Fluid Dynamics

For Graduate & Post Graduate Students

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Unit-I

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Fluid Dynamics

UNIT-1: *Kinematics- Velocity at a point of a fluid, Eulerian and Lagrangian methods, Steam lines, Path lines and Streak lines, Velocity potential, Irrotational and rotational motions, Vorticity and circulation, Equations of continuity, Boundary surfaces, Acceleration at a point of a fluid, Components of acceleration in cylindrical and spherical polar co-ordinates.*

1.Fluid dynamics is the science of treating the study of fluids in motion. By the term fluid, we mean a substance that flows i.e. which is not a solid. Fluids may be divided into two categories

(i) Liquids which are incompressible i.e. their volume do not change when the pressure changes

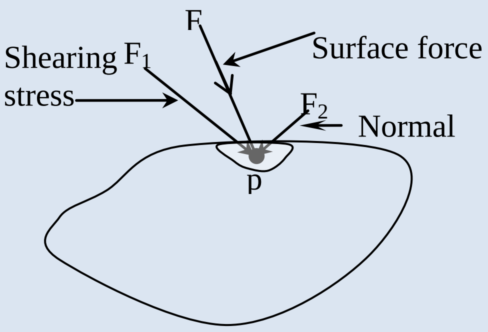
(ii) Gases which are compressible i.e. they undergo change in volume whenever the pressure changes. The term hydrodynamics is often applied to the science of moving incompressible fluids. However, there is no sharp distinction between the three states of matter i.e. solids, liquids and gases.

In the microscopic view of fluids, matter is assumed to be composed of molecules which are in random relative motion under the action of intermolecular forces. In solids, the spacing of the molecules is small, spacing persists even under strong molecule forces. In liquids, the spacing between molecules is greater even under weaker molecule forces and in gases, the gaps are even larger.

If we imagine that our microscope, with which we have observed the molecular structure of matter, has a variable focal length, we could change our observation of matter from the fine detailed microscopic view point to a longer range macroscopic viewpoint in which we would not see the gaps between the molecules and the matter would appear to be continuously distributed. We shall take this macroscopic view of fluids in which physical quantities associated with the fluids within a given volume V are assumed to be distributed continuously and, within a given sufficiently small volume δV uniformly. This observation is known as the Continuum hypothesis. It implies that at each point of a fluid, we can prescribe a unique velocity, a unique pressure, a unique density etc. Moreover, for a continuous or ideal fluid, we can define a fluid particle as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

1.1 Stresses: In general two types of forces act on a fluid element. One of them is body force and the other is surface force. The body force is proportional to the mass of the body on which it acts while the surface force is proportional to the surface area and acts on the boundary of the body.

Suppose $\vec{F}$ is the surface force acting on an elementary surface area dS at a point P of the surface S .



Let F_1 and F_2 be resolved parts of $\vec{F}$ in the directions of tangent and normal at P . The normal force per unit area is called the **normal stress** and is also called **pressure.** The tangential force per unit area is called the **shearing stress.**

1.2 Viscosity: It is the internal friction between the particles of the fluid which offers resistance to the deformation of the fluid. The friction is in the form of tangential and shearing forces (stresses). Fluids with such property are called inviscid or ideal or perfect fluids.

Actually, all fluids are real, but in many cases, when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored. By the definition of body force and shearing stress, it is clear that body force per unit area at every point of the surface of an ideal fluid acts along the normal to the surface at that point. Thus ideal fluid does not exert any shearing stress.

Thus, we conclude that viscosity of a fluid is that property by virtue of which it is able to offer resistance to shearing stress. It is kind of molecular frictional resistance.

1.3 Velocity of Fluid at a Point: Suppose that at time t, a fluid particle is at the point P having position vector $\vec{r}$ (i.e. $O\vec{P} = \vec{r}$)

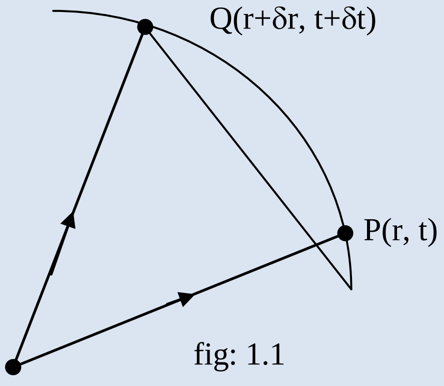


fig: 1.1

and at the time $t+\delta t$ the same particle at point Q has a position vector $\vec{r}+\delta\vec{r}$. the particle velocity $\vec{q}$ at the point $\vec{P}$ is $\delta\vec{r}$

$$\vec{q}=\lim_{\delta t\to 0}\frac{(\vec{r}+\delta\vec{r})-\vec{r}}{\delta t}=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}=\frac{d\vec{r}}{dt}$$

where the limit is assumed to exist uniquely. Clearly $\vec{q}$ is in general dependent on both $\vec{r}$ and t, so we may write

$$\vec{q}=\vec{q}(\vec{r},t)=\vec{q}(x,y,z,t),$$
$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}\ (P\ has\ co-ordinate(x,y,z))$$

Suppose,

$$\vec{q}=u\hat{i}+v\hat{j}+w\hat{k}$$

and since

$$\vec{q}=\frac{d\vec{r}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k},$$

Therefore

$$u=\frac{dx}{dt},\ v=\frac{dy}{dt},\ w=\frac{dz}{dt}$$

Remarks. (i) A point where $\vec{q}=\vec{0}$, is called a stagnation point.

(ii) where the flow is such that the velocity at each point is independent of time i.e. the flow is such that the velocity at each point is independent of time i.e. the flow pattern is same at each instant, then the motion is termed as steady motion, otherwise it is unsteady.

1.4 Flux across any surface: The flux i.e. the rate of flow across any surface S is defined by the integral

$$\int_S \rho(\vec{q}.\hat{n})dS$$

where ρ is density, $\vec{q}$ is the velocity of the fluid and $\hat{n}$ is the outward unit normal at any point of S.

Also, we define

$$Flux = density\times normal\,velocity\times area\,of\ the\,surface.$$

2. Eulerian and Lagrangian Methods (Local and Total range of change):

We have two methods for studying the general problem of fluid dynamics

2.1 Eulerian Method: In this method, we fix a point in the space occupied by the fluid and observation is made of whatever changes of velocity, density, pressure etc take place at that point. i.e. point is fixed and fluid particles are allowed to pass through it. If $P(x,y,z)$ is the point is fixed and fluid particles are do not depend upon the time parameter t, therefore $\dot{x},\dot{y},\dot{z}$ do not exist (dot denotes derivative w.r.t. time). Let $f(x,y,z,t)$ be a scalar function associated with some property of the fluid (e.g. its density) i.e. $f(x,y,z,t)=f(\vec{r},t)$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is the position vector of the point P, then

$$\frac{\partial f}{\partial t}=\lim_{\delta t\to 0}\frac{f(\vec{r},t+\delta t)-f(\vec{r},t)}{\delta t} \qquad (1)$$

Here, $\frac{\partial f}{\partial t}$ is called the local time rate of change.

2.2 Lagrangian Method:- In this case, observations are made at each point and each instant, i.e., any particle of the fluid is selected and observation is made of its particular motion and it is pursued throughout its course.

Let a fluid particle be initially at the point (a,b,c). After lapse of time t, let the same fluid particle be at (x,y,z). It is obvious that x,y,z are functions of t. But since the particles which have initially different positions occupy different positions after the motion is allowed. Hence the co-ordinates of the final position i.e. (x,y,z) depend on (a,b,c) also. Thus

$$x=f_1(a,b,c,t),\ y=f_2(a,b,c,t),\ z=f_3(a,b,c,t)$$

For this case, if $f(x,y,z,t)$ be scalar function associated with the fluid, then

$$\frac{df}{dt}=\frac{\lim\limits_{\delta t\to 0} f(\vec{r}+\delta\vec{r},t+\delta t)-f(\vec{r},t)}{\delta t} \qquad (2)$$

where $\dot{x},\dot{y},\dot{z}$ exist.

Here $\frac{df}{dt}$ is called an individual time rate or total rate or particle rate of change. Now, we establish the relation between these two-time rates (1) & (2).

We have

$$f=f(x,y,z,t)$$

Therefore,

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}+\frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right) . \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) + \frac{\partial f}{\partial t}$$

$$= \nabla f . \vec{q} + \frac{\partial f}{\partial t}$$

where

$$\vec{q} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = (u, v, w)$$

Thus

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{q}.\nabla f \qquad (3)$$

Remarks. (i) The relation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{q}.\nabla f$$

$$\Rightarrow \qquad \frac{df}{dt} = \left(\frac{\partial}{\partial t} + \vec{q}.\nabla \right) f$$

$$\Rightarrow \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{q}.\nabla$$

The operator $\frac{d}{dt}\left(also\, denoted\, by \frac{D}{Dt} \right)$ is called the Lagrangian operator or material derivative i.e. time rate of change in Lagrangian view. Sometimes, it is called 'differentiation following the fluid'

(ii) Similarly, for a vector function $\vec{F}(x, y, z, t)$ associated with some property of the fluid(e.g. its velocity, acceleration), we can show that

$$\frac{d\vec{f}}{dt} = \frac{\partial \vec{F}}{\partial t} + \vec{q}.\nabla \vec{f}$$

Hence the relation (3) holds for both scalar and vector functions associated with the moving fluid.

(iii) The Eulerian method is sometimes also called the flux method.

(iv) Both Lagrangian and Eulerian methods were used by Euler for studying fluid dynamics.

(v) Lagrangian methods resemble very much with to dynamics of a particle.

(vi) The two methods are essentially equivalent, but depending upon the problem, one has to judge whether the Lagrangian method is more useful or the Eulerian.

3. Streamlines, Pathlines and Streaklines

3.1 Streamlines: It is a curve drawn in the fluid such that the direction of the tangent to it at any point coincides with the direction of the fluid velocity vector $\vec{q}$ at that point. At any time t, let $\vec{q} = (u, v, w)$ be the velocity at each point $P(x, y, z)$ of the fluid. The direction ratios of the tangent to the curve at $P(x, y, z)$ are $d\vec{r} = (dx, dy, dz)$ since the tangent and the velocity at P have the same direction, therefore $\vec{q} \times d\vec{r} = \vec{0}$

i.e. $$(u\hat{i} + v\hat{j} + w\hat{k}) \times \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \vec{0}$$

i.e. $$vdz - wdy = 0 = wdx - udz = udy - vdx$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

These are the differential equations for the streamlines.

i.e. their solution gives the streamlines.

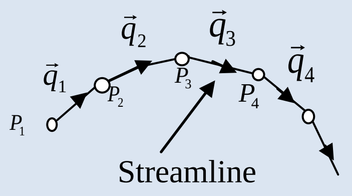


fig: 1.2

In the figure, if $\vec{q}_1, \vec{q}_2, \vec{q}_3, ...$denote the velocities at neighbouring points $P_1P_2, P_2P_3, P_3P_4.......$ collectively give the approximate form of the streamlines.

3.2. Pathlines: When the fluid motion is steady so that the pattern of flow does not vary with time, the paths of the fluid particles coincide with the streamlines. But in case of unsteady motion, the flow pattern varies with time and the paths of the particles do not coincide with the streamlines. However, the streamline through any point P does touch the pathline through P. Pathlines are, the curves described by the fluid particles during their motion i.e. these are the paths of the particles.

The differential equations for pathlines are

$$\frac{d\vec{r}}{dt} = \vec{q} \text{ i.e. } \frac{d\vec{x}}{dt} = u, \frac{d\vec{y}}{dt} = v, \frac{d\vec{z}}{dt} = w \qquad (1)$$

Where now (x,y,z) are the Cartesian co-ordinates of the fluid particle and not a fixed point of space. The equation of the pathline which passes through the point (x_0, y_o, z_0), fixed in space, at time $t = 0$ say, is the solution of (1) which satisfy initial condition that $x = x_0, y = y_0, z = z_0$ when $t = 0$. The solution gives a set of equations of the form

$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t) \\ y &= y(x_0, y_0, z_0, t) \\ z &= z(x_0, y_{0,}, z_0, t) \end{aligned}\right\} \qquad (2)$$

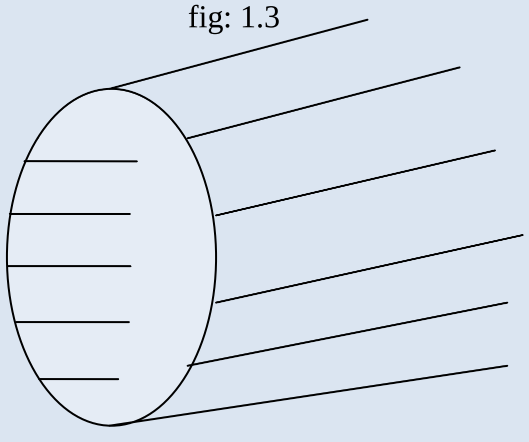
Which, as t takes all values greater than zero, will trace out the required pathline.

Remarks: (i) Streamlines give the motion of each particle at a given instant whereas pathlines give the motion of a given particle at each instant.

We can make these observations by using a suspension of aluminium dust in the liquid.

(ii) If we draw the streamlines through every point of a closed curve in the fluid, we obtain a **stream tube**. A stream tube of very small cross-section is called a **stream filament**.

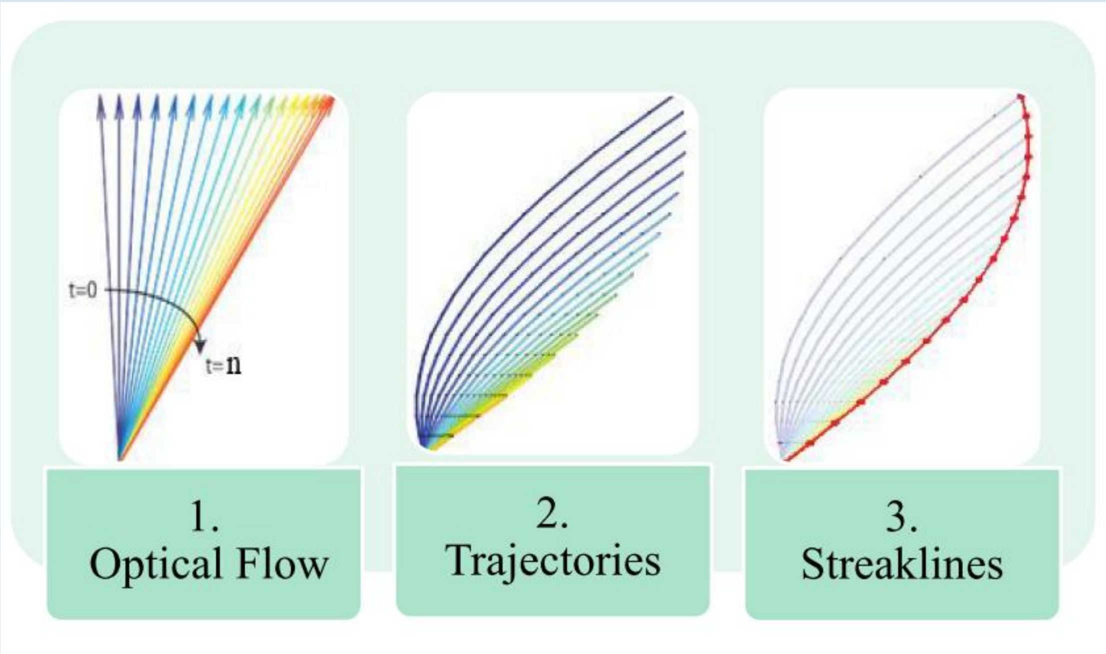
fig: 1.3



(iii) The components of velocity at the right angle to the streamline is always zero. This shows that there is no flow across the streamlines. This, if we replace the boundary of the stream tube by a rigid boundary, the flow is not affected. The principle of conservation of mass then gives that the flux across any cross-section of the stream tube should be the same.

3.3 Streaklines: In addition to streamlines and pathlines, it is useful for observational purposes define a streakline. This is the curve of all fluid particles which at some time have coincided with a particle which at some time have coincided with a particular fixed point of space. Thus, a streakline is observed when a neutrally buoyant marker fluid is continuously injected into the flow at a fixed point of space from time $\tau = -\infty$. The marker fluid may be smoke if the main flow involves a gas such as air, or a dye such as potassium permanganate ($KMnO_4$) if the main flow involves a liquid such as water.

fig.:1.4



If the co-ordinates of a particle of marker fluid are (x, y, z) at time t and the particle coincided with the injection point (x_0, y_0, z_0) at some time τ, where $\tau \le t$, then the time-history (streakline) of this particle is obtained by solving the equations for a pathline, subject to the initial condition that $x = x_0, y = y_0, z = z_0$ at $t = \tau$. As τ takes all possible values in the angle $-\infty \le \tau \le t$, the locations of all fluid particles on the streakline through (x_0, y_0, z_0) are obtained. Thus, the equation of the streakline at time t is given by

$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t) \\ y &= y(x_0, y_0, z_0, t) \\ z &= z(x_0, y_{0,}, z_0, t) \end{aligned}\right\} \qquad (-\infty \le \tau \le t)$$

Remark: (i) For a steady flow, streaklines also coincide with streamlines and pathlines.

(ii) Streamlines, pathlines and streaklines are termed as flowlines for a fluid.

4. Velocity Potential

Suppose that $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ is the velocity at any time t at each point $P(x, y, z)$ of the fluid. Also suppose that the expression $udx + vdy + wdz$ is an exact differential, say $-d\phi$

Then, $\quad -d\phi = udx + vdy + wdz$

i.e. $\quad -\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \frac{\partial\phi}{\partial t}dt\right) = udx + vdy + wdz$

where $\phi = \phi(x, y, z, t)$ is some scalar function, uniform throughout the entire field of flow.

Therefore,

$$u = \frac{-\partial\phi}{\partial x}, v = \frac{-\partial\phi}{\partial y}, w = \frac{-\partial\phi}{\partial z}, \frac{\partial\phi}{\partial t} = 0$$

but

$$\frac{\partial\phi}{\partial t} = 0 \qquad \Rightarrow \phi = \phi(x, y, z, t)$$

hence

$$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k} = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = -\nabla\phi$$

where ϕ is termed as the velocity potential and the flow of such type is called flow of potential kind.

In the above definition, the negative sign $\vec{q} = -\nabla\phi$ is a convention and it ensures that flow takes place from higher to lower potentials. The level surfaces $\phi(x, y, z, t) = const$, are called equipotentials or equipotential surfaces.

4.1. Theorem: At all points of the field of flow the equipotentials (i.e. equipotential surfaces) are cut orthogonally by the streamlines.

Proof. If the fluid velocity at any time t be $\vec{q}=(u,v,w)$, then the equations of streamlines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \qquad (1)$$

The surfaces are given by

$$\vec{q}.d\vec{r}=0 \text{ i.e. } udx+vdy+wdz=0 \qquad (2)$$

Are such that the velocity is at right angles to the tangent planes. The curves (1) and the surfaces (2) cut each other orthogonally. Suppose that the expression on the left-hand side (2) is an exact differential. Say, $-d\phi$, then

$$-d\phi=udx+vdy+wdz \qquad (3)$$

where ϕ is velocity potential.

The necessary and sufficient condition for the relations.

$$u=\frac{-\partial\phi}{\partial x}, v=\frac{-\partial\phi}{\partial y}, w=\frac{-\partial\phi}{\partial z}$$

i.e. $\vec{q}=-\nabla\phi$ to hold is

$$curl\,\vec{q}=curl\,(-\nabla\phi)=\vec{0} \qquad (4)$$

The solution of (2) i.e. $d\phi=0$ is

$$\phi(x,y,z)=const \qquad (5)$$

The surfaces (5) are called equipotentials. Thus the equipotentials are cut orthogonally by the stream lines.

Note: When $curl\,\vec{q}=\vec{0}$, the flow is said to be irrotational or of a potential kind, otherwise it is rotational. For irrotational flow, $\vec{q}=-\nabla\phi$

4.3. Example. The velocity potential of a two-dimensional flow is $\phi=cxy$. Find the stream lines

Solution. The stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

Where $\vec{q} = (u, v, w)$

For an irrotational motion (i.e. motion of potential kind)

We have

$$curl\,\vec{q} = 0 = curl\,(-\nabla\phi)$$

i.e. $\vec{q} = -\nabla\phi$, where ϕ is the velocity potential.

From here,

$$(u, v, w) = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = -(cy, cx, 0)$$

i.e. $u = -cy,\ v = -cx,\ w = 0$

Therefore, streamlines are

$$\frac{dx}{-cy} = \frac{dy}{-cx} = \frac{dz}{0}$$

i.e. $xdx - ydy = 0,\ dz = 0$

i.e $x^2 - y \ = a^2,\ z = K$

which are rectangular hyperbolae.

4.4. Example. If the speed of the fluid is everywhere the same, the streamlines are straight.

Solution. The streamlines are given by the differential equations.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Where u, v, and w are constants. The solutions are

$$vx - uy = const, \qquad vz - wy = const,$$

The intersections of these planes are necessarily straight lines. Hence the result.

4.5. Example. Find the stream lines and path lines of the particles for the two dimensional velocity field.

$$u = \frac{x}{1+t},\ v = y,\ w = 0$$

Solution. For streamlines, the differential equations are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

Therefore,

$$(1+t)\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{0}$$

Here $t=\text{constant}=t_0$ (at given instant), therefore the solutions are

$$(1+t_0)\log x=\log y+c_1,\quad z=c_2$$

Or $$\underline{\log x^{1+t_0}=\log y+\log a,}\ z=c_2$$

Or $$x^{1+t_0}=ay,\quad z=c_2$$

Which are the required stream lines.

For path lines, we have

$$\frac{dx}{dt}=u,\ \frac{dy}{dt}=v,\ \frac{dz}{dt}=w$$

Therefore,

$$\frac{dx}{dt}=\frac{x}{1+t},\ \frac{dy}{dt}=y,\ \frac{dz}{dt}=0$$

$\Rightarrow$ $$\frac{dx}{x}=\frac{dt}{1+t},\ \frac{dy}{dt}=y,\ dz=0$$

$\Rightarrow$ $$\log x=\log(1+t)+\log a,\quad \log y=t+\log b,\quad z=c$$

$\Rightarrow$ $$x=a(1+t),\quad y=be^t,\ z=c$$

$\Rightarrow$ $$y=be^{\frac{x-a}{a}},z=c$$

Which are the required path lines.

4.6. Note. In case of path lines, t must be eliminated since these give the motion at each instant (i.e. independent of t).

4.7 Example. Obtain the equations of the streamlines, path lines and streaklines which pass through $(l,l,0)$ at $t=0$ for the two dimensional flow

$$u = \frac{x}{t_0}\left(1+\frac{t}{t_0}\right),\ v = \frac{y}{t_0},\ w = 0$$

Where l and t_0 are constants having respectively the dimensions of length and time.

Solution. We define the dimensionless co-ordinates X, Y, Z and time T by writing

$$X = \frac{x}{l},\ Y = \frac{y}{l},\ Z = \frac{z}{l},\ T = \frac{t}{t_0}$$

Such that $$dX = \frac{1}{l}dx,\ dY = \frac{1}{l}dy,\ dZ = \frac{1}{l}dz,\ dT = \frac{1}{t_0}dt$$

And $$u = \frac{Xl}{t_0}(1+T),\ v = \frac{Yl}{t_0},\ w = 0$$

Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \qquad \frac{t_0 ldX}{Xl(1+T)} = \frac{t_0 ldY}{Yl} = \frac{ldZ}{0}$$

$$\Rightarrow \qquad \frac{dX}{X(1+T)} = \frac{dY}{Y} = \frac{dZ}{0}$$

Integrating these, we get

$$Z = \text{constant} = C_1(say) \qquad (1)$$

And $\log X = (1+T)\log Y + \log C_2$, where C_2 is constant

$$\Rightarrow \qquad X = C_2 Y^{(1+T)} \qquad (2)$$

As variables X, Y, Z and T are independent and C_1 and C_2 constants, equations (1) & (2) give the complete family of stream lines at all times $t = t_0 T$. In particular, $X = 1 = Y, Z = 0$ and $T = 0 \Rightarrow C_1 = 0, C_2 = 1$ and we get stream line as $Y = X$ i.e. $y = x$ and $z = 0$

Pathlines are given by $\frac{dX}{dT} = X(1+t),\ \frac{dY}{dT} = Y,\ \frac{dZ}{dT} = 0$

Now, X, Y, Z are the dimensionless co-ordinates of a fluid particle and are functions of T.

Therefore, $\dfrac{dX}{dT} = (1+t)dT \quad \Rightarrow \log X = \left(T + \dfrac{T^2}{2}\right) + \log K_1$

$$\Rightarrow \qquad X = K_1 e^{\frac{T+T^2}{2}} \qquad (3)$$

$$\frac{dY}{dT} = Y \;\Rightarrow \frac{dY}{dT} = dT \Rightarrow \log Y = T + \log K_2$$

$$\Rightarrow \qquad Y = K_2 e^T \qquad (4)$$

$$dZ = 0 \Rightarrow Z = const = K_3 \qquad (5)$$

These are the parametric equations of path lines. The path line through $P(1,1,0)$

i.e. $X = 1 = Y, Z = 0, T = 0$ is obtained when $K_1 = K_2 = 1, K_3 = 0$

$$\Rightarrow \qquad X = e^{T+\frac{T^2}{2}}, Y = e^T, Z = 0$$

Elimination of T gives.

$$X = e^{T\left(1+\frac{T}{2}\right)} = \left[e^T\right]^{\left(1+\frac{T}{2}\right)} = Y^{\left(1+\frac{1}{2}\log Y\right)}, Z = 0$$

The pathline which passes through $X = Y = 1, Z = 0$ when $T = \tau$ is given by

$$X = \exp.\left[T + \frac{1}{2}T^2 - \tau - \frac{1}{2}\tau^2\right],$$

$$Y = \exp.(T - \tau), Z = 0$$

These are the parametric equations of the streaklines true for all values of T. At $T = 0$, the equations give

$$X = \exp\left(-\tau - \frac{\tau^2}{2}\right), \; Y = \exp(-\tau), \; Z = 0$$

Eliminating τ, we have.

$$-\tau = \log Y \text{ i.e. } \tau = -\log Y$$

Therefore,

$$X = \exp\left(-\tau\left(1 + \frac{\tau}{2}\right)\right) = \left[e^{-\tau}\right]^{\left(1+\frac{\tau}{2}\right)} = (Y)^{\left(1+\frac{\tau}{2}\right)} = Y^{\left(1-\frac{\log Y}{2}\right)}, Z = 0$$

4.8. Article. To obtain the differential equations for streamlines in cylindrical and spherical co-ordinates.

We know that the streamlines are obtained from the differential equations

$$\vec{q} \times d\vec{r} = \vec{0} \qquad (1)$$

Where $\vec{q}$ is the velocity vector and $\vec{r}$ is the position vector of a liquid particle.

If the motion is irrotational, then

$$\vec{q} = -\nabla \phi$$

Therefore, the differential equations (1) become

$$\nabla \phi \times d\vec{r} = \vec{0} \qquad (2)$$

(i) In cylindrical co-ordinates (r, θ, z), we have

$$d\vec{r} = (dr, rd\theta, dz)$$

And

$$\nabla \phi = grad\phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

Thus, the different equations (2) become

$$\left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right) \times (dr, rd\theta, dz) = \vec{0}$$

$$\Rightarrow \qquad \frac{dr}{\partial \phi / \partial r} = \frac{rd\theta}{1/r \cdot \frac{\partial \phi}{\partial \theta}} = \frac{dz}{\partial \phi / \partial z} \qquad (3)$$

(ii) In spherical co-ordinates (r, θ, ψ), we have

$$d\vec{r} = (dr, rd\theta, d\psi)$$

And $$\nabla \phi = grad\phi = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \psi} \right)$$

The differential equations (2) become.

$$\left(\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial z} \right) \times (dr, rd\theta, r\sin\theta d\psi) = \vec{0}$$

$$\Rightarrow \qquad \frac{dr}{\partial\phi/\partial r} = \frac{r\,d\theta}{\frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta\,d\psi}{\frac{1}{r\sin\theta}\,\partial\phi/\partial\psi} \qquad (4)$$

Equations (3) & (4) are required differential equations.

4.9. Example. Show that if the velocity potential of an irrotational fluid motion is $\phi = \frac{A}{r^2}\psi\cos\theta,$ where (r,θ,ψ) are the spherical polar co-ordinates of any point, the lines of flow lie on the surface $r = k\sin^2\theta,$ k being a constant.

Solution. The differential equations for lines of flow (streamlines) are

$$\frac{dr}{\partial\phi/\partial r} = \frac{r\,d\theta}{\frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta\,d\psi}{\frac{1}{r\sin\theta}\,\partial\phi/\partial\psi}$$

From first two members, we have

$$\frac{dr}{-\psi\,2A/r^3\cos\theta} = \frac{r\,d\theta}{\frac{1}{r}\left(-\psi\frac{A}{r^2}\sin\theta\right)}$$

$$\Rightarrow \qquad \frac{dr}{\cos\theta} = \frac{2r\,d\theta}{\sin\theta} \qquad \Rightarrow \frac{dr}{r} = 2\frac{\cos\theta\,d\theta}{\sin\theta}$$

$$\Rightarrow \qquad \log r = 2\log\sin\theta + \log k \quad \Rightarrow r = k\sin^2\theta$$

Hence the result.

4.10. Note. In the above example, the velocity potential, in Cartesian co-ordinates, can be written as

$$\phi = A(x^2+y^2+z^2)^{-3/2}z.\tan^{-1}\left(\frac{y}{x}\right),$$

Where

$$x = r\sin\theta\cos\psi, \quad y = r\sin\theta\sin\psi, \quad z = r\cos\theta$$

Are spherical polar substitutions.

Also, the streamlines $r = k\sin^2\theta$ can be written as $r^3 = kr^2\sin^2\theta$

$$\Rightarrow \qquad (x^2+y^2+z^2)^{3/2} = k(x^2+y^2)$$

$$\Rightarrow \qquad x^2 + y^2 + z^2 = k^{2/3}(x^2 + y^2)^{2/3}$$

Which are the streamlines in Cartesian co-ordinates.

4.11. Example. At the point in an incompressible fluid having spherical polar co-ordinates (r, θ, ψ), are velocity components are $(2Mr^{-3}\cos\theta, Mr^{-2}\sin\theta, 0)$ where M is a constant. Show that velocity is of potential kind. Find the velocity potential and the equations of streamlines.

Solutions. Here $d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\psi\ \hat{\psi}$

$$\vec{q} = 2Mr^{-3}\cos\theta\,\hat{r} + Mr^{-2}\sin\theta\,\hat{\theta}, 0\,\hat{r})$$

Then,

$$curl\,\vec{q} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\psi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ 2Mr^{-3}\cos\theta & Mr^{-2}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}[\hat{r}.0 + r\hat{\theta}.0 + r\sin\theta\hat{\psi}(-2Mr^{-3}\sin\theta + 2Mr^{-3}\sin\theta)] = \hat{0}$$

Therefore, the flow is of potential kind.

Now, using the relation $\vec{q} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\hat{\psi}\right)$, we have

$$2Mr^{-3}\cos\theta\,\hat{r} + Mr^{-2}\sin\theta\,\hat{\theta} = \left(\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\hat{\psi}\right)$$

From here

$$\frac{\partial\phi}{\partial r} = 2Mr^{-3}\cos\theta,\ -\frac{\partial\phi}{\partial\theta} = Mr^{-2}\sin\theta,\ \frac{\partial\phi}{\partial\psi} = 0$$

Therefore,

$$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\psi}d\psi$$

$$= (-2Mr^{-3}\cos\theta)dr - (Mr^{-2}\sin\theta)d\theta$$

$$= d\left(Mr^{-2}\cos\theta\right)$$

Integrating, we get

$$\phi = Mr^{-2}\cos\theta$$

This is the required velocity potential.

The streamlines are given by

$$\frac{dr}{\partial\phi/\partial r} = \frac{r d\theta}{1/r \cdot \frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta d\psi}{\frac{1}{r\sin\theta}\,\partial\phi/\partial\psi}$$

Or $$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r d\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta d\psi}{0}$$

From the last term, $\psi = const$

From the first two terms, we get

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta = 2\cot\theta\, d\theta$$

Integrating, we get

$$\log r = \log\sin^2\theta + const$$

$$\Rightarrow \qquad r = A\sin^2\theta, \quad \psi = const$$

The equation $\psi = const$ shows that the streamlines lie in planes which pass through the axis of symmetry $\theta = 0$.

Check Yourself:

Problem 1. Determine the streamlines and the path of the particles

$$u = \frac{x}{1+t},\ v = \frac{y}{1+t},\ w = \frac{z}{1+t}$$

Answer: ($x = Ay$, $x = Bz$) & ($x = a(1+t), y = b(1+t), z = c(1+t)$)

Problem 2. The velocity $\vec{q}$ in a three-dimensional flow field for an incompressible fluid is given by $\vec{q} = 2x\hat{i} - y\hat{j} - z\hat{k}$ .Determine the equations of the stream lines passing through the point $(1,1,1)$.

Answer: $xy^2 = 1$ & $xz^2 = 1$

Problem 3. Find the equation of the stream lines for the flow $\vec{q} = -\hat{i}(3y^2) - \hat{j}(6x)$ at the point $(1,1)$.

Answer: $3x^2 = y^3 + 2$

Problem 4. The velocity field at a point in a fluid is given as $\vec{q} = \left(\frac{x}{t}, y, 0\right)$. Obtain path lines and streak lines.

Solution. Here $\vec{q} = \left(\frac{x}{t}, y, 0\right)$.

The differential equations of path lines are given by

$$\vec{q} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \frac{x}{t}\hat{i} + y\hat{j}$$

$$\Rightarrow \qquad \frac{dx}{dt} = \frac{x}{t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0.$$

By integrating (1), we have

$$\frac{dx}{dt} = \frac{x}{t} \Rightarrow \int \frac{dx}{x} = \int \frac{dt}{t} \Rightarrow \log x = \log t + \log A \Rightarrow \; x = At. \qquad (4)$$

Let (x_0, y_0, z_0) be the co-ordinates of the chosen fluid particle at time $t = t_{0,}$

Then

$$x_0 = At_0 \Rightarrow \qquad A = \frac{x_0}{t_0}.$$

From (4), we have

$$\int \frac{dy}{y} = \int dt$$

Or $$\log y = t + \log B \Rightarrow \; y = Be^t$$

At $$y = y_0, t = t_0 \Rightarrow \quad B = y_0 e^{-t_0} \qquad (5)$$

From (5), we have

$$y = y_0 e^{t - t_0}$$

By integrating (3), we have

$$\frac{dz}{dt} = 0 \Rightarrow z = c \text{ i.e., } z \text{ is independent of } t \Rightarrow z = z_{0.}$$

Hence the path lines are given by

$$x = \left(\frac{x_0}{t_0}\right)t,\ \ y = y_0 e^{t-t_0},\ z = z_0 \qquad (6)$$

Let the fluid particle (x_0, y_0, z_0) pass through a fixed point (x_1, y_1, z_1) at an instant in time $t = T$,

Where $t_0 \le T \le t$. Then the relation (6) reduces to

$$x_1 = \left(\frac{x_0}{t_0}\right)T,\ \ y_1 = y_0 e^{T-t_0},\ \ z_1 = z_0$$

Or

$$x_0 = \left(\frac{x_1}{T}\right)t_0,\ \ y_0 = y_1 e^{t_0-T},\ \ z_0 = z_1 \qquad (7)$$

where T is the parameter. Substituting the relation (7) into (6), we have

$$x = \left(\frac{x_1}{T}\right)t,\ y = y_1 e^{t-T},\ z = z_1$$

Which gives the equation of streak lines passing through the point (x_1, y_1, z_1).

Problem 5. Consider the velocity field given by $\vec{q} = (1 + At)\hat{i} + x\hat{j}$. Find the equation of stream line at $t = t_0$ passing through the point (x_0, y_0). Also obtain the equation of the path line of a fluid element which comes to (x_0, y_0) at $t = t_0$. Show that, if $A = 0$ (i.e. steady flow), the stream lines and path lines coincide.

Answer: Equation of stream line; $x^2 - x_0{}^2 = 2(1 + At_0)(y - y_0)$

Equation of path line; $x - x_0 = (t - t_0) + \frac{A}{2}(t^2 - t_0{}^2)$

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0) + \frac{A}{6}\left\{t^2 + tt_0 - 2t_0{}^2\right\}\right]$$

5. Irrotational and Rotational Motion, Vortex Lines

5.1. Vorticity. If $\vec{q}=(u,v,w)$ be the velocity vector of a fluid particle, then the vector $\vec{\xi}$ is defined by

$$\vec{\xi}=curl\,\vec{q}=\nabla\times\vec{q}$$

Is called the vortex vector or vorticity and it's component are $(\xi_1,\xi_2,\xi_3,)$ given by

$$\xi_1=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\ \xi_2=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\ \ \xi_3=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

5.2. Vortex Motion (or Rotational Motion). The fluid motion is said to be rotational if $\vec{\xi}=curl\,\vec{q}\neq\vec{0}$

5.3. Irrotational Motion. If $\vec{\xi}=curl\,\vec{q}=\vec{0}$, then the fluid motion is said to be irrotational or of potential kind and then $\vec{q}=-\nabla\phi.$

5.4 Vortexline. It is a curve in the fluid such that the tangent at any point on the curve has the direction of the vorticity vector $\vec{\xi}$.

The differential equations of vortexlines are given by $\vec{\xi}\times d\vec{r}=\vec{0}$

i.e. $$\frac{dx}{\xi_1}=\frac{dy}{\xi_2}=\frac{dz}{\xi_3}$$

5.5. Vortex Tube. It is the locus of vortex line drawn at each point of a closed curve i.e. vortex tube is the surface formed by drawing vortex lines through each point of a closed curve in the fluid.

A vortex tube with small cross section is called a vortex filament.

5.6 Flow. Let A and B be two points in the fluid.

Then $\int_B^A \vec{q}.d\vec{r}$ is called the flow along any path from A to B

If motion is irrotational then $\vec{q}=-\nabla\phi$ and flow$=-\int_A^{\mathbf{B}}\nabla\phi.d\vec{r}=-\int_A^B d\phi=\phi(A)-\phi(B)$

5.7. Circulation. It is the flow round a closed curve. If C be the closed curve in a moving fluid then circulations $\Gamma=\oint_C \vec{q}.d\vec{r}=\oint_S \hat{n}.curl\,\vec{q}dS=\int_S \hat{n}.\vec{\xi}dS.$

If the motion is irrotational, then $\vec{q}=-\nabla\phi$ and thus,

$$\Gamma = -\oint_C \nabla\phi.d\vec{r} = \oint_S d\phi = \phi(A) - \phi(B) = 0,$$

Where A is any point on the curve C. This shows that for an irrotational motion, circulation is zero.

5.8. Theorem:- The necessary and sufficient condition such that the vortex lines are at right angles to the stream lines, is

$$(u,v,w) = u\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

i.e. $\vec{q} = \mu\nabla\phi$, where μ and ϕ are functions of x, y, z and t

Proof. Necessary condition:- We know that the differential equation

$\vec{q}.d\vec{r} = \vec{0}$ is integrable if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + ...+ = 0$$

$\vec{q}.curl\,\vec{q} = 0$ (exactness condition)

i.e. $\vec{q}.\vec{\xi} = \vec{0}, \ \vec{\xi} = curl\,\vec{q}$

This shows that the streamlines are at right angles to the vortex lines. Thus the streamlines and vortex lines are at right angles to each other if the differential equation $\vec{q}.d\vec{r} = \vec{0}$ is integrable.

The exactness condition $\vec{q}.curl\,\vec{q} = 0$ implies that $\vec{q} = -\nabla\phi.$

But $curl\,\vec{q} = curl(-\nabla\phi) = \vec{0}.$ Thus the vortex lines do not exist. The equations $\vec{q}.d\vec{r} = 0$ are therefore not exact.

So, there exists an integrating factor μ (function of x, y, z, t) such that

$$\mu^{-1}\vec{q}.d\vec{r} = 0 \text{ is integrable.}$$

If this differential equation is integrable, then we can write

$\mu^{-1}\vec{q}.d\vec{r} = d\phi$, where ϕ is scalar function of x, y, z, t

$\Rightarrow \mu^{-1}\vec{q}.d\vec{r} = \nabla\phi.d\vec{r}$

$\Rightarrow \vec{q} = \mu\nabla\phi$

Sufficient condition:- Let us take $\vec{q} = \mu\nabla\phi \Rightarrow \nabla\phi = \mu^{-1}\vec{q}$

Then, $curl\vec{q} = curl(\mu\nabla\phi)$

$$\Rightarrow \vec{\xi} = \nabla\times(\mu\nabla\phi) = \mu(\nabla\times\nabla\phi) + \nabla\mu\times\nabla\phi = \nabla\mu\times\nabla\phi$$

Therefore,

$$\vec{q}.\vec{\xi} = (\nabla\mu\times\nabla\phi).\vec{q} = \nabla\mu.(\nabla\phi\times\vec{q})$$

$$= \nabla\mu.(\mu^{-1}\vec{q}\times\vec{q}) = 0$$

This shows that the directions of streamlines and vortex lines are at right angles to each other.

6. Equation of Continuity

6.1. Equation of Continuity by Euler's Method

(Equation of conservation of mass): Equation of continuity is obtained by using the fact that the mass contained inside a given volume of fluid remains constant throughout the motion. Consider a region of fluid in which there is no inlets or outlets through which the fluid can enter or leave the region. Let S be the surface enclosing volume V of this region and let $\hat{n}$ denotes the unit vector normal to an element δS of S drawn outwards.

Let, $\vec{q}$ be the fluid velocity and ρ be the fluid density.

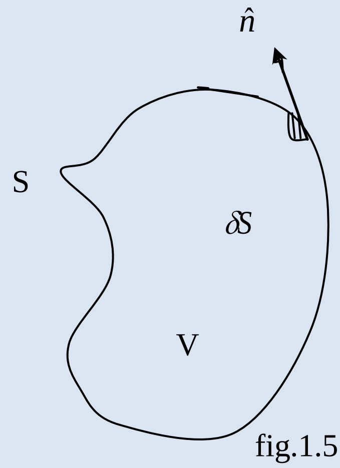


fig.1.5

First, we consider the mass of fluid which leaves V by flowing across an element δS of S in time δt. This quantity is exactly that which is contained in a small cylinder of cross-section δS of length $(\vec{q}.\hat{n})\delta t$.

Thus, mass of the fluid is $= density \times volume = \rho(\vec{q}.\hat{n})\delta t.\delta S$. Hence the rate at which fluid leaves V by flowing across the element δS is $\rho(\vec{q}.\hat{n})\delta S$.

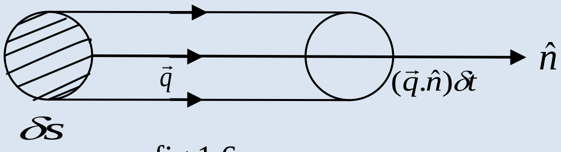


fig.1.6

Summing over all such elements δS, we obtain the rate of flow of fluid coming out of V across the entire surface S. Hence, the rate at which mass flow out of the region V is

$$\int_S \rho(\vec{q}.\hat{n})dS = \int_S (\rho\vec{q}).\hat{n}dS$$

$$= \int_V div(\rho\vec{q})dV \qquad (1)$$

Now, the mass M of the fluid possessed by the volume V of the fluid is

$$M = \int_V \rho dV$$

Where $\rho = \rho(x, y, z, t)$ with (x, y, z) the Cartesian co-ordinate of a general point of V, a fixed region of space, Since the space co-ordinate are independent of time t, therefore the rate of increase of mass within V is

$$\frac{dM}{dt} = \frac{d}{dt}\left(\int_V \rho dV\right) = \int_V \frac{\partial\rho}{\partial t}dV \qquad (2)$$

But the considered region is free from source or sink i.e. the mass is neither created nor be destroyed, therefore the total rate of change of mass is zero and thus from $(1)\,\&\,(2)$, we get

$$\int_V \frac{\partial\rho}{\partial t}dV + \int_V div(\rho\vec{q})dV = 0$$

$$\Rightarrow \qquad \int_V \left[\frac{\partial\rho}{\partial t} + div(\rho\vec{q})\right]dV = 0$$

Since V is arbitrary, we conclude that at any point of the fluid which is neither a source nor a sink,

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{q}) = 0 \qquad (3)$$

Equation (3) is known as equation of continuity.

Corollary (1). We know that

$$div(\rho \vec{q}) = \rho div \vec{q} + \vec{q}.(grad \rho)$$

Therefore, (3) takes the form

$$\frac{\partial \rho}{\partial t} + \rho(\nabla.\vec{q}) + (\vec{q}.\nabla)\rho = 0 \qquad (4)$$

Corollary (2). We know that the differential operator $\frac{D}{Dt}$ is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\nabla)$$

Therefore, from (4), we obtain the equation of continuity as $\frac{D\rho}{Dt} + \rho(\nabla.\rho) = 0$

Corollary (3). Equation (5) can be written as

$$\frac{1}{\rho}\frac{D\rho}{Dt} + div\vec{q} = 0$$

$$\Rightarrow \qquad \frac{D}{Dt}(\log \rho) + div\,\vec{q} = 0 \qquad (6)$$

Corollary (4). When the motion of fluid is steady, then $\frac{\partial \rho}{\partial t} = 0$ and thus the equation of continuity (3) becomes

$$div(\rho \vec{q}) = 0$$

Here ρ is not a function of time i.e. $\rho = \rho(x, y, z)$ (7)

Corollary (5). When the fluid is incompressible, then ρ =const and thus $\frac{D\rho}{Dt} = 0$.

The equation of continuity becomes

$$div\,\vec{q} = 0 \qquad (8)$$

Which is same for homogeneous and incompressible fluid.

Corollary (6). If in addition to homogeneity and incompressibility, the flow is of potential kind such that $\vec{q} = -\nabla\phi$, then the equation of continuity becomes single word

$$div(-\nabla\phi) = 0 \quad \Rightarrow \quad \nabla.(\nabla\phi) = 0 \quad \Rightarrow \quad \nabla^2\phi = 0$$

(9)

Which is known as the Laplace equation

6.2. Equation of continuity in Cartesian co-ordinates:- Let (x, y, z) be the rectangular Cartesian co-ordinates.

Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ (1)

And $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ (2)

Then, the equation of continuity $\frac{\partial\rho}{\partial t} + div(\rho\vec{q}) = 0$ can be written as

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (3)$$

i.e. $$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w\right) = 0 \quad (4)$$

which is the required equation of continuity in Cartesian co-ordinates.

Corollary (1) If the fluid motion is steady, then $\frac{\partial\rho}{\partial t} = 0$ and the equation (3) becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (5)$$

Corollary (2) If the fluid is incompressible, then $\rho = const$ and the equation of continuity is $\nabla.\vec{q} = 0$

i.e. $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6)$$

Corollary (3) If the fluid is incompressible and of potential kind, then equation of continuity is

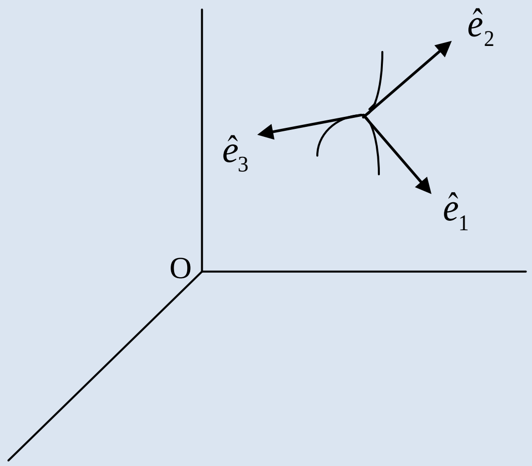
$$\nabla^2\phi = 0$$

i.e.

$$\frac{\partial\phi}{\partial x^2}+\frac{\partial\phi}{\partial y^2}+\frac{\partial\phi}{\partial z^2}=0, \quad \text{where} \qquad \vec{q}=-\nabla\phi$$

6.3. Equation of continuity in orthogonal curvilinear co-ordinate: Let (u_1,u_2,u_3) be the orthogonal curvilinear co-ordinates and $\hat{e}_1,\hat{e}_2,\hat{e}_3$ be the unit vector tangent to the co-ordinate curves.

Let $\qquad \vec{q}=q_1\hat{e}_1+q_2\hat{e}_2+q_3\hat{e}_3 \qquad (1)$



The general equation of continuity is

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho.\vec{q})=0 \qquad (2)$$

We know from vector calculus that for any vector point function $\vec{f}=(f_1, f_2, f_3)$,

$$\nabla.f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3f_1)+\frac{\partial}{\partial u_2}(h_3h_1f_2)+\frac{\partial}{\partial u_3}(h_1h_2f_3)\right] \qquad (3)$$

Where h_1, h_2, h_3 are scalars.

Using (3), the equation of continuity (2) becomes

$$\frac{\partial \rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3\rho q_1)+\frac{\partial}{\partial u_2}(h_3h_1\rho q_2)+\frac{\partial}{\partial u_3}(h_1h_2\rho q_3)\right] \tag{4}$$

Corollary (1). When motion of fluid is steady, then equation (4) becomes

$$\left[\frac{\partial}{\partial u_1}(h_2h_3\rho q_1)+\frac{\partial}{\partial u_2}(h_3h_1\rho q_2)+\frac{\partial}{\partial u_3}(h_1h_2\rho q_3)\right]=0 \tag{5}$$

Corollary (2). When motion of fluid is incompressible, the equation of continuity is $(\rho = const)$

$$\left[\frac{\partial}{\partial u_1}(h_2h_3q_1)+\frac{\partial}{\partial u_2}(h_3h_1q_2)+\frac{\partial}{\partial u_3}(h_1h_2q_3)\right]=0 \tag{6}$$

Corollary (3). When fluid is incompressible and irrotational then $(\rho = const)$

$\vec{q}=-\nabla\phi=\left(\frac{1}{h_1}\frac{\partial}{\partial u_1},\frac{1}{h_2}\frac{\partial}{\partial u_2},\frac{1}{h_3}\frac{\partial}{\partial u_3}\right)\phi$ and the equation of continuity becomes

$$\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)=0 \tag{7}$$

Now, we shall write equation (4) in cylindrical & spherical polar co-ordinate.

6.4. Equation of continuity in cylindrical co-ordinates (r,θ,z).

Here,

$u_1\equiv r, u_2\equiv\theta, u_3\equiv z$ and $h_1=1, h_2=r, h_3=1$

The equation of continuity becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(r\rho q_3)\right]=0 \tag{8}$$

Corollary (1). When the fluid motion is steady, then equation (8) becomes

$$\left[\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(r\rho q_3)\right]=0 \tag{9}$$

Corollary (2). For incompressible fluid, equation of continuity is

$$\left[\frac{\partial}{\partial r}(rq_1)+\frac{\partial}{\partial\theta}(q_2)+\frac{\partial}{\partial z}(rq_3)\right]=0 \tag{10}$$

Corollary (3). When the fluid is incompressible and is of potential kind, then equation (8) takes the from

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial \phi}{\partial z}\right)=0 \qquad (11)$$

Where $\vec{q}=-\nabla\phi$; ∇ is expressed in cylindrical co-ordinates.

6.5. Equation of continuity in spherical co-ordinates (r,θ,ψ).

Here,

$$(u_1,u_2,u_3)=(r,\theta,\psi). \text{ and } \quad h_1=1, h_2=r, h_3=r\sin\theta$$

The equation of continuity becomes

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\rho q_1)+\frac{\partial}{\partial \theta}(r\sin\theta\rho q_2)+\frac{\partial}{\partial \psi}(r\rho q_3)\right]=0$$

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial \theta}(\sin\theta\rho q_2)+r\frac{\partial}{\partial \psi}(\rho q_3)\right]=0 \qquad (12)$$

Corollary (1). For steady case, equation (12) becomes

$$\left[\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial \theta}(\sin\theta\rho q_2)+r\frac{\partial}{\partial \psi}(\rho q_3)\right]=0 \qquad (13)$$

Corollary (2). For incompressible fluid, we have

$$\left[\sin\theta\frac{\partial}{\partial r}(r^2 q_1)+r\frac{\partial}{\partial \theta}(\sin\theta q_2)+r\frac{\partial q_3}{\partial \psi}\right]=0 \qquad (14)$$

Corollary (3). When fluid is incompressible and of potential kind, then equation of continuity is

$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial \phi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{\partial}{\partial \psi 3}\left(\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \psi}\right)=0 \qquad (15)$$

Where $\vec{q}=-\nabla\phi$; ∇ is expressed in spherical co-ordinates.

6.6. Symmetrical forms of motion and equation of continuity for them. We have the following three types of symmetry which are special cases of cylindrical polar co-ordinates (r,θ,z), every physical quantity is independent of both θ and z so that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0 \text{ and } \vec{q} = \vec{q}(r,t)$$

For this case, the equation of continuity in cylindrical co-ordinates, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r) = 0 \qquad (1)$$

If the flow is steady, then equation (1) becomes

$$\frac{\partial}{\partial r}(\rho q_1 r) = 0 \qquad \Rightarrow (\rho q_1 r) = const = F(t), \text{(say)}.$$

Further, if the fluid is incompressible then $q_1 r = const = G(t),$ (say).

(ii) Spherical Symmetry:- In this case, the motion of fluid is symmetrical about the centre and thus with the choice of spherical polar co-ordinates (r,θ,ψ), every physical quantity is independent of both $\theta \,\&\, \psi$. so that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \psi} = 0 \text{ and } \vec{q} = \vec{q}(r,t)$$

The equation of continuity, for such symmetry, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \qquad (2)$$

For steady motion, it becomes

$$\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \quad \Rightarrow \rho q_1 r^2 = const = F(t), \text{ (say)}$$

And for incompressible fluid, it has the from $q_1 r^2 = const = G(t),$ say

(iii) Axial Symmetry:- (a) In cylindrical co-ordinates (r,θ,z), axial symmetry means that every physical quantity is independent of θ i.e. $\frac{\partial}{\partial \theta} = 0$ and thus the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{\partial}{\partial z}(\rho q_3 r)\right] = 0$$

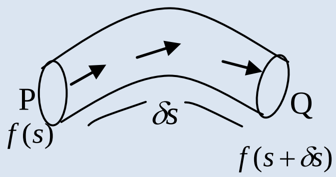
(b) In spherical co-ordinates (r,θ,ψ), axial symmetry means that every physical quantity is independent of ψ i.e. $\frac{\partial}{\partial \psi} = 0$ and the equation of continuity, for this case, reduce to

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)=0$$

6.7. Example. If $\sigma(s)$ is the cross-sectional area of a stream filament, prove that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho\sigma)+\frac{\partial}{\partial s}(\rho\sigma q)=0.$$ where δs is an element of arc of the filament and q is the fluid speed.

Solution. Let P and Q be the points on the end section of the stream filament.



The rate of flow of fluid out of volume of filament is

$$(\rho q\sigma)_Q+(\rho\sigma q)_P=\frac{\partial}{\partial s}(\rho q\sigma)_P\,\delta s \qquad (1)$$

Where we have retained the terms upto first order only, since δs is infinitesimally small

Now, the fluid speed is along the normal to the cross-section. At time t, the mass within the segment of filament is $\rho\sigma\delta s$ and its rate of increase is

$$\frac{\partial}{\partial t}(\rho\sigma\delta s)=\frac{\partial}{\partial t}(\rho\sigma)\delta s$$

Using law of conservation of mass, we have from $(1)\,\&\,(2)$

$$\frac{\partial}{\partial t}(\rho\sigma)\delta s+\frac{\partial}{\partial s}(\rho q\sigma)\delta s=0 \qquad (\text{total rate}=0)$$

i.e. $$\frac{\partial}{\partial t}(\rho\sigma)+\frac{\partial}{\partial s}(\rho q\sigma)=0 \qquad (3)$$

which is the required equation at any point P of the filament.

6.8. Deduction:- For steady incompressible flow, $\frac{\partial}{\partial t}(\rho\sigma)=0$ and equation (3) reduces to

$$\frac{\partial}{\partial s}(\rho q \sigma) = 0 \Rightarrow \frac{\partial}{\partial s}(\rho q) = 0 \Rightarrow \rho q = const.$$

Which shows that for steady incompressible flow product of velocity and cross-section of stream filament is constant. This result means that the volume of fluid a crossing every section per unit time is constant

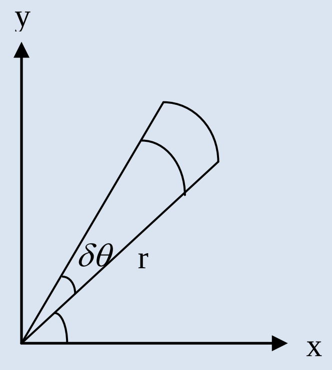
$$\left(\sigma q = c \Rightarrow \sigma \frac{dist.}{1} = c \Rightarrow \frac{vol.}{t} = c\right)$$

6.9. Example. A mass of a fluid moves in such a way that each particle describe a circle in one plane about a fixed axis, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho \omega) = 0$$

Where ω is the velocity of a particle whose azimuthal angle is θ at time t.

Solution. Here, the motion is in a plane i.e. we have a two dimensional case and the particle describe a circle



Therefore, $z = const., r = const.$

$$\Rightarrow \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial r} = 0 \qquad (1)$$

i.e. there is only rotation.

We know that the equation of continuity in cylindrical co-ordinates (r, θ, z) is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho q_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3) = 0 \qquad (2)$$

Using (1), we get

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)=0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho r\omega)=0, \text{ where } q=q_2=r\omega.$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(\rho\omega)=0$$

Hence the result.

6.10. Example. A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders, show that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Where v_θ, v_z are the velocities perpendicular and parallel to z.

Solution. We know that the equation of continuity in cylindrical co-ordinates (r,θ,z) is given by

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0, \quad \text{where } \vec{q}=(v_r,v_\theta,v_z)$$

Since the lines of motion (path lines) lie on the surface of cylinder, therefore the component of velocity in the direction of $d\vec{r}$ is zero i.e. $v_r=0$

Thus, the equation of continuity in the present case reduces to

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial v}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Hence the result.

6.11. Example. The particles of a fluid move symmetrically in space with regard to a fixed centre, prove that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2u)=0$$

Where u is the velocity at a distance r

Solution. First, derive the equation of continuity in spherical co-ordinates. Now, the present case is the case of spherical symmetry, since the motion is symmetrical w.r.t. a fixed centre.

Therefore, the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2)=0 \qquad (\because \frac{\partial}{\partial \theta}=\frac{\partial}{\partial \psi}=0)$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2)=0, \text{ where } q_1 \equiv u$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}ur^2+\frac{1}{r^2}\rho\frac{\partial}{\partial r}(ur^2)=0$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t}+u.\frac{\partial \rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(ru^2)=0$$

Hence the result

6.12. Example. If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho q_r)+\frac{2\rho}{r}q_r+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial \psi}(\rho q_\psi)=0$$

Solution. First derive the equation of continuity in spherical co-ordinates (r,θ,ψ) as

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(\rho q_1 r^2)+r\frac{\partial}{\partial \theta}(\rho q_2\sin\theta)+r\frac{\partial}{\partial \psi}(\rho q_3)\right]=0$$

In the present case, it is given that lines of motion lie on the surfaces of cones, therefore velocity perpendicular to the surface is zero i.e. $q_2=0$

Therefore, the equation of continuity becomes.

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_r r^2)+\frac{1}{r\sin\theta}q_r.\frac{\partial}{\partial \psi}(\rho q_\psi)=0$$ where $(q_1,q_2,q_3 \equiv (q_r,q_\theta,q_\psi)$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}\left[r^2\frac{\partial}{\partial r}(\rho q_r)+\rho q_r(2r)\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial \psi}(\rho q_\psi)=0$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho q_r)+\frac{2\rho}{r}q_r+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial \psi}(\rho q_\psi)=0$$

Hence the result.

6.13. Example. Show that polar form of equation of continuity for a two dimensional incompressible fluid is

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial \theta}=0$$

If $u=\frac{-\mu\cos\theta}{r^2}$, then find v and the magnitude of the velocity $\vec{q}$, where $\vec{q}=(u,v)$

Solution. First derive the equation of continuity in polar co-ordinates (r,θ) in two dimensional as

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_1)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2)=0 \qquad (z=0)$$

In the present case $\rho=const.$

Therefore, the equation of continuity reduces to

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru)+\frac{\rho}{r}\frac{\partial}{\partial \theta}(v)=0, \text{ where } \vec{q}=(q_1,q_2,q_3)\equiv(u,v,w)$$

i.e. $$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial \theta}=0$$

hence the result.

Now $u=\frac{-\mu\cos\theta}{r^2} \Rightarrow \frac{\partial}{\partial r}\left(\frac{-\mu\cos\theta}{r^2}r\right)+\frac{\partial v}{\partial \theta}=0$

$$\Rightarrow \frac{\mu\cos\theta}{r^2}+\frac{\partial v}{\partial \theta}=0 \Rightarrow \frac{\partial v}{\partial \theta}=\frac{-\mu\cos\theta}{r^2}$$

Integrating w.r.t. θ, we get

$$v=\frac{-\mu\sin\theta}{r^2} \quad \text{and thus} \quad |\vec{q}|=q=(u^2+v^2)^{1/2}=\frac{\mu}{r^2}$$

6.14. Equation of Continuity by Lagrange's Method. Let initially a fluid element be at (a,b,c) at time $t=t_0$ when its volume is dV_0 and density is ρ_0. After time t, let the same fluid element be at (x,y,z) when its volume is dV and density is ρ. Since mass of the fluid element remains invariant during its motion, we have

$$\rho_0 dV_0=\rho dV \quad \text{i.e.} \quad \rho_0 da\,db\,dc=\rho dx\,dy\,dz$$

or $$\rho_0 da\,db\,dc=\rho\frac{\partial(x,y,z)}{\partial(a,b,c)}da\,db\,dc$$

or $$\rho J = \rho_0$$

where $J = \dfrac{\partial(x, y, z)}{\partial(a, b, c)}$

which is the required equation of continuity.

6.15. Remark. By simple property of Jacobians, we get

$$\frac{dJ}{dt} = J\nabla.\vec{q}$$

Thus (1) gives $\dfrac{d}{dt}(\rho J) = 0 \Rightarrow \dfrac{d\rho}{dt}J + \rho\dfrac{dJ}{dt} = 0$

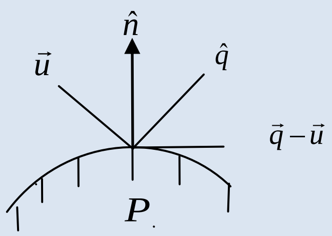
$$\Rightarrow \frac{d\rho}{dt}J + \rho J\nabla.\vec{q} = 0 \quad \Rightarrow \frac{d\rho}{dt}J + \rho\nabla.\vec{q} = 0 \quad \text{or} \quad \Rightarrow \frac{D\rho}{Dt} + \rho\nabla.\vec{q} = 0$$

Which is the Euler's equation of continuity.

7. Boundary Surfaces

Physical condition that should be satisfied on given boundaries of the fluid in motion, are called boundary conditions. The simplest boundary condition occurs where an ideal and incompressible fluid is in contact with rigid impermeable boundary, e.g., wall of a container or the surface of a body which is moving through the fluid.

Let P be any point on the boundary surface where the velocity of fluid is $\vec{q}$ and velocity of the boundary surface is $\vec{u}$.



The velocity at the point of contact of the boundary surface and the liquid must be tangential to the surface otherwise the fluid will break its contact with the boundary surface. Thus, if $\hat{n}$ be the unit normal to the surface at the point of contact, then

$$(\vec{q} - \vec{u}).\hat{n} = 0 \Rightarrow \vec{q}.\hat{n} = \vec{u}.\hat{n} \qquad (1)$$

In particular, if the boundary surface is at rest, then $\vec{u} = 0$ and the condition becomes

$$\vec{q}.\hat{n} = 0 \qquad (2)$$

Another type of boundary condition arrives at a free surface where liquid borders a vacuum e.g. the interface between liquid and air is usually regarded as free surface. For this free surface, pressure p satisfies

$$P = \Pi \qquad (3)$$

Where Π denotes the pressure outside the fluid i.e. the atmospheric pressure. Equation (3) is a dynamic boundary condition.

Third type of boundary condition occurs at the boundary between two immissible ideal fluids in which the velocities are $\vec{q}_1 \,\&\, \vec{q}_2$ and pressures are $p_1 \,\&\, p_2$ respectively.

Now, we find the condition that a given surface satisfies to be a boundary surface.

7.1. Article. To obtain the differential equation satisfied by boundary surface of a fluid in motion.

Or

To find the condition that the surface.

$$F(\vec{r}, t) = F(x, y, z, t) = 0$$

May represent a boundary surface :-

If $\vec{q}$ be the velocity of fluid and $\vec{u}$ be the velocity of the boundary surface at a point P of contact, then

$$(\vec{q} - \vec{u}).\hat{n} = 0 \Rightarrow \vec{q}.\hat{n} = \vec{u}.\hat{n} \qquad (1)$$

Where $\vec{q} - \vec{u}$ is the relative and $\hat{n}$ is a unity vector normal to the surface at P .

The equation of the given surface is

$$F(\vec{r}, t) = F(x, y, z, t) = 0 \qquad (2)$$

We know that a unit vector normal to the surface (2) is given by

$$\hat{n} = \frac{\nabla F}{|\nabla F|}$$

Thus, from (1), we get $\quad \vec{q}.\nabla F = \vec{u}.\nabla F \qquad (3)$

Since the boundary surface is itself in motion, therefore at time $(t + \delta t)$, its equation is given by

$$F(\vec{r} + \delta\vec{r}, t + \delta t) = 0 \qquad (4)$$

From $(2)\,\&\,(4)$, we have

$$F(\vec{r}+\delta\vec{r},t+\delta t)-F(\vec{r},t)=0$$

i.e. $F(\vec{r}+\delta\vec{r},t+\delta t)-F(\vec{r},t+\delta t)+F(\vec{r},t+\delta t)-F(\vec{r},t)=0$

By Taylor's series, we can have

$$(\delta\vec{r}.\nabla)F(\vec{r},t+\delta t)+\delta t\frac{\partial}{\partial t}\{F(\vec{r},t)\}+0$$

$$\because F(x+\delta x,y+\delta y,z+\delta z)=F(x,y,z)+\delta x\frac{\partial F}{\partial x}+\delta y\frac{\partial F}{\partial y}+\delta z\frac{\partial F}{\partial z}+.......=F(x,y,z)+\delta\vec{r}.\nabla F$$

$$\Rightarrow \qquad \left(\frac{\delta\vec{r}}{\delta t}.\nabla\right)F(\vec{r},t+\delta t)+\frac{\partial F}{\partial t}+0$$

Taking limit as $\delta t\to 0$, we get

$$\left(\frac{d\vec{r}}{dt}.\nabla\right)F+\frac{\partial F}{\partial t}=0$$

$$\Rightarrow \qquad \frac{\partial F}{\partial t}+(\vec{q}.\nabla)F=0 \quad \text{i.e. } \frac{DF}{Dt}=0 \qquad (5)$$

Which is the required condition for any surface F to be a boundary surface

Corollary (1) If $\vec{q}=(u,v,w)$, then the condition (5) becomes

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

In case, the surface is rigid and does not move with time, then $\frac{\partial F}{\partial t}=0$ and the boundary condition is $u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$ i.e. $(\vec{q}.\nabla)F=0$

Corollary (2) The boundary condition

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

is a linear equation and its solution gives

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \qquad \left| \frac{D}{Dt}\equiv\frac{d}{dt} \text{ in Lagrangian view}\right.$$

$$\Rightarrow \qquad \frac{dx}{dt}=u, \frac{dy}{dt}=v, \frac{dz}{dt}=w$$

Which are the equations of path lines.

Hence once a particle is in contact with the surface, it never leaves the surface.

Corollary (3) From equation (5), we have

$$\vec{q}.\nabla F=\frac{-\partial F}{\partial t}$$

$$\Rightarrow \qquad \vec{q}.\frac{\nabla F}{|\nabla F|}=\frac{-\partial F/\partial t}{|\nabla F|}$$

$$\Rightarrow \qquad \vec{q}.\hat{n}=\frac{-\partial F/\partial t}{|\nabla F|}$$

Which gives the normal velocity.

also from (1), we get

$$\vec{u}.\hat{n}=\frac{-\partial F/\partial t}{|\nabla F|} \qquad \left| \because \vec{q}.\hat{n}=\vec{u}.\hat{n}\right.$$

Which gives the normal velocity of the boundary surface.

7.2.Example. Show that the ellipsoid

$$\frac{x^2}{a^2k^2t^{2n}}+kt^n\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=1$$

Is a possible form of the boundary surface of a liquid.

Solution. The surface $F(x,y,z,t)=0$ can be a possible boundary surface, if it satisfies the boundary condition.

$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \qquad (1)$$

Where u,v,w satisfy the equation of continuity

$$\nabla.\vec{q}=0 \text{ i.e. } \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (2)$$

Here, $$F(x,y,z,t) \equiv \frac{x^2}{a^2k^2t^{2n}} + kt^n\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] - 1 = 0$$

Therefore, $$\frac{\partial F}{\partial t} = -\frac{x^2.2n}{a^2k^2t^{2n+1}} + nkt^{n-1}\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right]$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2k^2t^{2n}}, \quad \frac{\partial F}{\partial y} = \frac{2kt^n y}{b^2}, \quad \frac{\partial F}{\partial z} = -\frac{2kt^n z}{c^2}$$

Thus, from (1), we get

$$-\frac{x^2.2n}{a^2k^2t^{2n+1}} + nkt^{n-1}\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] + \frac{2xu}{a^2k^2t^{2n}} + \frac{2kt^n yv}{b^2} + \frac{2kt^n zw}{c^2} = 0$$

Or $$\left(u - \frac{nx}{t}\right)\frac{2x}{a^2k^2t^{2n}} + \left(v + \frac{ny}{2t}\right)\frac{2kt^n y}{b^2} + \left(w + \frac{nz}{2t}\right)\frac{2kt^n z}{c^2} = 0$$

Which will hold. if we take

$$\left(u - \frac{nx}{t}\right) = 0, \left(v + \frac{ny}{2t}\right) = 0, \left(w + \frac{nz}{2t}\right) = 0$$

i.e. $$u = \frac{nx}{t}, \quad v = -\frac{ny}{2t}, \quad w = -\frac{nz}{2t} \qquad (3)$$

It will be a justifiable step if equation (2) is satisfied.

i.e. $$\frac{n}{t} + \frac{-n}{2t} + \frac{-n}{2t} = 0$$

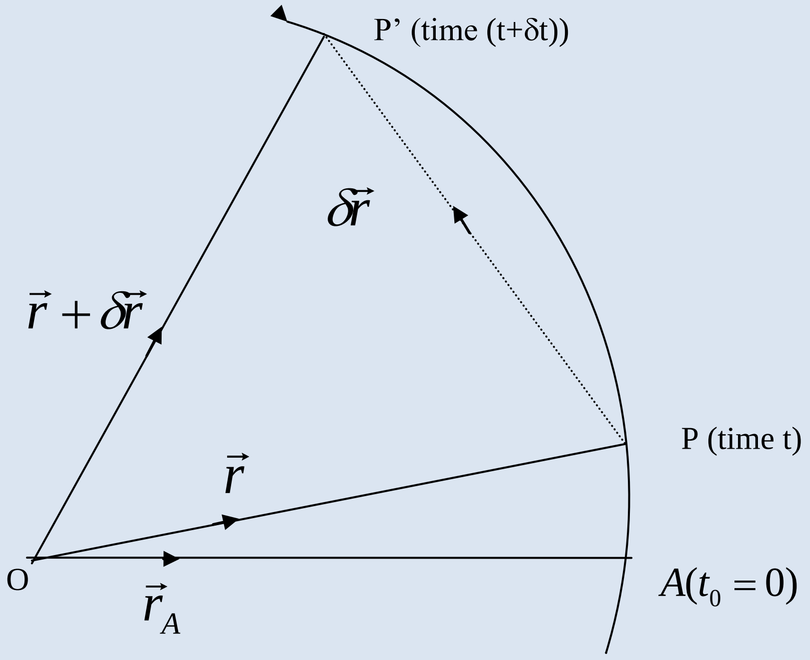
which is true.

Hence the given ellipsoid is a possible form of boundary surface of a liquid.

8.Acceleration at a Point of a Fluid

Suppose that a fluid particle is moving along a curve C, initially it being at point $A(t_0 = 0)$ with position vector $\vec{r}_A$. Let P and P' be its position at time t and $t + \delta t$ with position vector $\vec{r}$ and $\vec{r} + \delta\vec{r}$ respectively.

Therefore, $\delta\vec{r} = \overrightarrow{PP'}$



The points A, P, P' are geometrical points of region occupied by fluid and they coincide with the locations of the same fluid particle at time $t_{0,}\ t, t+\delta t$ respectively. Let $\vec{f}$ be the acceleration of the particle at time t when it coincide with P. By definition

$$\vec{f} = \lim_{\delta t \to 0} \frac{(change\, in\; particle\, velocity\; in\, time\, \delta t)}{\delta t} \qquad (1)$$

But the particle velocity at time t is $\vec{q}(\vec{r}, t)$ and at time $t+\delta t$ it is $\vec{q}(\vec{r}+\delta\vec{r}, t+\delta t)$.

Thus (1) becomes

$$\vec{f} = \lim_{\delta t \to 0} \frac{[\vec{q}(\vec{r}+\delta\vec{r}, t+\delta t) - \vec{q}(\vec{r}, t)]}{\delta t} \qquad (2)$$

Now,

$$\frac{[\vec{q}(\vec{r}+\delta\vec{r}, t+\delta t) - \vec{q}(\vec{r}, t)]}{\delta t} = \frac{\vec{q}(\vec{r}+\delta\vec{r}, t+\delta t) - \vec{q}(\vec{r}, t+\delta t)}{\delta t} + \frac{\vec{q}(\vec{r}, t+\delta t) - \vec{q}(\vec{r}, t)}{\delta t}$$

Since $\vec{r}$ is independent of time t, therefore

$$\frac{\partial \vec{q}}{\partial t} = \lim_{\delta t \to 0} \frac{\vec{q}(\vec{r}, t+\delta t) - \vec{q}(\vec{r}, t)]}{\delta t} = \frac{\partial \vec{q}}{\partial t} \qquad (4)$$

Using Taylor's expansion, we get

$$\vec{q}(\vec{r}+\delta\vec{r},t+\delta t)-\vec{q}(\vec{r},t+\delta t)=(\delta\vec{r}.\nabla)\vec{q}(\vec{r},t+\delta t)+\in \qquad (5)$$

Where $|\in|=0[(\delta r)^2]$

$$[\because F(x+\delta x,y+\delta y,z+\delta z)-F(x,y,z)=\left(\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}+\delta z\frac{\partial}{\partial z}\right)F(x,y,z)$$

$$+\frac{1}{2!}\left(\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}+\delta z\frac{\partial}{\partial z}\right)^2 F(x,y,z)+.........$$

And

$$\delta x\frac{\partial}{\partial x}+\delta y\frac{\partial}{\partial y}+\delta z\frac{\partial}{\partial z}=(\delta\vec{r}.\nabla F),\text{ where}$$

$$\delta\vec{r}=\delta x\hat{i}+\delta y\hat{j}+\delta z\hat{k},\quad \nabla\equiv\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}]$$

But $\delta\vec{r}$ is merely the displacement of the fluid particle in time δt, therefore,

$$\delta\vec{r}=\vec{q}(\vec{r},t)\delta t \qquad (6)$$

Thus, from (5), we obtain

$$\lim_{\delta t\to 0}\frac{\vec{q}(\vec{r}+\delta\vec{r},t+\delta t)-\vec{q}(\vec{r},t+\delta t)]}{\delta t}=(\vec{q}.\nabla)\vec{q} \qquad (7)$$

Where R.H.S. of (4) & (7) are evaluated at $P(\vec{r},t)$. Hence, from (2), the acceleration of fluid at P in vector form is given by

$$\vec{f}=(\vec{q}.\nabla)\vec{q} \qquad (8)$$

8.1. Remark. We have obtained the acceleration i.e. rate of change of velocity $\vec{q}$. The same procedure can be applied to find the rate of change of any physical property associated with the fluid, such as density. Thus, if $F=F(\vec{r},t)$ is any scalar or vector quantity associated with the fluid, it's rate of change at time t is given by

$$\frac{Df}{Dt}=\frac{\partial F}{\partial t}+(\vec{q}.\nabla)F$$

The operator $\frac{D}{Dt}=\frac{\partial}{\partial t}+(\vec{q}.\nabla)$ is Lagrangian and operator on R.H.S. are Eulerian since $\vec{r}$ is independent of t. $\frac{D}{Dt}$ is also called material derivative.

In particular, if $F=\rho$, the density of the fluid, then

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+(\vec{q}.\nabla)\rho$$

Which is the general equation of motion for unsteady flow.

8.2. Components of Acceleration in Cartesian co-ordinates. Let u,v,w be the Cartesian components of $\vec{q}$ and f_1, f_2, f_3 that of $\vec{f}$ i.e. $\vec{q}=(u,v,w)$, $\vec{f}=(f_{1,}\, f_2, f_3)$

Then from equation

$$\vec{f}=\frac{\partial\rho}{\partial t}+(\vec{q}.\nabla)\rho \qquad (1)$$

We get

$$f_1=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

$$f_2=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}$$

$$f_3=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}$$

Which are the required Cartesian components of $\vec{f}$.

In tensor form with co-ordinates x_i and velocity components $q_i(i=1,2,3)$, the above set of equations can be written as

$$f_i=\frac{\partial q_i}{\partial t}+q_j q_{i,j}\,, \quad \text{where } q_{i,j}=\frac{\partial q_i}{\partial x_j}$$

8.3. Components of Acceleration Curvilinear co-ordinates. Before obtaining the acceleration components in curvilinear co-ordinates; we obtain a more suitable form of equation (1). as

$$\vec{f}=\frac{\partial\vec{q}}{\partial t}+\nabla\left(\frac{1}{2}\vec{q}^{\,2}\right)-\vec{q}\times(\nabla\times\vec{q})$$

$$=\frac{\partial\vec{q}}{\partial t}+\nabla\left(\frac{1}{2}\vec{q}^{\,2}\right)-\vec{\xi}\times\vec{q}, \quad \text{where } \xi=curl\,\vec{q}=\nabla\times\vec{q}$$

We have

$$(\vec{q}.\nabla)\vec{q} = (\vec{q}.\hat{i})\frac{\partial \vec{q}}{\partial x} + (\vec{q}.\hat{j})\frac{\partial \vec{q}}{\partial y} + (\vec{q}.\hat{k})\frac{\partial \vec{q}}{\partial z} \qquad (2)$$

For any three vectors $\vec{A}, \vec{B}, \vec{C}$, we have

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

i.e.
$$(\vec{A}.\vec{B})\vec{C} = (\vec{A}.\vec{C})\vec{B} - \vec{A}\times(\vec{B}\times\vec{C})$$

In particular, taking $\vec{A} = \vec{q}, \vec{B} = \hat{i}, \vec{C} = \frac{\partial \vec{q}}{\partial x}$, we get

$$(\vec{q}.\hat{i})\frac{\partial \vec{q}}{\partial x} = \left(\vec{q}.\frac{\partial \vec{q}}{\partial x}\right)\hat{i} - \vec{q}\times\left(\hat{i}\times\frac{\partial \vec{q}}{\partial x}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\left(\hat{i}\times\frac{\partial \vec{q}}{\partial x}\right) \qquad (3)$$

Similary,

$$(\vec{q}.\hat{j})\frac{\partial \vec{q}}{\partial y} = \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\left(\hat{j}\times\frac{\partial \vec{q}}{\partial y}\right) \qquad (4)$$

$$(\vec{q}.\hat{k})\frac{\partial \vec{q}}{\partial z} = \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\left(\hat{j}\times\frac{\partial \vec{q}}{\partial z}\right) \qquad (5)$$

Adding (3), (4), (5), we get

$$(\vec{q}.\nabla)\vec{q} = \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\sum\left(\hat{j}\times\frac{\partial \vec{q}}{\partial x}\right)$$

$$= \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times(\nabla\times\vec{q})$$

Thus, from (1), we obtain

$$\vec{f} = \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times(\nabla\times\vec{q})$$

$$= \frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^2\right) + \vec{\xi}\times\vec{q} \qquad (6)$$

Now, let (u_1, u_2, u_3) denote the orthogonal curvilinear co-ordinates.

Also let $\vec{q}=(q_1,q_2,q_3),\ \vec{f}=(f_1,f_2,f_3),\ \vec{\xi}=(\xi_1,\xi_2,\xi_3),$ where the terms have their usual meaning. We know that the expression for the operator ∇ in curvilinear co-ordinates is

$$\nabla \equiv \left(\frac{1}{h_1}\frac{\partial}{\partial u_1},\frac{1}{h_2}\frac{\partial}{\partial u_2},\frac{1}{h_3}\frac{\partial}{\partial u_3}\right),$$

where $h_{1,}h_2,h_3$ are scalar factors.

The components of

$$\left.\begin{aligned}
\xi_1 &= \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3q_3)-\frac{\partial}{\partial u_3}(h_2q_2)\right]\\
\xi_2 &= \frac{1}{h_3h_1}\left[\frac{\partial}{\partial u_2}(h_1q_1)-\frac{\partial}{\partial u_1}h_3q_3)\right]\\
\xi_3 &= \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2q_2)-\frac{\partial}{\partial u_2}(h_1q_1)\right]
\end{aligned}\right\} \qquad (7)$$

Using these results in (6), we find that

$$\left.\begin{aligned}
f_1 &= \frac{\partial q_1}{\partial t}+\frac{1}{2h_1}\frac{\partial}{\partial u_1}\left(q_1{}^2+q_2{}^2+q_3{}^2\right)+(\xi_2q_3-\xi_3q_2)\\
f_2 &= \frac{\partial q_2}{\partial t}+\frac{1}{2h_2}\frac{\partial}{\partial u_2}\left(q_1{}^2+q_2{}^2+q_3{}^2\right)+(\xi_3q_1-\xi_1q_3)\\
f_1 &= \frac{\partial q_1}{\partial t}+\frac{1}{2h_1}\frac{\partial}{\partial u_1}\left(q_1{}^2+q_2{}^2+q_3{}^2\right)+(\xi_2q_3-\xi_3q_2)
\end{aligned}\right\} \qquad (8)$$

Which are the components of acceleration in curvilinear co-ordinates.

Now, we write the components of acceleration in cylindrical (r,θ,z) and spherical (r,θ,ψ) co-ordinates.

8.4. Components of Acceleration in Cylindrical Co-ordinates (r,θ,z).

Therefore, $$\nabla=\left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta},\frac{\partial}{\partial z}\right)$$

And

$$\xi_1=\frac{1}{2}\left[\frac{\partial q_3}{\partial\theta}-\frac{\partial}{\partial z}(rq_2)\right]=\frac{1}{r}\frac{\partial q_3}{\partial\theta}-\frac{\partial q_2}{\partial z}$$

$$\xi_2=\frac{\partial q_1}{\partial z}-\frac{\partial q_3}{\partial r}$$

$$\xi_3 = \frac{1}{r}\left[\frac{\partial}{\partial r}(rq_2) - \frac{\partial q_1}{\partial \theta}\right] = \frac{\partial q_2}{\partial r} + \frac{q_2}{r} - \frac{1}{r}\frac{\partial q_1}{\partial \theta}$$

Thus,

$$f_1 = \frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}\left(q_1^{\,2} + q_2^{\,2} + q_3^{\,2}\right) + \left(q_3\frac{\partial q_1}{\partial z} - q_3\frac{\partial q_3}{\partial r}\right) - \left(q_2\frac{\partial q_2}{\partial r} + \frac{q_2^2}{r} - \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}\right)$$

$$= \frac{\partial q_1}{\partial t} + q_1\frac{\partial q_1}{\partial r} + q_2\frac{\partial q_2}{\partial r} + q_3\frac{\partial q_3}{\partial z} - q_3\frac{\partial q_3}{\partial r} - q_2\frac{\partial q_2}{\partial r} - \frac{q_2^2}{r} + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}$$

$$= \frac{\partial q_1}{\partial t} + q_1\frac{\partial q_1}{\partial r} + q_3\frac{\partial q_1}{\partial z} - \frac{q_2^2}{r} + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}$$

If we define the differential operator

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + q_1\frac{\partial}{\partial r} + q_3\frac{\partial}{\partial z} + \frac{q_2}{r}\frac{\partial}{\partial \theta}, \text{ then}$$

Similarly,

$$\left.\begin{aligned} f_1 &= \frac{Dq_1}{Dt} - \frac{q_2^2}{r} \equiv \frac{Du}{Dt} - \frac{v^2}{r} \\ f_2 &= \frac{Dq_2}{Dt} + \frac{q_1 q_2}{r} \equiv \frac{Dv}{Dt} + \frac{uv}{r} \\ f_3 &= \frac{Dq_3}{Dt} \equiv \frac{Dw}{Dt} \end{aligned}\right\} \quad (9)$$

where $(q_1, q_2, q_3) \equiv (u, v, w)$

Equation (9) gives the required components of acceleration in cylindrical co-ordinates.

8.5. Components of Acceleration in Spherical Co-ordinates (r, θ, ψ).

$u_1 = r, u_2 = \theta, u_3 = \psi$ and $h_1 = 1, h_2 = r, h_3 = r\sin\theta$

Therefore, $$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \psi}\right)$$

And $$\xi_1 = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial \theta}(r\sin\theta q_3) - \frac{\partial}{\partial \psi}(rq_2)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[r\left(\cos\theta + \sin\theta \frac{\partial q_3}{\partial \theta} \right) - r\frac{\partial q_2}{\partial \psi} \right]$$

$$= \frac{1}{r \sin\theta}\left[q_3 \cos\theta + \sin\theta \frac{\partial q_3}{\partial \theta} - \frac{\partial q_2}{\partial \psi} \right]$$

$$\xi_2 = \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \psi}(q_1) - \frac{\partial}{\partial r}(r \sin\theta q_3) \right]$$

$$= \frac{1}{r \sin\theta}\left[\frac{\partial q_1}{\partial \psi} - \sin\theta q_3 - r\sin\theta \frac{\partial q_3}{\partial r} \right]$$

$$\xi_3 = \frac{1}{r}\left[\frac{\partial}{\partial r}(r q_2) - \frac{\partial}{\partial \theta}(q_1) \right] = \frac{1}{r}\left[q_2 + r\frac{\partial q_2}{\partial r} - \frac{\partial q_1}{\partial \theta} \right]$$

Thus,

$$f_1 = \frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}\left(q_1^{\,2} + q_2^{\,2} + q_3^{\,2} \right) + \frac{q_3}{r\sin\theta}\left[\frac{\partial q_1}{\partial \psi} - q_3 \sin\theta - r\sin\theta \frac{\partial q_3}{\partial r} \right] - \frac{q_2}{r}\left[q_2 + r\frac{\partial q_2}{\partial r} - \frac{\partial q_1}{\partial \theta} \right]$$

$$= \frac{\partial q_1}{\partial t} + q_1 \frac{\partial q_1}{\partial r} + q_2 \frac{\partial q_2}{\partial r} + q_3 \frac{\partial q_3}{\partial z} + \frac{q_3}{r\sin\theta}\frac{\partial q_1}{\psi} - \frac{q_3^{\,2}}{r} - q_3\frac{\partial q_3}{r} - \frac{q_2^2}{r} - q_2\frac{\partial q_3}{\partial r} - + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}$$

$$= \frac{Dq_1}{Dt} - \frac{q_2^2}{r} - \frac{q_3^2}{r}, \quad \text{where} \quad \frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + q_1\frac{\partial}{\partial r} + \frac{q_2}{r}\frac{\partial}{\partial \theta} + \frac{q_3}{r\sin\theta}\frac{\partial}{\partial \psi}$$

i.e.

$$f_1 = \frac{Dq_1}{Dt} - \frac{q_2^2 + q_3^2}{r} \equiv \frac{Du}{Dt} - \frac{v^2 + w^2}{r}$$

Similarly,

$$f_1 = \frac{Dq_2}{Dt} + \frac{q_1 q_{2-} q_3^2 \cot\theta}{r} \equiv \frac{Dv}{Dt} + \frac{uv - w^2\cot\theta}{r}$$

$$f_3 \equiv \frac{Dq_3}{Dt} + \frac{q_1 q_3 + q_2 q_3 \cot\theta + w^2}{r} \equiv \frac{Dw}{Dt} + \frac{w(u + v\cot\theta)}{r} \qquad (10)$$

Equation (10) gives the required comps of acceleration in spherical co-ordinates.

End of the Unit -1