The Riemann - Stieltjes Integral

Defn let $[a,b]$ be a given interval. By a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, x_2, \ldots, x_n$, where

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = b$$

Let α be a monotonically increasing function on $[a,b]$. Then α is bounded on $[a,b]$ since $\alpha(a)$ and $\alpha(b)$ are finite. Corresponding to each partition P of $[a,b]$, we write

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

Then $\Delta \alpha_i \geq 0$ since α is monotonically increasing.

For any real function f which is bounded on $[a,b]$, we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$$

where $M_i = \text{lub } f(x) \quad (x_{i-1} \leq x \leq x_i)$

and $m_i = \text{glb } f(x) \quad (x_{i-1} \leq x \leq x_i)$

Then the sums $U(P,f,\alpha)$ and $L(P,f,\alpha)$ are respectively called the <u>upper Stieltjes</u> and <u>lower Stieltjes</u> sums of the function f w.r.t. α corresponding to the partition P.

We further define

$$\overline{\int_a^b} f\,d\alpha = \text{glb } U(P, f, \alpha)$$

and

$$\underline{\int_a^b} f\,d\alpha = \text{lub } L(P, f, \alpha)$$

where the glb and the lub are taken over all possible partitions P of $[a, b]$. The integrals $\overline{\int_a^b} f\,d\alpha$ and $\underline{\int_a^b} f\,d\alpha$ are called the upper and lower Riemann-Stieltjes integrals of f with respect to α over $[a, b]$, respectively.

If the upper and lower integrals are equal, we say that f is integrable with respect to α, in the Riemann sense and write $f \in \mathcal{R}(\alpha)$. Further in this case, common value of upper and lower integral is denoted by

$$\int_a^b f\,d\alpha \quad \text{or} \quad \int_a^b f(x)\,d\alpha(x). \qquad (1)$$

and is called the Riemann-Stieltjes integral of f w.r.t. α over $[a, b]$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann Stieltjes integral.

We shall now investigate the existence of integral (1). Without saying so every time, f will be assumed real and bounded, and α monotonically increasing on $[a, b]$.

Defn. let P and P^* be two partitions of $[a,b]$.

We say that the partition P^* is a refinement of P if every point of P is a point of P^*.

Given two partitions P_1 and P_2 of $[a,b]$, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$.

Theorem If P^* is a refinement of P, then

$$L(f,P,\alpha) \leq L(f,P^*,\alpha)$$

and $$U(f,P^*,\alpha) \leq U(f,P,\alpha).$$

Proof. Suppose first that P^* contains just one point more than P. let this extra point be x^* and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P.

Put
$$w_1 = \text{glb } f(x) \qquad (x_{i-1} \leq x \leq x^*)$$
$$w_2 = \text{glb } f(x) \qquad (x^* \leq x \leq x_i)$$
$$m_i = \text{glb } f(x) \qquad (x_{i-1} \leq x \leq x_i)$$

Then $w_1 \geq m_i$ and $w_2 \geq m_i$.

Hence $L(P^*,f,\alpha) - L(P,f,\alpha)$

$$= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)]$$
$$- m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \geq 0.$$

$\therefore \; L(P^*, f, \alpha) \geq L(P, f, \alpha)$

If P^* contains k points more than P, we repeat this reasoning k times to get the reqd result.

(ii) Suppose first that P^* contains just one point more than P. Let this extra point be x^* and suppose $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P.

Put $\quad W_1 = \text{lub } f(x) \qquad (x_{i-1} \leq x \leq x^*)$

$\quad\quad W_2 = \text{lub } f(x) \qquad (x^* \leq x \leq x_i)$

$\quad\quad M_i = \text{lub } f(x) \qquad (x_{i-1} \leq x \leq x_i)$

Then $\quad W_1 \leq M_i$ and $W_2 \leq M_i$.

$\therefore \; U(P, f, \alpha) - U(P^*, f, \alpha)$

$$= M_i[\alpha(x_i) - \alpha(x_{i-1})] - W_1[\alpha(x^*) - \alpha(x_{i-1})] - W_2[\alpha(x_i) - \alpha(x^*)]$$

$$= (M_i - W_2)[\alpha(x_i) - \alpha(x^*)] + (M_i - W_1)[\alpha(x^*) - \alpha(x_{i-1})] \geq 0$$

$\Rightarrow \; U(P, f, \alpha) \geq U(P^*, f, \alpha)$.

If P^* contains k points more than P, we repeat the above reasoning k times to get the reqd result.

<u>Theo</u>. For any two partitions P_1 and P_2 of $[a, b]$,

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha).$$

<u>Pf</u>. Let $P = P_1 \cup P_2$ so that P is a common refinement of P_1 & P_2.

$\therefore \; L(P_1, f, \alpha) \leq L(P, f, \alpha)$ ($\because$ P is refinement of P_1).

$\leq U(P, f, \alpha)$

$\leq U(P_2, f, \alpha)$ ($\because$ P is refinement of P_2)

<u>Theorem</u>. $$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

<u>Proof</u>. let P_1 and P_2 be any two partitions of $[a,b]$ and $P^* = P_1 \cup P_2$. Then

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha).$$

Hence $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

for any partitions P_1 and P_2 of $[a,b]$.

Keeping P_2 fixed and taking lub over all partitions P_1, we get

$$\underline{\int_a^b} f\,d\alpha \leq U(P_2, f, \alpha).$$

Taking glb over all partitions P_2, we get

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

<u>Example</u> let $\alpha(x) = x$ and define f on $[0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0,1]$.

Then for each $i = 1, 2, 3, \ldots, n$;

$$M_i = \text{lub } f(x) \text{ in } [x_{i-1}, x_i] = 1$$

and $$m_i = \text{glb } f(x) \text{ in } [x_{i-1}, x_i] = 0$$

because each subinterval $[x_{i-1}, x_i]$ contains rational as well as irrational numbers.

Therefore $$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i = 0$$

and $$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1 - 0 = 1.$$

This holds for every partition P of $[0,1]$.

So $$\underline{\int_0^1} f\,d\alpha = 0 \quad \text{and} \quad \overline{\int_0^1} f\,d\alpha = 1.$$

Hence $$\underline{\int_0^1} f\,d\alpha < \overline{\int_0^1} f\,d\alpha$$

<u>Theorem</u> $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

<u>Proof.</u> Suppose first that given condition holds.

Let $\epsilon > 0$. Then $\exists$ a partition P of $[a,b]$ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Now $$L(P, f, \alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P, f, \alpha)$$

So $$0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha)$$

that is,

$$0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon$$

Since $\epsilon > 0$ is arbitrary small, this gives

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = 0$$

Hence $$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$ and so $f \in \mathcal{R}(\alpha)$

Conversely suppose that $f \in \mathcal{R}(\alpha)$.

Then $$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

let $\epsilon > 0$ be given.

Since $\underline{\int_a^b} f\,d\alpha$ is lub of all lower sums, there exists a partition P_1 of $[a,b]$ s.t.

$$L(P_1,f,\alpha) > \underline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2} = \int_a^b f\,d\alpha - \frac{\epsilon}{2}$$

Similarly there exists a partition P_2 of $[a,b]$ s.t.

$$U(P_2,f,\alpha) < \overline{\int_a^b} f\,d\alpha + \frac{\epsilon}{2} = \int_a^b f\,d\alpha + \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$ so that P is a common refinement of P_1 and P_2. Then

$$U(P,f,\alpha) \leq U(P_2,f,\alpha)$$

$$< \int_a^b f d\alpha + \frac{\epsilon}{2}$$

$$< L(P_1,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\leq L(P,f,\alpha) + \epsilon$$

Hence

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

<u>Theorem</u> If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Proof. let $\epsilon > 0$ be given. Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \epsilon.$$

Since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. So there exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta \quad \text{whenever } x,t \in [a,b]$$
$$\text{with } |x-t| < \delta. \qquad (1)$$

If P is any partition of $[a,b]$ such that $|P| < \delta$ then (1) implies

$$M_i - m_i \leq \eta \qquad (i = 1,2,3,\ldots,n)$$

where $M_i = \text{lub } f(x)$ in $[x_{i-1}, x_i]$ and

$m_i = \text{glb } f(x)$ in $[x_{i-1}, x_i]$.

Hence $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$

$$\leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta\,[\alpha(b) - \alpha(a)] < \epsilon$$

and so $f \in \mathcal{R}(\alpha)$.

Theorem let f be a continuous real function on the interval $[a,b]$ If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$ then there exists a point $x \in (a,b)$ such that $f(x) = c$.

Theorem If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$.

(We still assume, of course, that α is monotonic)

Proof. Let $\epsilon > 0$ be given.

For any positive integer n, choose a partition $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ of $[a,b]$ s.t.

$$\alpha(x_1) = \alpha(a) + \frac{\alpha(b) - \alpha(a)}{n}$$

$$\alpha(x_2) = \alpha(a) + \frac{2\,[\alpha(b) - \alpha(a)]}{n}$$

$$\cdots\cdots$$

$$\alpha(x_{n-1}) = \alpha(a) + \left(\frac{n-1}{n}\right)[\alpha(b) - \alpha(a)].$$

This is possible since α is continuous on $[a,b]$.

We suppose that f is monotonically increasing on $[a,b]$.

Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$ $(i = 1,2,3,\dots,n)$

So that

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$$

$$= \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

$$= \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n} [f(b) - f(a)] < \epsilon$$

If n is taken large enough.

Hence $f \in \mathcal{R}(\alpha)$.

<u>Theorem</u> If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \mathcal{R}(\alpha)$

and $$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

<u>Proof</u>. Let $\epsilon > 0$ be given.

Since $f_1 \in \mathcal{R}(\alpha)$, $f_2 \in \mathcal{R}(\alpha)$, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$U(P_1,f_1,\alpha) - L(P_1,f_1,\alpha) < \frac{\epsilon}{2}$$

$$U(P_2,f_2,\alpha) - L(P_2,f_2,\alpha) < \frac{\epsilon}{2}.$$

Let $f = f_1 + f_2$ and $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be common refinement of P_1 and P_2. Then $U(P,f_1,\alpha) - L(P,f_1,\alpha) < \epsilon/2$ and $U(P,f_2,\alpha) - L(P,f_2,\alpha) < \frac{\epsilon}{2}$.

Suppose further that M_i', m_i' ; M_i'', m_i'' and M_i, m_i are the bounds of f_1, f_2 and f respectively in the subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \ldots, n$. Then

$$m_i' + m_i'' \leq m_i \leq M_i \leq M_i' + M_i'' \quad \text{for } i = 1, 2, 3, \ldots, n.$$

Multiplying by $\Delta \alpha_i$ and adding for $i = 1, 2, 3, \ldots, n$; we get

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

Therefore

$$U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $f \in \mathcal{R}(\alpha)$, that is, $f_1 + f_2 \in \mathcal{R}(\alpha)$.

With this same P, we have

$$U(P, f_1, \alpha) \leq U(P_1, f_1, \alpha)$$

$$< L(P_1, f_1, \alpha) + \frac{\epsilon}{2}$$

$$\leq \underline{\int_a^b} f_1 \, d\alpha + \frac{\epsilon}{2} = \int_a^b f_1 \, d\alpha + \frac{\epsilon}{2}$$

that is, $U(P, f_1, \alpha) < \int_a^b f_1 \, d\alpha + \frac{\epsilon}{2}$

Similarly, $U(P, f_2, \alpha) < \int_a^b f_2 \, d\alpha + \frac{\epsilon}{2}$

Therefore $\int_a^b f \, d\alpha \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$

$$< \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + \epsilon$$

Since ϵ is arbitrary positive number, we have

$$\int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad (1)$$

Replacing f_1 and f_2 in (1) by $-f_1$ and $-f_2$ respectively,

we have

$$\int_a^b -f d\alpha \leq \int_a^b -f_1 d\alpha + \int_a^b -f_2 d\alpha$$

i.e. $$-\int_a^b f d\alpha \leq -\int_a^b f_1 d\alpha - \int_a^b f_2 d\alpha$$

i.e. $$\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \leq \int_a^b f d\alpha \qquad (2)$$

From (1) & (2), $$\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Hence $$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

<u>Theorem</u> If $f \in R(\alpha)$ then $cf \in R(\alpha)$ for every constant c

and $$\int_a^b cf d\alpha = c\int_a^b f d\alpha.$$

<u>Proof</u>. Clearly result is true for $c = 0$.

So assume $c \neq 0$ let $\epsilon > 0$ be given

Since $f \in R(\alpha)$, $\exists$ a partition P of $[a, b]$ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{|c|}. \qquad (1)$$

Let the partition P be $\{a = x_0, x_1, x_2, \ldots, x_n = b\}$

Suppose $M_i = \text{lub } f(x)$ in $[x_{i-1}, x_i]$,

$m_i = \text{glb } f(x)$ in $[x_{i-1}, x_i]$,

$M_i' = \text{lub } (cf)(x)$ in $[x_{i-1}, x_i]$

and $m_i' = \text{glb } (cf)(x)$ in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

Then $M_i' = \begin{cases} c\,M_i & \text{if } c > 0 \\ c\,m_i & \text{if } c < 0 \end{cases}$

and $m_i' = \begin{cases} c\,m_i & \text{if } c > 0 \\ c\,M_i & \text{if } c < 0 \end{cases}$

So $U(P, cf, \alpha) = \sum_{i=1}^{n} M_i' \Delta\alpha_i = \begin{cases} c\,U(P, f, \alpha) & \text{if } c > 0 \\ c\,L(P, f, \alpha) & \text{if } c < 0 \end{cases}$

and $L(P, cf, \alpha) = \sum_{i=1}^{n} m_i' \Delta\alpha_i = \begin{cases} c\,L(P, f, \alpha) & \text{if } c > 0 \\ c\,U(P, f, \alpha) & \text{if } c < 0 \end{cases}$

Thus $U(P, cf, \alpha) - L(P, cf, \alpha) = \begin{cases} c\big(U(P, f, \alpha) - L(P, f, \alpha)\big) & \text{if } c > 0 \\ -c\big(U(P, f, \alpha) - L(P, f, \alpha)\big) & \text{if } c < 0. \end{cases}$

that is, $U(P, cf, \alpha) - L(P, cf, \alpha) = |c| \left[U(P, f, \alpha) - L(P, f, \alpha)\right]$

Hence $U(P, cf, \alpha) - L(P, cf, \alpha) < \epsilon$ [using (1)]

and so $cf \in \mathcal{R}(\alpha)$.

It remains to show $\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$.

For $c = 0$, result is trivial.

So let $c > 0$.

Since $U(P, f, \alpha) < L(P, f, \alpha) + \frac{\epsilon}{c} \leq \underline{\int_a^b} f\,d\alpha + \frac{\epsilon}{c} = \int_a^b f\,d\alpha + \frac{\epsilon}{c}$,

we have

$$\int_a^b cf\,d\alpha = \overline{\int_a^b} cf\,d\alpha \leq U(P, cf, \alpha) = c\,U(P, f, \alpha) < c\int_a^b f\,d\alpha + \epsilon$$

that is,

$$\int_a^b cf\,d\alpha < c\int_a^b f\,d\alpha + \epsilon.$$

But $\epsilon > 0$ is arbitrary small. So we must have

$$\int_a^b cf\,d\alpha \leq c\int_a^b f\,d\alpha \qquad (2)$$

Again from (1),

$$L(P, f, \alpha) > U(P, f, \alpha) - \frac{\epsilon}{c} \geq \overline{\int_a^b} f\,d\alpha - \frac{\epsilon}{c} = \int_a^b f\,d\alpha - \frac{\epsilon}{c}$$

So $$\int_a^b cf\,d\alpha = \underline{\int_a^b} cf\,d\alpha \geq L(P, cf, \alpha) = c\,L(P, f, \alpha) > c\int_a^b f\,d\alpha - \epsilon.$$

Since this holds for each $\epsilon > 0$, we have

$$\int_a^b cf\,d\alpha \geq c\int_a^b f\,d\alpha \qquad (3)$$

Hence $$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha.$$

Similarly we can prove the result if $c < 0$.

<u>Theorem</u> If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1+\alpha_2)$

and $$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2.$$

<u>Proof</u> :-

Let $\epsilon > 0$ be given.

Since $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, $\exists$ partitions P_1 and P_2 of $[a,b]$ s.t.

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\epsilon}{2}$$

$$U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\epsilon}{2}.$$

These inequalities also hold if P_1 and P_2 are replaced by their common refinement P.

let the partition P be $\{a = x_0, x_1, x_2, -, x_n = b\}$.

If M_i and m_i are bounds of f in $[x_{i-1}, x_i]$ then

$$U(P, f, \alpha_1+\alpha_2) - L(P, f, \alpha_1+\alpha_2)$$

$$= \sum_{i=1}^{n} (M_i - m_i)[(\alpha_1+\alpha_2)(x_i) - (\alpha_1+\alpha_2)(x_{i-1})]$$

$$= \sum_{i=1}^{n} (M_i - m_i)[\alpha_1(x_i) - \alpha_1(x_{i-1})] + \sum_{i=1}^{n} (M_i - m_i)[\alpha_2(x_i) - \alpha_2(x_{i-1})]$$

$$= [U(P, f, \alpha_1) - L(P, f, \alpha_1)] + [U(P, f, \alpha_2) - L(P, f, \alpha_2)]$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $f \in \mathcal{R}(\alpha_1+\alpha_2)$.

It remains to show $\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$

We have

$$U(P,f,\alpha_1) < \int_a^b f\,d\alpha_1 + \frac{\epsilon}{2}$$

and $$U(P,f,\alpha_2) < \int_a^b f\,d\alpha_2 + \frac{\epsilon}{2}.$$

So $$\int_a^b f\,d(\alpha_1+\alpha_2) = \overline{\int_a^b} f\,d(\alpha_1+\alpha_2) \leq U(P,f,\alpha_1+\alpha_2)$$

$$= U(P,f,\alpha_1) + U(P,f,\alpha_2)$$

$$< \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 + \epsilon.$$

Since $\epsilon > 0$ is arbitrary small, this gives

$$\int_a^b f\,d(\alpha_1+\alpha_2) \leq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \qquad (1)$$

Replacing f by $-f$, we have

$$\int_a^b (-f)\,d(\alpha_1+\alpha_2) \leq \int_a^b (-f)\,d\alpha_1 + \int_a^b (-f)\,d\alpha_2$$

i.e. $$\int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \leq \int_a^b f\,d(\alpha_1+\alpha_2) \qquad (2)$$

Hence $$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2.$$

Theorem If $f \in \mathcal{R}(\alpha)$ and c is a positive constant, then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha$$

Proof :- First we observe that $c\alpha$ is monotonically increasing on $[a, b]$ since α is monotonically increasing and c is positive.

Let $\epsilon > 0$ be given.

Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{c}$$

Let M_i and m_i be the bounds of f in $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \ldots, n$

Then

$$U(P, f, c\alpha) - L(P, f, c\alpha) = \sum_{i=1}^{n} (M_i - m_i)[(c\alpha)(x_i) - (c\alpha)(x_{i-1})]$$

$$= \sum_{i=1}^{n} (M_i - m_i)[c\alpha(x_i) - c\alpha(x_{i-1})]$$

$$= c \sum_{i=1}^{n} (M_i - m_i)[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= c\,[U(P, f, \alpha) - L(P, f, \alpha)] < \epsilon$$

and so $f \in \mathcal{R}(c\alpha)$.

Further with the same partition P as above,

$$\int_a^b f\, d(c\alpha) = \overline{\int_a^b} f\, d(c\alpha) \leq U(P, f, c\alpha) = c\, U(P, f, \alpha)$$

$$< c\, L(P, f, \alpha) + \epsilon$$

$$\leq c \underline{\int_a^b} f\, d\alpha + \epsilon = c \int_a^b f\, d\alpha + \epsilon$$

Since $\epsilon > 0$ is arbitrary small, this gives

$$\overline{\int_a^b} f \, d(c\alpha) \leq c \int_a^b f \, d\alpha$$

Similarly,

$$\int_a^b f \, d(c\alpha) = \underline{\int_a^b} f \, d(c\alpha) \geq L(P, f, c\alpha) = c \, L(P, f, \alpha)$$

$$> c \, U(P, f, \alpha) - \epsilon$$

$$\geq c \overline{\int_a^b} f \, d\alpha - \epsilon = c \int_a^b f \, d\alpha - \epsilon$$

This holds for each $\epsilon > 0$. So

$$\int_a^b f \, d(c\alpha) \geq c \int_a^b f \, d\alpha$$

Hence
$$\int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha.$$
Proved.

Theorem If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ such that

$$f_1(x) \leq f_2(x) \text{ on } [a, b]$$

then

$$\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$$

Proof :- Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$.

Let $M_i = \text{lub } f_1(x)$ in $[x_{i-1}, x_i]$

and $M_i' = \text{lub } f_2(x)$ in $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \ldots, n$.

Since $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$, it follows that

$$M_i \leq M_i' \quad \text{for } i = 1, 2, 3, -, n.$$

$$\therefore U(P, f_1, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i' \Delta\alpha_i = U(P, f_2, \alpha)$$

Taking infimum over all partitions P of $[a,b]$, we have

$$\overline{\int_a^b} f_1 \, d\alpha \leq \overline{\int_a^b} f_2 \, d\alpha$$

This implies $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$ since $f_1 \in R(\alpha)$, $f_2 \in R(\alpha$

<u>Theorem</u> If $f \in R(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in R(\alpha)$ on $[a,c]$ and on $[c,b]$, and

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

Proof :- Since $f \in R(\alpha)$ so given $\epsilon > 0$, $\exists$ a partition P of $[a,b]$ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

let P^* be a refinement of P such that $P^* = P \cup \{c\}$

Then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

This gives $U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

let P_1 and P_2 denote the sets of points of P^* between $[a,c]$ and $[c,b]$ respectively Then P_1 and P_2 are partitions of

$[a, c]$ and $[c, b]$ respectively s.t. $P_1 \cup P_2 = P^*$.

Further $U(P^*, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$

$L(P^*, f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha)$

So

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) + U(P_2, f, \alpha) - L(P_2, f, \alpha)$$

$$= U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

Since each of the numbers $U(P_1, f, \alpha) - L(P_1, f, \alpha)$ and $U(P_2, f, \alpha) - L(P_2, f, \alpha)$ is non-negative, it follows that

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$$

$$U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon$$

Hence $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $[c, b]$.

It remains to show $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$.

let P_1 be any partition of $[a, c]$ and P_2 be any partition of $[c, b]$. Then $P = P_1 \cup P_2$ is a partition of $[a, b]$ such that

$$U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$$

and $L(P, f, \alpha) = L(P_1, f, \alpha) + L(P_2, f, \alpha)$.

Now

$$\int_a^b f d\alpha = \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$$

Keeping P_1 fixed and taking infimum over all partitions P_2 of $[c, b]$

we get $$\int_a^b f d\alpha \leq U(P_1, f, \alpha) + \overline{\int_c^b} f d\alpha$$

Now taking infimum over all partitions P_1 of $[a,c]$, we have

$$\int_a^b f d\alpha \leq \overline{\int_a^c} f d\alpha + \overline{\int_c^b} f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha \qquad (1).$$

since $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]$.

Again

$$\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha \geq L(P,f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha).$$

Keeping P_1 fixed and taking supremum over all partitions P_2 of $[c,b]$, we get

$$\int_a^b f d\alpha \geq L(P_1,f,\alpha) + \underline{\int_c^b} f d\alpha.$$

Now taking supremum over all partitions P_1 of $[a,c]$, we have

$$\int_a^b f d\alpha \geq \underline{\int_a^c} f d\alpha + \underline{\int_c^b} f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha \qquad (2).$$

From (1) & (2),

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

<u>Theorem</u>. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \leq K$ on $[a,b]$,

then $$\left|\int_a^b f d\alpha\right| \leq K\,[\alpha(b) - \alpha(a)].$$

<u>Proof</u>. Let m, M be the bounds of f on $[a,b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$.

Suppose $M_i = \text{lub } f(x)$ in $[x_{i-1}, x_i]$

and $m_i = \text{glb } f(x)$ in $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \ldots, n$.

Then
$$m \leq m_i \leq M_i \leq M \quad \text{for } i = 1, 2, 3, \ldots, n.$$

This implies
$$m[\alpha(b) - \alpha(a)] = \sum_{i=1}^{n} m \Delta\alpha_i \leq \sum_{i=1}^{n} m_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$\leq M \sum_{i=1}^{n} \Delta\alpha_i$$
$$= M[\alpha(b) - \alpha(a)]$$

that is,
$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)].$$

But
$$L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha)$$

So we have
$$m[\alpha(b) - \alpha(a)] \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq M[\alpha(b) - \alpha(a)] \qquad (1)$$

Since $f \in \mathcal{R}(\alpha)$ so $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$

Therefore (1) implies
$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]. \qquad (2).$$

Now $|f(x)| \leq K$ for all $x \in [a, b]$

So $-K \leq f(x) \leq K$ for all $x \in [a, b]$.

This implies $-K \leq m \leq M \leq K$.

Therefore
$$\begin{aligned} -K[\alpha(b)-\alpha(a)] &\leq m[\alpha(b)-\alpha(a)] \\ &\leq \int_a^b f\,d\alpha \\ &\leq M[\alpha(b)-\alpha(a)] \\ &\leq K[\alpha(b)-\alpha(a)]. \end{aligned}$$

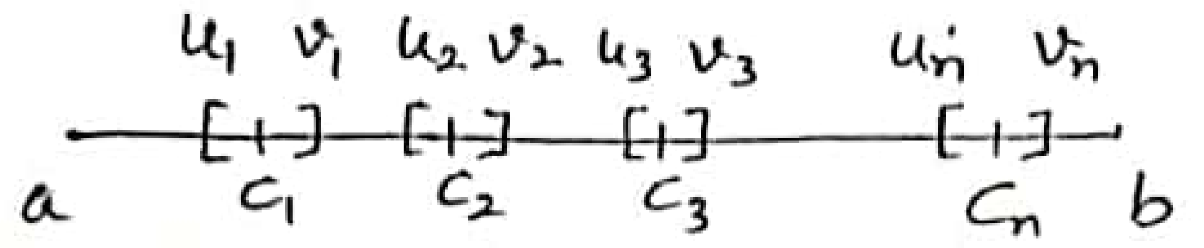
that is,
$$-K[\alpha(b)-\alpha(a)] \leq \int_a^b f\,d\alpha \leq K[\alpha(b)-\alpha(a)].$$

Hence $\left|\int_a^b f\,d\alpha\right| \leq K[\alpha(b)-\alpha(a)]$.

Theorem. Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

Proof.- Let $\epsilon > 0$ be given.

Put $M = \sup |f(x)|$ on $[a,b]$ and let E be the set of pts at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a,b]$ s.t. the sum of the corresponding differences $\alpha(v_j) - \alpha(u_j)$ is less than ϵ.



Furthermore, we can place these intervals in such a way that every point of $E \cap (a,b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from $[a,b]$. The remaining set K is compact. Hence f is uniformly continuous on K. So $\exists$ $\delta > 0$ s.t.

$$|f(s) - f(t)| < \epsilon \quad \text{if } s,t \in K,\ |s-t| < \delta.$$

Now form a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ as follows :-

Each u_j occurs in P. Each v_j occurs in P

No point of any segment (u_j, v_j) occurs in P.

If x_{i-1} is not one of the u_j then $\Delta x_i < \delta$.

Note that $M_i - m_i \leq 2M$ for every i and that $M_i - m_i \leq \epsilon$ unless x_{i-1} is one of the u_j.

Hence $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$

$$\leq \epsilon \sum_{i \in A} \Delta\alpha_i + 2M \sum_j [\alpha(v_j) - \alpha(u_j)]$$

where A is the set of those $i \in \{1,2,\ldots,n\}$ for which

$M_i - m_i \leq \epsilon$.

So $U(P,f,\alpha) - L(P,f,\alpha) \leq \epsilon[\alpha(b)-\alpha(a)] + 2M\epsilon$.

Since $\epsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}(\alpha)$.

<u>Example</u> Let f be a constant function on $[a,b]$ defined by $f(x) = R$ and α, a monotonically increasing function on $[a,b]$. Then $\int_a^b f\,d\alpha$ exists and $\int_a^b f\,d\alpha = R[\alpha(b)-\alpha(a)]$

<u>Soln</u>. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$.

Then M_i = lub $f(x)$ in $[x_{i-1}, x_i] = R$,

m_i = glb $f(x)$ in $[x_{i-1}, x_i] = R$ for $i = 1, 2, \ldots, n$.

$$\therefore\ U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i = R\sum_{i=1}^{n}\Delta\alpha_i = R[\alpha(b)-\alpha(a)]$$

$$\text{and } L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i = R\sum_{i=1}^{n}\Delta\alpha_i = R[\alpha(b)-\alpha(a)]$$

This holds for each partition P of $[a,b]$.

So

$$\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha) = R[\alpha(b)-\alpha(a)]$$

and

$$\underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha) = R[\alpha(b)-\alpha(a)].$$

Since $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$, f is Riemann-Stieltjes integrable w.r.t. α

and $\int_a^b f\,d\alpha = R[\alpha(b)-\alpha(a)]$.

Theorem Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$,

ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$.

Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

Proof :- Let $\epsilon > 0$ be given.

Since ϕ is continuous on $[m, M]$, it is uniformly continuous on $[m,M]$. So $\exists$ $\delta > 0$ s.t. $\delta < \epsilon$ and

$$|\phi(s) - \phi(t)| < \epsilon \quad \text{if } |s-t| \leq \delta \text{ and } s,t \in [m,M]. \quad (1)$$

Since $f \in \mathcal{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \quad (2)$$

let $M_i = \text{lub } f(x)$ in $[x_{i-1}, x_i]$,

$m_i = \text{glb } f(x)$ in $[x_{i-1}, x_i]$,

$M_i' = \text{lub } h(x)$ in $[x_{i-1}, x_i]$

and $m_i' = \text{glb } h(x)$ in $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \dots, n$.

Divide the numbers $1, 2, 3, \dots, n$ into two classes:

$i \in A$ if $M_i - m_i < \delta$,

$i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, our choice of δ implies $M_i' - m_i' \leq \epsilon$ [using (1)]

For $i \in B$, $M_i' - m_i' \leq 2K$ where $K = \sup |\phi(t)|$, $m \leq t \leq M$

Hence

$$\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

so that $\sum_{i \in B} \Delta \alpha_i < \delta$.

Therefore

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i=1}^{n} (M_i' - m_i') \Delta \alpha_i$$

$$= \sum_{i \in A} (M_i' - m_i') \Delta \alpha_i + \sum_{i \in B} (M_i' - m_i') \Delta \alpha_i$$

$$\leq \epsilon \sum_{i \in A} \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i$$

$$\leq \epsilon [\alpha(b) - \alpha(a)] + 2K\delta$$

$$< \epsilon [\alpha(b) - \alpha(a) + 2K] \qquad (\because \delta < \epsilon)$$

Since ϵ was arbitrary, we get $h \in \mathcal{R}(\alpha)$.

<u>Theorem</u> If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a,b]$, then

(a) $fg \in \mathcal{R}(\alpha)$;

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$.

<u>Proof</u>. Let m and M be bounds of f on $[a,b]$.

Define a function ϕ on $[m, M]$ as $\phi(t) = t^2$.

Then $h(x) = \phi(f(x)) = (f(x))^2 = f^2(x)$ on $[a,b]$

so that $h = f^2$.

By above theorem, $h \in \mathcal{R}(\alpha)$ Thus $f^2 \in \mathcal{R}(\alpha)$.

Since $f, g \in \mathcal{R}(\alpha)$ so $f+g \in \mathcal{R}(\alpha)$, $f-g \in \mathcal{R}(\alpha)$.

Therefore $(f+g)^2, (f-g)^2 \in \mathcal{R}(\alpha)$.

This implies $\frac{1}{4}[(f+g)^2 - (f+g)^2] \in \mathcal{R}(\alpha)$

that is, $fg \in \mathcal{R}(\alpha)$ which proves (a).

If we take $\phi(t) = |t|$ on $[m, M]$, we have

$$h(x) = \phi(f(x)) = |f(x)| = |f|(x) \text{ on } [a,b].$$

This gives $h = |f|$. Again by last therem, $|f| \in \mathcal{R}(\alpha)$.

Choose $c = \pm 1$ so that $c\int_a^b f\,d\alpha \geq 0$.

Then $$\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha = \int_a^b cf\,d\alpha \leq \int_a^b |f|\,d\alpha$$

since $cf \leq |f|$.

Hence the result.

Riemann-Stieltjes integral as limit of sum

Defn Let f be a bounded and α be monotonically increasing function on $[a,b]$. Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a,b]$ and $t_1, t_2, \dots, t_n$ be points such that

$$t_i \in [x_{i-1}, x_i] \quad \text{for } i = 1,2,3,\dots,n.$$

Then the sum $\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i$ is called a Riemann-Stieltjes sum of f with respect to α. and is denoted by $S(P,f,\alpha)$.

We say that

$$\lim_{|P|\to 0} S(P,f,\alpha) = I$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$|P| < \delta$ implies $\quad |S(P,f,\alpha) - I| < \epsilon$

Theorem If $\lim_{|P|\to 0} S(P,f,\alpha)$ exists, then $f \in \mathcal{R}(\alpha)$ and

$$\lim_{|P|\to 0} S(P,f,\alpha) = \int_a^b f\,d\alpha$$

Proof. Suppose that $\lim_{|P|\to 0} S(P,f,\alpha)$ exists and is equal to I.

Then given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|P| < \delta$ implies

$$|S(P,f,\alpha) - I| < \frac{\epsilon}{2}$$

i.e. $$I - \frac{\epsilon}{2} < S(P,f,\alpha) < I + \frac{\epsilon}{2} \qquad (1).$$

Choose a partition $P = \{x_0, x_1, x_2, \dots, x_n = b\}$ s.t. $|P| < \delta$.

If we let the points t_i range over the intervals $[x_{i-1}, x_i]$ and take the lub and glb of the numbers $S(P, f, \alpha)$ so obtained, then (1) gives

$$I - \frac{\epsilon}{2} \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq I + \frac{\epsilon}{2} \qquad (1)$$

that is,

$$U(P, f, \alpha) - L(P, f, \alpha) \leq \epsilon$$

Hence $f \in \mathcal{R}(\alpha)$. and $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$

Since $L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha)$, we have

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha) \qquad (2)$$

Using (2) in (1), we get

$$I - \frac{\epsilon}{2} \leq \int_a^b f d\alpha \leq I + \frac{\epsilon}{2}$$

$$\Rightarrow \left| \int_a^b f d\alpha - I \right| \leq \frac{\epsilon}{2} < \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\int_a^b f d\alpha = I$$

Hence $\int_a^b f d\alpha = \lim_{|P| \to 0} S(P, f, \alpha).$

The next theorem tells us that the symbol $d\alpha(x)$ can be replaced by $\alpha'(x)dx$ in the Riemann Stieltjes integral $\int_a^b f(x)d\alpha$. This is the situation in which Riemann Stieltjes integral reduces to Riemann integral.

<u>Theorem</u> If $f \in \mathcal{R}$ and $\alpha' \in \mathcal{R}$ on $[a,b]$ then $f \in \mathcal{R}(\alpha)$

and
$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx$$

Proof :- Since $f \in \mathcal{R}$, $\alpha' \in \mathcal{R}$, it follows that $f\alpha' \in \mathcal{R}$.

Since f is bounded, $\exists\ M > 0$ s.t.

$$|f(x)| \leq M \qquad \forall\ x \in [a,b]. \qquad (0)$$

Let $\epsilon > 0$ be given. Since $f\alpha' \in \mathcal{R}$ and $\alpha' \in \mathcal{R}$, there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\left| \sum_{i=1}^n f(t_i)\,\alpha'(t_i)\,\Delta x_i - \int_a^b f\alpha' dx \right| < \epsilon \qquad (1)$$

for all partitions P with $|P| < \delta_1$ and all $t_i \in [x_{i-1}, x_i]$

and
$$\left| \sum_{i=1}^n \alpha'(t_i)\,\Delta x_i - \int_a^b \alpha' dx \right| < \epsilon \qquad (2)$$

for all partitions P with $|P| < \delta_2$ and all $t_i \in [x_{i-1}, x_i]$.

Varying t_i in (2), we have

$$\left| \sum_{i=1}^n [\alpha'(t_i) - \alpha'(s_i)]\,\Delta x_i \right| \leq \left| \sum_{i=1}^n \alpha'(t_i)\,\Delta x_i - \int_a^b \alpha' dx \right| + \left| \int_a^b \alpha' dx - \sum \alpha'(s_i)\Delta x_i \right|$$

$$< \epsilon + \epsilon = 2\epsilon \quad \text{if } |P| < \delta_2,\ s_i, t_i \in [x_{i-1}, x_i]$$

This implies
$$\sum_{i=1}^n |\alpha'(t_i) - \alpha'(s_i)|\,\Delta x_i < 2\epsilon \qquad (3)$$

whenever $|P| < \delta_2$, $s_i, t_i \in [x_{i-1}, x_i]$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Choose a partition P with $\|P\| < \delta$.

and choose $t_i \in [x_{i-1}, x_i]$. By mean value theorem of Differential Calculus, $\exists\, s_i \in [x_{i-1}, x_i]$ s.t.

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(s_i)(x_i - x_{i-1}) = \alpha'(s_i)\Delta x_i$$

Therefore

$$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i = \sum_{i=1}^{n} f(t_i)\alpha'(s_i)\Delta x_i$$

$$= \sum_{i=1}^{n} f(t_i)\alpha'(t_i)\Delta x_i + \sum_{i=1}^{n} f(t_i)[\alpha'(s_i) - \alpha'(t_i)]\Delta x_i$$

So

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\alpha' dx \right|$$

$$= \left| \sum_{i=1}^{n} f(t_i)\alpha'(t_i)\Delta x_i + \sum_{i=1}^{n} f(t_i)[\alpha'(s_i) - \alpha'(t_i)]\Delta x_i - \int_a^b f\alpha' dx \right|$$

$$\leq \left| \sum_{i=1}^{n} f(t_i)\alpha'(t_i)\Delta x_i - \int_a^b f\alpha' dx \right| + \sum_{i=1}^{n} |f(t_i)|\,|\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i$$

$< \epsilon + 2\epsilon M$, using (0), (1) and (3).

This implies $\displaystyle\lim_{|P|\to 0} \sum_{i=1}^{n} f(t_i)\Delta\alpha_i = \int_a^b f\alpha' dx$

Hence $\displaystyle\int_a^b f\,d\alpha = \int_a^b f\alpha' dx.$

Example Evaluate

(i) $\int_0^2 x^2\, dx^2$ (ii) $\int_0^2 [x]\, dx^2$

Soln. We know that

$$\int_a^b f\, d\alpha = \int_a^b f(x)\, \alpha'(x)\, dx$$

So $$\int_0^2 x^2 dx^2 = \int_0^2 x^2 . 2x\, dx = 2\int_0^2 x^3 dx = 2\left|\frac{x^4}{4}\right|_0^2 = 8.$$

and $$\int_0^2 [x]\, dx^2 = \int_0^2 [x]\, 2x\, dx$$

$$= 2\int_0^1 [x]\, x\, dx + 2\int_1^2 [x]\, x\, dx$$

$$= 2\int_1^2 x\, dx \qquad \left[\because [x] = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2. \end{cases} \right.$$

$$= \left| x^2 \right|_1^2 = 4 - 1 = 3.$$

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We now establish a connection between the <u>integrand</u> and the <u>integrator</u> in a Riemann - Stieltjes integral.

We show that the existence of $\int f d\alpha$ implies the existence of $\int \alpha df$.

<u>Partial Integration Formula.</u>

If $f \in R(\alpha)$ on $[a,b]$ then $\alpha \in R(f)$ on $[a,b]$ and

$$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x).$$

<u>Proof</u> :- Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$. Choose $t_1, t_2, \dots, t_n$ s.t $x_{i-1} \leq t_i \leq x_i$ for $i = 1, 2, 3, \dots, n$. and take $t_0 = a$, $t_{n+1} = b$. Let the partition $\{t_0, t_1, \dots, t_{n+1}\}$ be denoted by Q. Then

$$\sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] = f(b)\alpha(b) - f(a)\alpha(a) - \sum_{i=1}^{n+1} \alpha(x_{i-1})[f(t_i) - f(t_{i-1})]$$

so that

$$S(P, f, \alpha) = f(b)\alpha(b) - f(a)\alpha(a) - S(Q, \alpha, f)$$

since $t_{i-1} \leq x_{i-1} \leq t_i$.

Taking limits as $|P| \to 0$, we have

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \lim_{|Q| \to 0} S(Q, \alpha, f)$$

This shows $\lim\limits_{|Q|\to 0} S(Q,\alpha,f)$ exists. So $\alpha \in R(f)$.

Hence

$$\int_a^b f\,d\alpha = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \int_a^b \alpha\,df.$$

Mean Value theorems for Riemann-Stieltjes Integrals

First mean value theorem for Riemann-Stieltjes integrals.

If f is continuous on $[a,b]$ and α is monotonically increasing on $[a,b]$ then $\exists$ a point c in $[a,b]$ s.t.

$$\int_a^b f\,d\alpha = f(c)\,[\alpha(b) - \alpha(a)].$$

Proof :- If $\alpha(a) = \alpha(b)$ then theorem holds trivially since each side equals '0' in that case.

So assume $\alpha(b) > \alpha(a)$.

Let $m =$ glb $f(x)$ in $[a,b]$ and $M =$ lub $f(x)$ in $[a,b]$.

Then $m \leq f(x) \leq M$ for all $x \in [a,b]$.

So

$$\int_a^b m\,d\alpha \leq \int_a^b f\,d\alpha \leq \int_a^b M\,d\alpha$$

that is,

$$m\,[\alpha(b) - \alpha(a)] \leq \int_a^b f\,d\alpha \leq M\,[\alpha(b) - \alpha(a)].$$

Thus

$$m \leq \frac{1}{[\alpha(b) - \alpha(a)]} \int_a^b f\,d\alpha \leq M.$$

Put $\lambda = \frac{1}{\alpha(b)-\alpha(a)} \int_a^b f\,d\alpha$ so that $m \le \lambda \le M$.

Then
$$\int_a^b f\,d\alpha = \lambda\,[\alpha(b)-\alpha(a)] \qquad (1)$$

Now $\lambda \in [m, M]$ and f assumes every value between its bounds. So $\exists\ c \in [a,b]$ s.t. $f(c) = \lambda$.

Hence
$$\int_a^b f\,d\alpha = f(c)\,[\alpha(b)-\alpha(a)]$$

Proved.

Second mean value theorem

let f be monotonic on $[a,b]$ and α be real valued monotonically increasing continuous function on $[a,b]$. Then $\exists$ a point $c \in [a,b]$ s.t.
$$\int_a^b f\,d\alpha = f(a)\,[\alpha(c)-\alpha(a)] + f(b)\,[\alpha(b)-\alpha(c)].$$

Proof. By partial integration formula, we have
$$\int_a^b f\,d\alpha = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \int_a^b \alpha\,df. \qquad (1)$$

Since α is continuous on $[a,b]$, by first mean value theorem for Riemann-Stieltjes integral there exists a point $c \in [a,b]$

s.t.
$$\int_a^b \alpha\,df = \alpha(c)\,[f(b)-f(a)].$$

Putting in (1), we have

$$\int_a^b f\,d\alpha = f(b)\,\alpha(b) - f(a)\,\alpha(a) - \alpha(c)\,[f(b) - f(a)]$$

$$= f(b)\,[\alpha(b) - \alpha(c)] + f(a)\,[\alpha(c) - \alpha(a)].$$

<u>Defn</u>. Let f be a bounded real valued function defined on $[a,b]$ and α be a function of bounded variation over $[a,b]$. Then $\exists$ non-decreasing functions β & r on $[a,b]$ s.t. $\alpha = \beta - r$. We define

$$\int_a^b f\,d\alpha = \int_a^b f\,d\beta - \int_a^b f\,dr$$

provided two integrals on RHS exist.

<u>Example</u> Show that $\int_0^3 x\,d([x]-x) = \frac{3}{2}$.

<u>Soln</u>. Here $\alpha = [x] - x$ is of bounded variation being difference of two non-decreasing functions $[x]$ and x.

So $$\int_0^3 x\,d([x]-x) = \int_0^3 x\,d([x]) - \int_0^3 x\,dx \qquad (1)$$

But $$\int_0^3 x\,d([x]) = f(3)\,\beta(3) - f(0)\,\beta(0) - \int_0^3 [x]\,dx$$

where $f(x) = x$, $\beta(x) = [x]$.

and $$\int_0^3 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx.$$

$$= 3$$

Hence $$\int_0^3 x\,d([x]-x) = 9 - 3 - \int_0^3 x\,dx = 6 - \frac{9}{2} = \frac{3}{2}$$ <u>Ans</u>.

Theorem (Change of variable)

Suppose ϕ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by

$$\beta(y) = \alpha(\phi(y)), \quad g(y) = f(\phi(y))$$

Then $g \in \mathcal{R}(\beta)$ and $\displaystyle\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof - To each partition $P = \{x_0, x_1, -, x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0, y_1, -, y_n\}$ of $[A,B]$ s.t. $x_i = \phi(y_i)$. All partitions of $[A,B]$ are obtained in this way since ϕ is invertible.

let $M_i = \text{lub } f(x)$ in $[x_{i-1}, x_i]$

and $m_i = \text{glb } f(x)$ in $[x_{i-1}, x_i]$.

Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we have

$$\text{lub } g(y) \text{ in } [y_{i-1}, y_i] = \text{lub } f(x) \text{ in } [x_{i-1}, x_i] = M_i$$

and $\text{glb } g(y)$ in $[y_{i-1}, y_i] = \text{glb } f(x)$ in $[x_{i-1}, x_i] = m_i$

So

$$U(Q, g, \beta) = \sum_{i=1}^{n} M_i \left[\beta(y_i) - \beta(y_{i-1})\right]$$

$$= \sum_{i=1}^{n} M_i \left[\alpha(\phi(y_i)) - \alpha(\phi(y_{i-1}))\right]$$

$$= \sum_{i=1}^{n} M_i [\alpha(x_i) - \alpha(x_{i-1})] = U(P, f, \alpha).$$

Similarly, $L(Q, g, \beta) = L(P, f, \alpha)$.

Since $f \in \mathcal{R}(\alpha)$, there is a partition P of $[a,b]$ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

This implies

$$U(Q, g, \beta) - L(Q, g, \beta) < \epsilon.$$

Hence $g \in \mathcal{R}(\beta)$.

Also $$L(P,f,\alpha) \leq \int_a^b f\,d\alpha \leq U(P,f,\alpha)$$

and $$L(Q,g,\beta) \leq \int_A^B g\,d\beta \leq U(Q,g,\beta).$$

These give

$$-[U(P,f,\alpha) - L(P,f,\alpha)] \leq \int_a^b f\,d\alpha - \int_A^B g\,d\beta \leq U(P,f,\alpha) - L(P,f,\alpha)$$

that is,

$$\left| \int_a^b f\,d\alpha - \int_A^B g\,d\beta \right| \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Since ϵ is arbitrary, we have

$$\int_a^b f\,d\alpha = \int_A^B g\,d\beta.$$

<u>Note</u> Taking $\alpha(x) = x$, we have $\int_a^b f(x)\,dx = \int_A^B f(\phi(y))\,d\phi(y)$

Integration and Differentiation.

Defn. If $f \in \mathcal{R}$ on $[a,b]$ then the function F defined on $[a,b]$ by
$$F(x) = \int_a^x f(t)\,dt$$
is called integral function of f.

Theorem Let $f \in \mathcal{R}$ on $[a,b]$.

For $a \leq x \leq b$, put
$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0 and
$$F'(x_0) = f(x_0).$$

Pf Since $f \in \mathcal{R}$ on $[a,b]$, it is bounded on $[a,b]$.

So $\exists$ $M > 0$ s.t. $|f(t)| \leq M$ for $a \leq t \leq b$.

Let $\epsilon > 0$ be given.

If $a \leq x < y \leq b$, then
$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$
$$= \left| \int_x^y f(t)\,dt \right| \leq M(y-x) < \epsilon$$

when $|y-x| < \frac{\epsilon}{M}$.

So taking $\delta = \frac{\epsilon}{M}$, we have $|F(y) - F(x)| < \epsilon$

whenever $|y-x| < \delta$.

This shows F is uniformly continuous on $[a, b]$ and hence continuous on $[a, b]$.

Now suppose f is continuous at x_0.

Given $\epsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \epsilon$$

if $|t - x_0| < \delta$ and $a \leq t \leq b$.

Hence if $x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta$ and $a \leq s < t \leq b$,

we have

$$\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t f(u)\,du - f(x_0) \right|$$

$$= \left| \frac{1}{t-s} \int_s^t f(u)\,du - \frac{1}{t-s} \int_s^t f(x_0)\,du \right|$$

$$= \left| \frac{1}{(t-s)} \int_s^t [f(u) - f(x_0)]\,du \right|$$

$$= \frac{1}{(t-s)} \left| \int_s^t [f(u) - f(x_0)]\,du \right|$$

$$< \frac{1}{(t-s)} \cdot \epsilon(t-s) = \epsilon$$

$\left[\because |f(u) - f(x_0)| < \epsilon \text{ for all } u \in [s, t] \right.$

This implies $F'(x_0)$ exists and

$$F'(x_0) = f(x_0).$$

The fundamental theorem of calculus

If $f \in \mathcal{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $\epsilon > 0$ be given. Since $f \in \mathcal{R}$ on $[a,b]$, $\exists$ a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \epsilon. \qquad (1)$$

By Lagrange's mean value theorem, $\exists\ t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1})\, F'(t_i)$$

$$= (x_i - x_{i-1})\, f(t_i)$$

for $i = 1, 2, \ldots, n$.

Thus $$\sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = F(b) - F(a). \qquad (2)$$

Now $$L(P,f) \leq \sum_{i=1}^{n} f(t_i)\,\Delta x_i \leq U(P,f)$$

and $$L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f).$$

This implies
$$L(P,f) - U(P,f) \leq \sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)\,dx \leq U(P,f) - L(P,f)$$

that is,

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)\,dx \right| \leq U(P,f) - L(P,f) < \epsilon$$

Using (2), this gives

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \epsilon$$

Since this holds for every $\epsilon > 0$, we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>Theorem</u> (<u>integration</u> by <u>parts</u>)

Suppose F and G are differentiable functions on $[a,b]$.

$F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$. Then

$$\int_a^b F(x)\,g(x)\,dx = F(b)\,G(b) - F(a)\,G(a) - \int_a^b f(x)\,G(x)\,dx.$$

<u>Proof</u> Put $H(x) = F(x)\,G(x)$

Then

$$H'(x) = F'(x)\,G(x) + F(x)\,G'(x)$$

$$= f(x)\,G(x) + F(x)\,g(x)$$

Since $f, g, F, G \in \mathcal{R}$ so $H' \in \mathcal{R}$. By fundamental theorem of calculus,

$$\int_a^b H'(x)\,dx = H(b) - H(a)$$

Hence

$$\int_a^b [f(x)\,G(x) + F(x)\,g(x)]\,dx = F(b)\,G(b) - F(a)\,G(a).$$

Integration of vector-valued functions.

Defn. Let $f_1, f_2, \ldots, f_k$ be real functions on $[a,b]$ and let $\bar{f} = (f_1, f_2, \ldots, f_k)$ be the corresponding mapping of $[a,b]$ into R^k.

Let α be a monotonically increasing function $[a,b]$.

We say $\bar{f} \in R(\alpha)$ if $f_j \in R(\alpha)$ for $j = 1, 2, 3, \ldots, k$.

In this case, integral of $\bar{f}$ is defined as

$$\int_a^b \bar{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \int_a^b f_2\, d\alpha, \ldots, \int_a^b f_k\, d\alpha \right)$$

Thus $\int_a^b \bar{f}\, d\alpha$ is the point in R^k whose jth coordinate is $\int_a^b f_j\, d\alpha$.

Theorem Let $\bar{f}, \bar{g}$ map $[a,b]$ into R^k and $\bar{f} \in R(\alpha)$, $\bar{g} \in R(\alpha)$

Then $\bar{f} + \bar{g} \in R(\alpha)$ and

$$\int_a^b (\bar{f} + \bar{g})\, d\alpha = \int_a^b \bar{f}\, d\alpha + \int_a^b \bar{g}\, d\alpha.$$

Proof:- Let $\bar{f} = (f_1, f_2, \ldots, f_k)$ and $\bar{g} = (g_1, g_2, \ldots, g_k)$

where f_j, g_j are real functions on $[a,b]$ for $j = 1, 2, \ldots, k$.

Since $\bar{f} \in R(\alpha)$ so $f_j \in R(\alpha)$ for $j = 1, 2, \ldots, k$.

Similarly, since $\bar{g} \in R(\alpha)$, we have $g_j \in R(\alpha)$ for $j = 1, 2, \ldots, k$

So $f_j + g_j \in R(\alpha)$ and $\int_a^b (f_j + g_j)\, d\alpha = \int_a^b f_j\, d\alpha + \int_a^b g_j\, d\alpha$

for $j = 1, 2, \ldots, k$.

Since $\bar{f}+\bar{g} = (f_1+g_1, f_2+g_2, \ldots, f_k+g_k)$ and

$f_j+g_j \in \mathcal{R}(\alpha)$ for $j=1,2,\ldots,k$; we have $\bar{f}+\bar{g} \in \mathcal{R}(\alpha)$.

Hence
$$\int_a^b (\bar{f}+\bar{g})\,d\alpha = \left(\int_a^b (f_1+g_1)\,d\alpha, \int_a^b (f_2+g_2)\,d\alpha, \ldots, \int_a^b (f_k+g_k)\,d\alpha\right)$$

$$= \left(\int_a^b f_1\,d\alpha + \int_a^b g_1\,d\alpha, \ldots, \int_a^b f_k\,d\alpha + \int_a^b g_k\,d\alpha\right)$$

$$= \left(\int_a^b f_1\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right) + \left(\int_a^b g_1\,d\alpha, \ldots, \int_a^b g_k\,d\alpha\right)$$

$$= \int_a^b \bar{f}\,d\alpha + \int_a^b \bar{g}\,d\alpha$$

Proved.

Theorem. Let $\bar{f}$ map $[a,b]$ into R^k and $\bar{f} \in \mathcal{R}(\alpha)$ for some monotonically increasing function α on $[a,b]$. If $a<c<b$ then $\bar{f} \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ s.t.
$$\int_a^b \bar{f}\,d\alpha = \int_a^c \bar{f}\,d\alpha + \int_c^b \bar{f}\,d\alpha.$$

Proof. Let $f_1, f_2, \ldots, f_k$ be the components of $\bar{f}$. Since $\bar{f} \in \mathcal{R}(\alpha)$ on $[a,b]$, it follows that $f_j \in \mathcal{R}(\alpha)$ on $[a,b]$ for $j=1,2,3,\ldots,k$. So $f_j \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ for $j=1,2,\ldots,k$; and
$$\int_a^b f_j\,d\alpha = \int_a^c f_j\,d\alpha + \int_c^b f_j\,d\alpha. \qquad (1)$$

Now $f_j \in \mathcal{R}(\alpha)$ on $[a,c]$ for $j = 1,2,\ldots,k$; so $\bar{f} \in \mathcal{R}(\alpha)$ on $[a,c]$

Similarly, $\bar{f} \in \mathcal{R}(\alpha)$ on $[c,b]$.

Further

$$\int_a^c \bar{f}\,d\alpha + \int_c^b \bar{f}\,d\alpha = \left(\int_a^c f_1\,d\alpha, \ldots, \int_a^c f_k\,d\alpha\right) + \left(\int_c^b f_1\,d\alpha, \ldots, \int_c^b f_k\,d\alpha\right)$$

$$= \left(\int_a^c f_1\,d\alpha + \int_c^b f_1\,d\alpha, \;\ldots\ldots, \int_a^c f_k\,d\alpha + \int_c^b f_k\,d\alpha\right)$$

$$= \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right) \quad [\text{using } (1)]$$

$$= \int_a^b \bar{f}\,d\alpha$$

Proved.

Theorem If $\bar{f}$ maps $[a,b]$ into R^k and $\bar{f} \in \mathcal{R}(\alpha_1)$, $\bar{f} \in \mathcal{R}(\alpha_2)$ then $\bar{f} \in \mathcal{R}(\alpha_1 + \alpha_2)$ and $\int_a^b \bar{f}\,d(\alpha_1 + \alpha_2) = \int_a^b \bar{f}\,d\alpha_1 + \int_a^b \bar{f}\,d\alpha_2$.

Proof :- Let $f_1, f_2, \ldots, f_k$ be the components of $\bar{f}$.

Since $\bar{f} \in \mathcal{R}(\alpha_1)$ so $f_j \in \mathcal{R}(\alpha_1)$ for $j = 1,2,\ldots,k$.

Also $\bar{f} \in \mathcal{R}(\alpha_2)$ so $f_j \in \mathcal{R}(\alpha_2)$ for $j = 1,2,\ldots,k$.

Therefore $f_j \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f_j\,d(\alpha_1 + \alpha_2) = \int_a^b f_j\,d\alpha_1 + \int_a^b f_j\,d\alpha_2$$

for $j = 1,2,3,\ldots,k$.

Now $f_j \in \mathcal{R}(\alpha_1+\alpha_2)$ for each $j = 1,2,\ldots,k$. So $\bar{f} \in \mathcal{R}(\alpha_1+\alpha_2)$

and

$$\int_a^b \bar{f}\, d(\alpha_1+\alpha_2) = \left(\int_a^b f_1\, d(\alpha_1+\alpha_2), \int_a^b f_2\, d(\alpha_1+\alpha_2), \ldots\ldots, \int_a^b f_k\, d(\alpha_1+\alpha_2)\right)$$

$$= \left(\int_a^b f_1\, d\alpha_1 + \int_a^b f_1\, d\alpha_2, \ldots\ldots\ldots, \int_a^b f_k\, d\alpha_1 + \int_a^b f_k\, d\alpha_2\right)$$

$$= \left(\int_a^b f_1\, d\alpha_1, \ldots\ldots, \int_a^b f_k\, d\alpha_1\right) + \left(\int_a^b f_1\, d\alpha_2, \ldots\ldots, \int_a^b f_k\, d\alpha_2\right)$$

$$= \int_a^b \bar{f}\, d\alpha_1 + \int_a^b \bar{f}\, d\alpha_2 .$$

Fundamental theorem of calculus for vector-valued functions

Theorem If $\bar{f}$ and $\bar{F}$ map $[a,b]$ into R^k, if $\bar{f} \in \mathcal{R}$ on $[a,b]$ and if $\bar{F}' = \bar{f}$, then

$$\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a).$$

Proof :- Let $\bar{f} = (f_1, f_2, \ldots, f_k)$ and $\bar{F} = (F_1, F_2, \ldots, F_k)$

so that f_j, F_j are real functions on $[a,b]$ for $j = 1,2,\ldots,k$

Since $\bar{f} \in \mathcal{R}$ on $[a,b]$, $f_j \in \mathcal{R}$ on $[a,b]$ for $j = 1,2,\ldots,k$.

Also $\bar{F}' = \bar{f}$ So $(F_1', F_2', \ldots, F_k') = (f_1, f_2, \ldots, f_k)$

This implies $F_j' = f_j$ on $[a,b]$ for $j = 1,2,\ldots,k$.

By fundamental theorem of calculus for real fns on $[a,b]$

we have

$$\int_a^b f_j(t)\,dt = F_j(b) - F_j(a) \qquad \text{for } j = 1, 2, -, n.$$

Thus

$$\int_a^b \bar{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, ---, \int_a^b f_k(t)\,dt \right)$$

$$= \left(F_1(b) - F_1(a), \; F_2(b) - F_2(a), ---, \; F_k(b) - F_k(a) \right)$$

$$= \left(F_1(b), F_2(b), --, F_k(b) \right) - \left(F_1(a), F_2(a), --, F_k(a) \right)$$

$$= \bar{F}(b) - \bar{F}(a).$$

<u>Theorem</u> If $\bar{f}$ maps $[a, b]$ into R^k and if $\bar{f} \in \mathcal{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|\bar{f}| \in \mathcal{R}(\alpha)$ and

$$\left| \int_a^b \bar{f}\,d\alpha \right| \leq \int_a^b |\vec{f}|\,d\alpha$$

<u>Proof</u> :-

If $f_1, f_2, --, f_k$ are the components of $\bar{f}$, then

$$|\bar{f}| = \left(f_1^2 + f_2^2 + --- + f_k^2 \right)^{1/2}$$

Since $\bar{f} \in \mathcal{R}(\alpha)$, each of the functions f_i belongs to $\mathcal{R}(\alpha)$.

So $f_i^2 \in \mathcal{R}(\alpha)$ for $i = 1, 2, -, k$. This implies

$$\left(f_1^2 + f_2^2 + --- + f_k^2 \right) \in \mathcal{R}(\alpha).$$

Since x^2 is a continuous function of x, the square root function is continuous on $[0, M]$ for every real M.

Using the result

" Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$."

by taking $\phi(x) = x^{1/2}$, we get $(f_1^2 + f_2^2 + \cdots + f_k^2)^{1/2} \in \mathcal{R}(\alpha)$.

Hence $|\bar{f}| \in \mathcal{R}(\alpha)$ on $[a,b]$.

To prove $\left| \int_a^b \bar{f}\, d\alpha \right| \leq \int_a^b |\bar{f}|\, d\alpha$,

we put $\bar{y} = (y_1, y_2, \ldots, y_k)$ where $y_j = \int_a^b f_j\, d\alpha$.

Then we have $\bar{y} = \int_a^b \bar{f}\, d\alpha$. and

$$|\bar{y}|^2 = y_1^2 + \cdots + y_k^2$$

$$= \sum_{j=1}^{k} y_j^2$$

$$= \sum_{j=1}^{k} y_j \int_a^b f_j\, d\alpha$$

$$= \int_a^b \left(\sum_{j=1}^{k} y_j f_j \right) d\alpha \qquad (1)$$

By Schwarz inequality,

$$\sum_{j=1}^{k} y_j f_j(t) \leq \left(\sum_{j=1}^{k} y_j^2 \right)^{1/2} \left(\sum_{j=1}^{k} f_j^2(t) \right)^{1/2} = |\bar{y}|\, |\bar{f}(t)|$$

$$(a \leq t \leq b)$$

So $$\int_a^b \left(\sum_{j=1}^{k} y_j \, f_j(t) \right) d\alpha(t) \leqslant |\bar{y}| \int_a^b |\bar{f}(t)| \, d\alpha(t)$$

i.e. $$\int_a^b \left(\sum_{j=1}^{k} y_j f_j \right) d\alpha \leqslant |\bar{y}| \int_a^b |\bar{f}| \, d\alpha \qquad (2)$$

From (1) and (2), we have

$$|\bar{y}|^2 \leqslant |\bar{y}| \int_a^b |\bar{f}| \, d\alpha \qquad (3)$$

If $\bar{y} = 0$ then $\int_a^b \bar{f} \, d\alpha = 0$ and so $\left| \int_a^b \bar{f} \, d\alpha \right| \leqslant \int_a^b |\bar{f}| \, d\alpha$

If $\bar{y} \neq 0$, dividing (3) by $|\bar{y}|$, we get

$$|\bar{y}| \leqslant \int_a^b |\bar{f}| \, d\alpha$$

that is, $$\left| \int_a^b \bar{f} \, d\alpha \right| \leqslant \int_a^b |\bar{f}| \, d\alpha .$$

Proved.

Rectifiable Curves

Defn A continuous mapping γ of an interval $[a,b]$ into R^k is called a curve in R^k.

If γ is one-to-one, γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

We associate to each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and to each curve γ on $[a,b]$, the number

$$\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

The ith term in this sum is the distance in R^k between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$. Hence $\Lambda(P,\gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely.

We define the length of γ as

$$\Lambda(\gamma) = \sup \Lambda(P,\gamma)$$

where the supremum is taken over all partitions of $[a,b]$.

If $\Lambda(\gamma) < \infty$, we say that γ is rectifiable.

In certain cases, $\Lambda(\gamma)$ is given by a Riemann integral. We shall prove this for continuously differentiable curves i.e. for curves γ whose derivative γ' is continuous.

<u>Theorem</u> let γ be a curve in R^k. If γ' is continuous on $[a,b]$ then γ is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof :- let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$.

Then using Fundamental theorem of calculus,

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\,dt\right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

for $i = 1, 2, 3, \dots, n$.

Therefore

$$\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt = \int_a^b |\gamma'(t)|\, dt$$

This holds for every partition P of $[a,b]$. Consequently,

$$\Lambda(\gamma) \leq \int_a^b |\gamma'(t)|\, dt \qquad (1)$$

To prove the opposite inequality, let $\epsilon > 0$ be given. Since γ' is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. So there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \epsilon \qquad \text{if } |s-t| < \delta.$$

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$ with $\Delta x_i < \delta$ for all i.

If $x_{i-1} \leqslant t \leqslant x_i$, it follows that

$$|\gamma'(t)| = |\gamma'(t) - \gamma'(x_i) + \gamma'(x_i)|$$

$$\leqslant |\gamma'(t) - \gamma'(x_i)| + |\gamma'(x_i)| < \epsilon + |\gamma'(x_i)|$$

Hence

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt \leqslant |\gamma'(x_i)|\, \Delta x_i + \epsilon\, \Delta x_i$$

$$= \int_{x_{i-1}}^{x_i} |\gamma'(x_i)|\, dt + \epsilon\, \Delta x_i$$

$$= \left| \int_{x_{i-1}}^{x_i} \gamma'(x_i)\, dt \right| + \epsilon\, \Delta x_i$$

$$= \left| \int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)]\, dt \right| + \epsilon\, \Delta x_i$$

$$\leqslant \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]\, dt \right| + \epsilon\, \Delta x_i$$

$$\leqslant |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon\, \Delta x_i \quad \text{for } i = 1, 2, -, n.$$

Adding these inequalities for $i = 1, 2, 3, -, n$; we get

$$\int_a^b |\gamma'(t)|\, dt \leqslant \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon\, (b-a)$$

$$= \Lambda(P, \gamma) + 2\epsilon\, (b-a).$$

$$\leqslant \Lambda(\gamma) + 2\epsilon\, (b-a).$$

Since ϵ was arbitrary, it follows that $\int_a^b |\gamma'(t)|\, dt \leqslant \Lambda(\gamma)$.

Hence $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

Ques Find the value of $\int_0^5 (x^2+1)\, d([x])$

Soln Here $f(x) = x^2+1$, $\alpha(x) = [x]$.

So
$$\int_0^5 (x^2+1)\, d([x]) = f(5)\alpha(5) - f(0)\alpha(0) - \int_0^5 [x]\, df(x)$$
$$= 26\times 5 - 1\times 0 - \int_0^5 [x]\, d(x^2+1)$$
$$= 130 - \int_0^5 [x] . 2x\, dx.$$
$$= 130 - 2\int_0^5 x[x]\, dx \qquad (1)$$

But
$$\int_0^5 [x]\, x\, dx = \int_0^1 [x]\, x\, dx + \int_1^2 [x]\, x\, dx + \int_2^3 [x]\, x\, dx + \int_3^4 [x]\, x\, dx + \int_4^5 [x]\, x\, dx$$
$$= \int_0^1 0\, dx + \int_1^2 x\, dx + \int_2^3 2x\, dx + \int_3^4 3x\, dx + \int_4^5 4x\, dx$$
$$= \left|\frac{x^2}{2}\right|_1^2 + \left|x^2\right|_2^3 + \left|\frac{3}{2}x^2\right|_3^4 + \left|2x^2\right|_4^5$$
$$= \frac{3}{2} + (9-4) + \frac{3}{2}(16-9) + 2(25-16)$$
$$= \frac{3}{2} + 5 + \frac{21}{2} + 18 = 35$$

Hence $\int_0^5 (x^2+1)\, d([x]) = 130 - 2\times 35 = 60$ Ans.