon}{2}$ whenever $|\alpha - \beta| < \delta$

Hence, f is uniformly continuous on $[a,b]$

Case-II α, β do not belong to the same sub-interval.

As $|\alpha - \beta| < \delta$, therefore α, β lie in the consective sub-interval.

Let x_r be the point of division of the two sub-intervals such that



$$x_{r-1} < \alpha < x_r < \beta < x_{r+1}$$

$$|f(\alpha) - f(\beta)| = |f(\alpha) - f(x_r) + f(x_r) - f(\beta)|$$
$$\leq |f(\alpha) - f(x_r)| + |f(x_r) - f(\beta)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow |f(\alpha) - f(\beta)| < \varepsilon$ whenever $|\alpha - \beta| < \delta\ \forall\ \alpha, \beta \in [a,b]$

Hence, f is uniformly continuous on $[a,b]$.

Q.1 $f(x) = x^2$ is uniformly cont. on $[-2, 2]$.

Q.2 $f(x) = 2x^2 + 3x - 4$ is uniformly cont. on $[-2, 2]$

Q.3 $f : (0, \infty) \to \mathbb{R}$ s.t. $f(x) = \frac{1}{x}$. Prove that f is uniformly continuous on $[a, \infty)$ where $a > 0$. But f is cont but not uniformly cont. on $(0, \infty)$

Q.4 $f(x) = \sin\frac{1}{x}$; $x \in \mathbb{R}^+$ is cont but not uniformly continuous.

Sol^n Let $g(x) = \frac{1}{x}$ and $h(x) = \sin x$ $\forall\ x \in \mathbb{R}^+$

So $h \circ g(x) = \sin\frac{1}{x}$.

h & g are continuous, Hence $h \circ g(x)$ is also continuous.

To show $\sin\frac{1}{x}$ is not uniformly continuous for all $\mathbb{R}^+$.

Let $\delta > 0$ choose a +ve integer n s.t.

$$\frac{1}{2n\left(n\pi + \frac{\pi}{2}\right)} < \delta$$

Let $x_1 = \frac{1}{n\pi}$; $x_2 = \frac{1}{n\pi + \frac{\pi}{2}}$, so that $x_1, x_2 \in \mathbb{R}^+$

$$|x_1 - x_2| = \left|\frac{1}{n\pi} - \frac{1}{n\pi + \frac{\pi}{2}}\right| = \left|\frac{1}{2n\left(n\pi + \frac{\pi}{2}\right)}\right| < \delta$$

Now $|f(x_1) - f(x_2)| = \left|\sin n\pi - \sin\left(n\pi + \frac{\pi}{2}\right)\right|$

$$= |\cos n\pi| = 1 > \varepsilon \text{ if } \varepsilon = \frac{1}{2}.$$

Thus we have for any $\delta > 0$, $\exists$ $\varepsilon = \frac{1}{2}$ s.t.

$$|f(x_1) - f(x_2)| > \varepsilon \quad \text{whenever} \quad |x_1 - x_2| < \delta.$$

$\therefore$ $f(x)$ is not uniformly continuous

Q.5 $f(x) = \sqrt{x}$ is uniformly cont. on $[1, 2]$

Q.6 $f(x) = x^2$ is uniformly cont on $[0, 1]$

Q.7 $f(x) = x^2 + 2x + 2$ is uniformly cont. on $[1, 2]$

Q.8 $f(x) = x^3$ is uniformly cont. on $[0, 3]$

Q.9 $f(x) = \frac{x}{x+1}$ is uniformly cont. on $[0, 2]$

Q.10 $f(x) = \frac{1}{x^2}$ is not uniformly cont on $(0, 1]$

Q.11 $f(x) = x^2$, $x \in \mathbb{R}$ is not uniformly cont. on $\mathbb{R}$

Q.12 Product of two uniformly continuous f^n need not be uniformly continuous.