M.Sc. (MATHEMATICS)

MECHANICS OF SOLIDS-I

MAL-633

DIRECTORATE OF DISTANCE EDUCATION

GURU JAMBHESHWAR UNIVERSITY OF SCIENCE & TECHNOLOGY

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CHAPTER-I

CARTESIAN TENSOR

1.1 Introduction

The concept of a tensor has its origin in the developments of differential geometry by Gauss, Riemann and Christoffel. The emergence of Tensor calculus, as a systematic branch of Mathematics is due to Ricci and his pupil Levi-Civita. In collaboration they published the first memoir on this subject: - '***Methods de calcul differential absolu et leurs applications***' **Mathematische Annalen, Vol. 54,** (**1901**). The investigation of relations which remain valid when we change from one coordinate system to any other is the chief aim of Tensor calculus. The laws of Physics cannot depend on the frame of reference which the physicist chooses for the purpose of description. Accordingly it is aesthetically desirable and often convenient to utilize the Tensor calculus as the mathematical background in which such laws can be formulated. In particular, Einstein found it an excellent tool for the presentation of his General Relativity theory. As a result, the Tensor calculus came into great prominence and is now invaluable in its applications to most branches of theoretical Physics; it is also indispensable in the differential geometry of hyperspace.

A physical state or a physical phenomenon of the quantity which is invariant, i.e remain unchanged, when the frame of reference within which the quantity is defined is changed that quantity is called **tensor.** In this chapter, we have to confine ourselves to Cartesian frames of reference.

As a Mathematical entity, a tensor has an existence independent of any coordinate system. Yet it may be specified in a particular coordinate system by a certain set of quantities, known as its components. Specifying the components of a tensor in one coordinate system determines the components in any other system according to some definite law of transformation.

Under a transformation of cartesian coordinate axes, a scalar quantity, such as the **density** or the **temperature**, remain unchanged. This means that a scalar is an invariant under a coordinate transformation. Scalars are called **tensors of zero rank.** All physical quantities having magnitude only are tensors of zero order. It is assumed that the reader has an elementary knowledge of determinants and matrices. **Rank/Order of tensor**

1) If the value of the quantity at a point in space can be described by a single number, the quantity is a scalar or a tensor of rank/order zero. For example, '5' is a scalar or tensor of rank/order zero.
2) If three numbers are needed to describe the quantity at a point in the space, the quantity is a tensor of rank one. For example vector is a tensor of rank/order one.
3) If nine numbers are needed to describe the quantity, the quantity is a tensor of rank three. The 3×3, 1×9 and 9×1, nine numbers describe the quantity is an example of tensor of rank/order 3.
4) In general, if 3^n numbers are needed to describe the value of the quantity at a point in space, the quantity is a tensor of rank/order n. A quantity described by 12 or 10 or 8 ………… numbers, then the quantity is not a tensor of any order/rank.

OR

Tensor: A set of members/numbers 3^n represents the physical quantity in the reference coordinates, then the physical quantity is called a tensor of order n.

1.1.1 Characteristics of the tensors

1) Tensors are the quantities describing the same phenomenon regardless of the coordinate system used; they provide an important guide in the formulation of the correct form of physical law. Equations describing physical laws must be tensorially homogenous, which means that every term of the equation must be a tensor of the same rank.
2) The tensor concept provides convenient means of transformation of an equation from one system of coordinates to another.
3) An advantage of the use of Cartesian tensors is that once the properties of a tensor of a certain rank have been established, they hold for all such tensors regardless of the physical phenomena they represent.

   Note: For example, in the study of strain, stress, inertia properties of rigid bodies, the common bond is that they are all symmetric tensors of rank two.

1.2 Notation and Summation Convention

Let us begin with the matter of notation. In tensor analysis one makes extensive use of indices. A set of n variables $x_1, x_2, ..., x_n$ is usually denoted as x_i , $i = 1, 2, 3, ... n$. Consider an equation describing a plane in a three-dimensional space

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = p \qquad (1.2.1)$$

where a_i and p are constants. This equation can be written as

$$\sum_{i=1}^{3} a_i x_i = p \qquad (1.2.2)$$

However, we shall introduce the summation convention and write the equation above in the simple form $a_i x_i = p$ (1.2.3)

The convention is as follow: The repetition of an index (*whether superscript or subscript*) in a term will denote a summation with respect to that index over its range. The range of an index i is the set of n integer values 1 to n. An index that is summed over is called a *dummy index*, and one that is not summed out is called a free index.

1.3 Law of Transformation

Let $P(x_1, x_2)$ be a physical quantity in $ox_1x_2x_3$ is the Cartesian coordinate systems before deformation and $P'(x_1', x_2')$ be corresponding to $P(x_1, x_2)$ in the new coordinate system $ox_1'x_2'x_3'$ after rotating the x_3-axis about itself at an angle θ, i.e., after deformation.

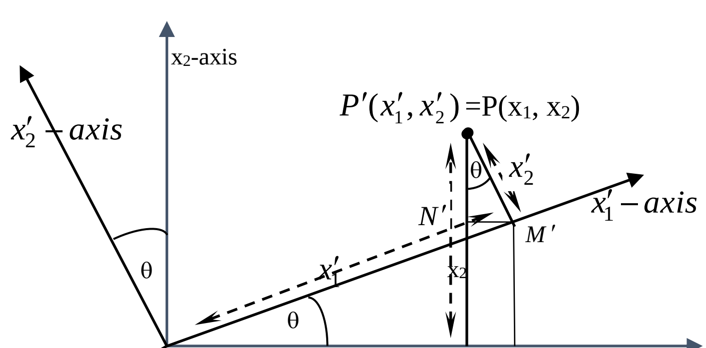
From the figure given below (Figure1.1)

$$\begin{aligned} x_1 &= OM \\ &= ON - MN \\ &= ON - M'N' \end{aligned}$$

$$= x_1' \cos\theta - x_2' \sin\theta \qquad (1.3.1)$$

$$\begin{aligned} x_2 &= PM \\ &= PN' + N'M \\ &= PN' + M'N \end{aligned}$$

$$= x_2' \cos\theta + x_1' \sin\theta \qquad (1.3.2)$$



Using the relation (1.3.1) and (1.3.2) we get

$$x_1' = x_1 \cos\theta + x_2 \sin\theta + 0x_3 \qquad (1.3.3)$$

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta + 0x_3 \qquad (1.3.4)$$

$$x_3' = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \qquad (1.3.5)$$

Relation (1.3.3), (1.3.4) and (1.3.5) can be written as

$$x_1' = x_1\ell_{11} + x_2\ell_{12} + x_3\ell_{13} \qquad (1.3.6)$$

$$x_2' = x_1\ell_{21} + x_2\ell_{22} + x_3\ell_{23} \qquad (1.3.7)$$

$$x_3' = x_1\ell_{31} + x_2\ell_{32} + x_3\ell_{33} \qquad (1.3.8)$$

where $\ell_{ij} = \cos(angle between x_i' \; and \; x_j);\; i, j = 1,2,3$ that is (1.3.9)

$$\ell_{11} = \cos(angle between x_1' \; and \; x_1) = \cos\theta$$

$$\ell_{12} = \cos(angle between x_1' \; and \; x_2) = \cos(90 - \theta) = \sin\theta$$

$$\ell_{13} = \cos(angle between x_1' \; and \; x_3) = \cos 90$$

$$\ell_{21} = \cos(angle between x_2' \; and \; x_1) = \cos(90 + \theta) = -\sin\theta$$

$$\ell_{22} = \cos(angle between x_2' \; and \; x_2) = \cos\theta$$

$$\ell_{23} = \cos(angle between x_2' \; and \; x_3) = \cos 90$$

$$\ell_{31} = \cos(angle between x_3' \; and \; x_1) = \cos 90$$

$$\ell_{32} = \cos(angle between x_3' \; and \; x_2) = \cos 90$$

$$\ell_{33} = \cos(angle\,between\,x_3'\,and\,x_3) = \cos 0 = 1$$

Law of transformation can be written in a tensor form of order one as follow

$$x_1' = \ell_{11}x_1 + \ell_{12}x_2 + \ell_{13}x_3 = \ell_{1j}x_j \quad ; j = 1,2,3$$

$$x_i' = \ell_{ij}x_j\,;\; i, j = 1,2,3 \qquad (1.3.10)$$

$$\frac{\partial x_i'}{\partial x_j} = \ell_{ij} \text{ and } \frac{\partial x_i}{\partial x_j'} = \ell_{ji}$$

Similarly, law of transformation for a tensor of order two

$$x_{pq}' = \ell_{pi}\ell_{qj}x_{ij}\,;\; i, j = 1,2,3;\text{ p, q are dummy variables} \qquad (1.3.11)$$

law of transformation for a tensor of order three

$$x_{pqr}' = \ell_{pi}\ell_{qj}\ell_{rk}x_{ijk}\,;\; i, j, k = 1,2,3;\text{ p, q, r are dummy variables} \quad (1.3.12)$$

and law of transformation of order n

$x_{pqr..........n\ terms}' = (\ell_{pi}\ell_{qj}\ell_{rk}..............n\,terms)x_{ijk.............n\,terms}$ (1.3.13) where $i, j, k,...........n\,terms = 1,2,3,...........n$; p, q, r, ……….n terms are dummy variables

Example.1. The x_i'-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ=30^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find it's coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Solution. The figure (1.2) shows how the x_i'-system is related to the x_i-system. The direction cosines for the given transformation is represented in relation (1.3.14)

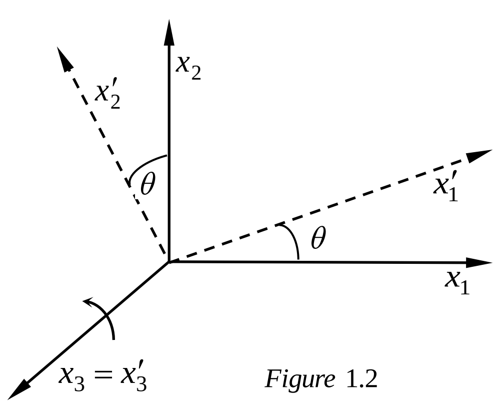


Figure 1.2

Hence, the matrix of the transformation by using (1.9) is

$$(\ell_{ij}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.3.14)$$

Using law of transformation for a tensor of order one, i.e, form (1.3.10), we get

$x_i' = \ell_{ij} x_j$; i, j=1, 2, 3

$x_1' = \ell_{11} x_1 + \ell_{12} x_2 + \ell_{13} x_3$

$\Rightarrow \quad x_1' = 2\cos\theta + 4\sin\theta + 1\times 0 = 2\times \sqrt{3}/2 + 4\times 1/2 + 1\times 0 = (\sqrt{3}+2)$

$\Rightarrow \quad x_2' = 2\sin\theta + 4\cos\theta + 1\times 0 = -2\times 1/2 + 4\times \sqrt{3}/2 + 1\times 0 = (2\sqrt{3}-1)$

$\Rightarrow \quad x_3' = 2\times 0 + 4\times 0 + 1\times 1 = 1 \qquad (1.3.15)$

Hence, (x_1', x_2', x_3') = ($\sqrt{3}+2, 2\sqrt{3}-1, 1$) is in new coordinate system.

Further for the second, (1, 3, 2) are the coordinate of a point in new coordinate system, i.e. ($x'_1 = 1, x'_2 = 3, x'_3 = 2$) to finding the corresponding coordinate in to old coordinate system i.e. (x_1, x_2, x_3). Using law of transformation (1.3.10),

we have $\quad x_i = \ell_{ji} x_j'$; i, j= 1,2,3 $\qquad$ (1.3.16)

or $\quad x_1 = \ell_{11} x_1' + \ell_{21} x_2' + \ell_{31} x_3'$

$x_2 = \ell_{12} x_1' + \ell_{22} x_2' + \ell_{32} x_3'$

$x_3 = \ell_{13} x_1' + \ell_{23} x_2' + \ell_{33} x_3'$

$\Rightarrow \quad x_1 = \cos\theta x_1' - \sin\theta\ x_2', \qquad x_3' = 1\times \sqrt{3}/2 - 3\times 1/2 + 2\times 0 = (\sqrt{3}/2 - 3/2)$

$\Rightarrow \quad x_2 = \sin\theta x_1' + \cos\theta\ x_2', \qquad x_3' = 1\times 1/2 + 3\times \sqrt{3}/2 + 2\times 0 = (1/2 - \sqrt{3}/2)$

$\Rightarrow \qquad x_3 = \cos 90^0\ x_1' + \sin 90^0\ \ x_2' + 1, \qquad\qquad x_3' = 1\times 0 + 3\times 0 + 2\times 1 = 2 \qquad (1.3.17)$

Hence, (x_1, x_2, x_3) = $(\sqrt{3}/2 - 3/2, 1/2 - \sqrt{3}/2, 2)$ in old coordinate system.

Practice 1. The x_i'-system is obtained by rotating the x_i-system about the x_2-axis through an angle θ=45^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Practice 2. The x_i'-system is obtained by rotating the x_i-system about the x_1-axis through an angle θ=60^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Practice 3. The x_i'-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ= 60^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Example2. The x_i'-system is obtained by rotating the x_i-system about the x_2-axis through an angle θ= 60^0 in the sense of right handed screw. Find the transformation matrix. If a tensor of rank/order two has components $[a_{ij}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

Solution. The figure (1.3) shows how the x_i'-system is related to the x_i-system. The direction cosines for the given transformation are represented in the (1.3.18) when x_2-axis is rotated at an angle 60^0 about itself in right handed screw, where a'_{pq} are the components of the tensor of order two in new coordinate system corresponding to a_{ij} in old coordinate system.

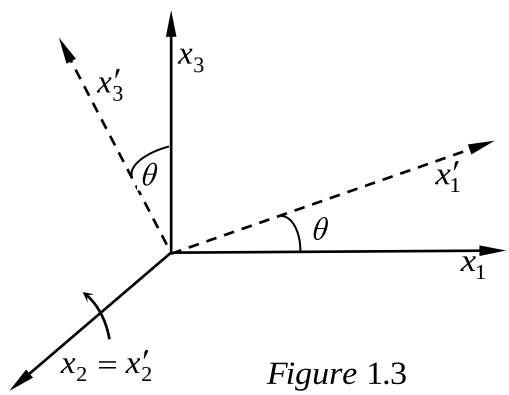


Figure 1.3

Hence, the matrix of the transformation is by using (1.3.9)

$$(\ell_{ij}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \qquad (1.3.18)$$

Using law of transformation (1.3.11) for a tensor of order two, i.e

$$x'_{pq} = \ell_{pi}\ell_{qj}x_{ij}$$

$$a'_{pq} = \ell_{pi}\ell_{qj}a_{ij}$$

$$\Rightarrow \quad a'_{11} = \ell_{1i}\ell_{1j}a_{ij}$$

$$= \ell_{1i}(\ell_{11}a_{i1} + \ell_{12}a_{i2} + \ell_{13}a_{i3})$$

$$= \ell_{11}(\ell_{11}a_{11} + \ell_{12}a_{12} + \ell_{13}a_{13})$$
$$+ \ell_{12}(\ell_{11}a_{21} + \ell_{12}a_{22} + \ell_{13}a_{23})$$
$$+ \ell_{13}(\ell_{11}a_{31} + \ell_{12}a_{32} + \ell_{13}a_{33})$$

using value of ℓ_{ij} from (1.3.18), we have

$$a'_{11} = 1/2\,(1/2\times 1 + 0\times 0 - \sqrt{3}/2\times 1)$$
$$+ 0(1/2\times 0 + 0\times 2 + \sqrt{3}/2\times 2)$$
$$+ \sqrt{3}/2\,(-1/2\times 2 + 0\times 0 + \sqrt{3}/2\times 1)$$

$$= \frac{1}{2}\left(\frac{1-\sqrt{3}}{2}\right) + 0 + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-2}{2}\right) = \left(\frac{4-3\sqrt{3}}{4}\right) \tag{1.3.19}$$

Similarly, $a'_{22} = 2, a'_{33} = \dfrac{4+3\sqrt{3}}{4}$

and $\quad a'_{23} = \ell_{2i}\ell_{3j}a_{ij}$

$$= \ell_{2i}(\ell_{31}a_{i1} + \ell_{32}a_{i2} + \ell_{33}a_{i3})$$

$$= \ell_{21}(\ell_{31}a_{11} + \ell_{32}a_{12} + \ell_{33}a_{13})$$
$$+ \ell_{22}(\ell_{31}a_{21} + \ell_{32}a_{22} + \ell_{33}a_{23})$$
$$+ \ell_{33}(\ell_{31}a_{31} + \ell_{32}a_{32} + \ell_{33}a_{33})$$

$$= 0\times(1/2\times 1 + 0\times 0 - \sqrt{3}/2\times 1)$$
$$+ 1\times(-\sqrt{3}/2\times 0 + 0\times 2 + 1/2\times 2)$$
$$+ 0\times(-1/2\times 2 + 0\times 0 + \sqrt{3}/2\times 1)$$

$$a'_{23} = 0 + 1 + 0 = 1 \tag{1.3.20}$$

Similarly, $a'_{31} = \dfrac{1}{4}, a'_{13} = \dfrac{5}{4}, a'_{12} = 0, a'_{21} = \sqrt{3}, a'_{32} = \dfrac{1}{2}$

Hence,

the tensor $[a_{ij}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ is transformed into $[a'_{pq}] = \begin{bmatrix} \left(\dfrac{4-3\sqrt{3}}{4}\right) & 0 & \dfrac{5}{4} \\ \sqrt{3} & 2 & 1 \\ \dfrac{1}{4} & \dfrac{1}{2} & \dfrac{4+3\sqrt{3}}{4} \end{bmatrix}$

Practice 4. The x'_i-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ= 45^0 in the sense of right handed screw. Find the transformation matrix. If a tensor of rank two has components $[a_{ij}] = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & 2 \\ 2 & -1 & 4 \end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

Practice 5. The x_i'-system is obtained by rotating the x_i-system about the x_1-axis through an angle θ= 30^0 in the sense of right handed screw. Find the transformation matrix. If a tensor of rank two has components $[a_{ij}] = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

1.4 Some Properties of Tensor

Zero Tensors: A tensor whose all components in one Cartesian coordinates system are 0 is called a zero. A tensor may have any order n.

Property 1.4.1 If all component of a tensor are '0' in one coordinate system then they are '0' in all coordinate systems.

Proof. Let $u_{ijk..........n\,terms}$ and $u'_{pqr..........n\,terms}$ the component of a n^th^ order tensor in two coordinates systems $ox_1x_2x_3$ and $ox_1'x_2'x_3'$.

Suppose $\quad u_{ijk..........n\ terms} = 0, \forall i, j, k.......$ (1.4.1)

We know the law of transformation of tensor of order n as

$$u'_{pqr..........n\,terms} = (\ell_{pi}\ell_{qj}\ell_{rk}..............n\,terms)u_{ijk..........n\,terms} \quad (1.4.2)$$

Using (1.4.10) into (1.4.11) we get

$u'_{pqr..........n\,terms} = 0, \ \forall\, p, q, r.......$. Hence, zero tensor of any order in one coordinate system remains always zero tensor of same order in all other coordinate systems.

Property 1.4.2 If the corresponding components of two tensors of the same order are equal in one coordinate system, then they are equal in all coordinate systems.

Property 1.4.3 Equality of Tensors: Two tensors of the same order whose corresponding components are equal in a coordinate system (and hence in all coordinates) are called equal tensors.

Thus, in order to show that two tensors are equal, it is sufficient to show that their corresponding components are equal in any one of the coordinate system.

Property 1.4.4 (**Scalar multiplication of a tensor**): If components of a tensor of order n are multiplied by a scalar α, then the resulting components form a tensor of the same order n.

Proof: Let $u_{ijk.........n terms}$ be a tensor of order n in $ox_1x_2x_3$ system. Let $u'_{pqr............n\ terms}$ be the corresponding components in the dashed ($ox'_1x'_2x'_3$) system. The transformation rule for a tensor of order n, (1.3.13) yields.

$$u'_{pqr.............n terms} = \ell_{pi}\ell_{qj}\ell_{rk}\text{.........} ...n\, terms(u_{ijk..............n\ terms}) \qquad (1.4.3)$$

Now $\alpha\, u'_{pqr.............n terms} = \ell_{pi}\ell_{qj}\ell_{rk}\text{.........} ...n\, terms(\alpha\, u_{ijk..............n\ terms})$ (1.4.4)

This shows that components $\alpha\, u_{ijk..............}$ form a tensor of rank n.

Property 1.4.5 (Sum and Difference of tensors) If $u_{ijk.........n terms}$ and $v_{ijk.........n terms}$ are tensors of the same rank n then their sum ($u_{ijk.........n terms} + v_{ijk.........n terms}$) is a tensor of the same order n.

Proof: Let $w_{ijk.........n terms} = u_{ijk.........n terms} + v_{ijk.........n terms}$ (1.4.5)

and let $u'_{pqr.........n terms}$ and $v'_{pqr.........n terms}$ be the components of the given tensors of order n relative to the new system $ox'_1x'_2x'_3$. Then transformation rules for these tensors are

$$u'_{pqr...........n terms} = \ell_{pi}\ell_{qj}\ell_{rk}\text{.........} ..n\, terms(u_{ijk.............n terms}) \qquad (1.4.6) \quad \text{and}$$

$$v'_{pqr...........n terms} = \ell_{pi}\ell_{qj}\ell_{rk}\text{.........} ..n\, terms(v_{ijk.............n terms}) \qquad (1.4.7)$$

where $\ell_{pi} = \cos(x'_p, x_i)$ (1.4.8)

let $w'_{pqr.........n terms} = u'_{pqr.........n terms} + v'_{pqr.........n terms}$ (1.4.9)

using relations (1.4.6 and 1.4.7) in the relation (1.4.9), we get

$$w'_{pqr.........n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}.........n\,terms(\,u_{ijk.........n\,terms} + v_{ijk.............n\,terms} \qquad (1.4.10)$$

$$w'_{pqr.........n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}.........n\,terms(\,w_{ijk.........n\,terms}\,) \qquad (1.4.11)$$

Thus quantities $w_{ijk.........n\,terms}$ obey the transformation rule of a tensor of order n. Therefore, they are components of a tensor of rank/order n.

Corollary: Similarly, their difference $u_{ijk.........n\,terms} - v_{ijk.............n\,terms}$ is also a tensor of rank n.

Property 1.4.6 (Tensor Multiplication)

The product of two tensors is also a tensor whose order is the sum of orders of the given tensors.

Proof: Let $u_{ijk.............m\,terms}$ and $v_{\alpha\beta\gamma.............n\,terms}$ be two tensors of order m and n respectively in the coordinate system $ox_1x_2x_3$ also $u'_{pqr............m\ terms}$ and $v'_{\sigma\tau\varsigma............n\ terms}$ are corresponding components of tensors in $ox'_1x'_2x'_3$ system.

We shall show that the product

$$w_{ijk.........m\,terms+\alpha\beta\gamma........n\,terms} = u_{ijk.........n\,terms} \times v_{\alpha\beta\gamma.........n\,terms} \qquad (1.4.5)$$

is tensor of order m+n. Using the law of transformation (1.3.13), we have

$$\left.\begin{aligned} u'_{pqr............m\ terms} &= \ell_{pi}\ell_{qj}\ell_{rk}..........m\,terms(u_{ijk..............m\,terms}) \\ v'_{\sigma\tau\varsigma............n\ terms} &= \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}..........n\,terms(v_{\alpha\beta\gamma..............m\,terms}) \end{aligned}\right\} \qquad (1.4.6)$$

where, ℓ_{ij} is having its standard meaning as defined in relation (1.3.9).

Let $$w'_{pqr.........m\,terms+\sigma\tau\varsigma........n\,terms} = u'_{pqr.........n\,terms} \times v'_{\sigma\tau\varsigma.........n\,terms} \qquad (1.4.7)$$

Using relation (1.4.6) in to (1.4.7), we get

$$w'_{pqr.........m\,terms+\sigma\tau\varsigma........n\,terms} =$$

$$\ell_{pi}\ell_{qj}\ell_{rk}....m\,terms(u_{ijk....m\,terms}) \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}...n\,terms(v_{\alpha\beta\gamma.....m\,terms})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}....m\,terms \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}...n\,terms(v_{\alpha\beta\gamma.....m\,terms} \times u_{ijk....m\,terms})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}....mterms \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}...nterms(w_{\alpha\beta\gamma.....mterms} +_{ijk....mterms}) \qquad (1.4.8)$$

This shows that components $w_{ijk.........mterms+\alpha\beta\gamma........nterms}$ obey the transformation rule of a tensor of order (m+n). Hence $u_{ijk.........nterms} \times v_{\alpha\beta\gamma.........nterms}$ are components of a (m+n)th order tensor.

Practice 6. If u_i and v_j are components of vectors, then show that $u_i v_j$ are components of a second-order tensor.

Practice 7. If u_{ij} and v_k are components of tensors of second-order and first-order, respectively, then prove that $u_{ij}v_k$ are components of a third order tensor.

Practice 8. If u_{ij} and v_{km} are components of second-order tensors, then prove that $u_{ij}v_{km}$ are components of a fourth order tensor.

Practice 9. If u_i and v_j are components of two tensors. Let $w_{ij} = u_i v_j + u_j v_i$ and $\alpha_{ij} = u_i v_j - u_j v_i$. Show that each of w_{ij} and α_{ij} is a second order tensor.

1.5 Contraction of a Tensor

> ***The operation or process of setting two suffixes equal in a tensor and then summing over the dummy suffix is called a contraction operation or simply a contraction.*** *The tensor resulting from a contraction operation is called a* ***contraction*** *of the original tensor. Contraction operations are applicable to tensor of all orders higher than 1 and each* ***such operation reduces the order of a tensor by 2.***

Property 1.5 Prove that the result of applying a contraction of a tensor of order n is a tensor of order (n-2).

Proof: Let $u_{ijk.........nterms}$ and $u'_{pqr.........nterms}$ be the components of the given tensor of order n relative to two Cartesian coordinate systems $ox_1x_2x_3$ and $ox'_1x'_2x'_3$. The rule of transformation of tensor of order n (1.3.13) is

$$u'_{pqr...........n\ \ terms} = (\ell_{pi}\ell_{qj}\ell_{rk}.............nterms)u_{ijk..............nterms} \qquad (1.5.1)$$

without loss of generality, we contract the given tensor by setting $i = j$ and summation convention. Let

$$v_{kl..........} = u_{iikl............} \quad (1.5.2)$$

Now $$u'_{pqr..........n\ terms} = (\ell_{pi}\ell_{qi})\ell_{rk}..........\ldots n\,terms \times u_{iik..............n\,terms} \quad (1.5.3)$$

$$= (\delta_{pq})\ell_{rk}..........\ldots n\,terms \times v_{kl................(n-2)\,terms}$$

$$u'_{ppr................} = \ell_{rk}..........(n-2)terms \times v_{kl..............(n-2)\,terms} \quad \because \delta_{pq} = \begin{cases} 1\ if\ \ p = q \\ 0\ if\ \ p \neq q \end{cases}$$

$$v'_{r................(n-2)\,terms} = \ell_{rk}..........(n-2)terms \times v_{kl..............(n-2)\,terms} \quad (1.5.4)$$

Hence, the resulting tensor is tensor of order n-2. So contraction applying once on a tensor of order greater then 1, the order of the tensor reduces by 2. Similarly contraction applying twice on a tensor of order n the order of that tensor reduces by 4.

1.6 Quotient law of Tensors

(Quotient law is the partial converse of the contraction law)

Property 1.6 If there is an entity represents by the set of 9 quantities u_{ij} relative to any given system of Cartesian axes, and if $u_{ij}v_j$ is a vector for an arbitrary vector v_j, then show that u_{ij} is a second order tensor.

Proof: $$w_i = u_{ij}v_j \quad (1.6.1)$$

Suppose that u'_{pq}, u'_p and w'_p be the corresponding components in the dashed system $ox'_1x'_2x'_3$. Then by using law of transformation and inverse law of transformation (1.3.10 and 11)

Now $$u'_{pq}v'_p = w'_p \quad (1.6.2)$$

$$= \ell_{pi}w_i$$

$$= \ell_{pi}(u_{ij}v_j)$$

$$= \ell_{pi}\,\ell_{qj}\,u_{ij}v'_q$$

$$\Rightarrow \qquad (u'_{pq} - \ell_{pi}\ell_{qj}u_{ij})v'_q = 0 \qquad (1.6.3)$$

for an arbitrary vector v'_q. Therefore, we must have

$$u'_{pq} = \ell_{pi}\ell_{qj}u_{ij} \qquad (1.6.4)$$

This rule shows that components u_{ij} obey the tensor law of transformation of a second order. Hence, u_{ij} is a tensor of order two.

Practice 10. Let α_i be an ordered triplet and β_i be a vector, referred to the $x_i -$axis. If $\alpha_i\beta_i$ is a scalar, show that α_i are component of a vector.

Example 3. If there is an entity representable by a set of 27 quantities u_{ijk} relative to $ox_1x_2x_3$ system and if $u_{ijk}v_{jk}$ is a tensor of order one for an arbitrary tensor v_{jk} if order 2, show that u_{ijk} is tensor of order 3.

Solution. Let $w_i = u_{ijk}v_{jk}$ (1.6.5)

It is given that v_{jk} is a tensor of order 2 and $u_{ijk}v_{jk}$ is a tensor of order one, and v'_{pq}, u'_{pqr} are corresponding to v_{jk}, u_{ijk} in new coordinate system $ox'_1x'_2x'_3$. Then by using transformation law and inverse transformation law (1.3.10 and 11) we get.

$$u'_{pqr}v'_{qr} = w'_p \qquad (1.6.6)$$

$$= \ell_{pi}w_i$$

$$= \ell_{pi}u_{ijk}v_{jk} \qquad \text{(by using 1.6.5)}$$

$$= \ell_{pi}u_{ijk}(\ell_{qj}\ell_{rk}v'_{qr})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk}v'_{qr}$$

$$\Rightarrow \qquad (u'_{pqr} - \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk})v'_{qr} = 0 \qquad (1.6.7)$$

for an arbitrary vector v'_{qr}. Therefore, we must have

$$u'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk} \qquad (1.6.8)$$

This rule shows that components u_{ijk} obey the tensor law of transformation of a second order. Hence, u_{ijk} is a tensor of order two.

Practice 11. If there is an entity representable by a set of 27 quantities u_{ijk} relative to $ox_1x_2x_3$ system and if $u_{ijk}v_k$ is a tensor of order two for an arbitrary tensor v_k of order one, show that u_{ijk} is tensor of order 3.

Practice 12. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_{jkl}$ is a tensor of order one for an arbitrary tensor v_{jkl} if order 3, show that u_{ijkl} is tensor of order 4.

Practice 13. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_l$ is a tensor of order three for an arbitrary tensor v_l if order one, show that u_{ijkl} is tensor of order 4.

Practice 14. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_{kl}$ is a tensor of order two for an arbitrary tensor v_{kl} of order 2, show that u_{ijkl} is tensor of order 4.

1.7 Symmetric & Skew symmetric tensors

1.7.1 A second order tensor u_{ij} is said to be symmetric if $u_{ij} = u_{ji} \quad \forall i, j$. For example unit matrix of order 3×3 is symmetric tensor of order two.

1.7.2 A second order tensor u_{ij} is said to be skew-symmetric if $u_{ij} = -u_{ji} \quad \forall i, j$. For example skew-symmetric matrix of order 3×3 is skew-symmetric tensor of order two.

Definition: (Gradient) if $u_{pqr.........n\,terms}$ is a tensor of order n in $ox_1x_2x_3$ system, then

$$v_{spqr.........(n+1)\,terms} = \frac{\partial}{\partial s} u_{pqr.........n\,terms}$$

$$= u_{pqr.........n\,terms,s} \tag{1.7.1}$$

is defined as the gradient of the tensor $u_{pqr.........n\,terms}$.

For example $u_{p,q} = \frac{\partial}{\partial x_q} u_p$ represents the gradient of vector u_p .

Property 1.7 Show that the gradient of a scalar point function is a tensor of order one.

Proof: Suppose that $U = U(x_1, x_2, x_3)$ be a scalar point function and

$$v_i = \frac{\partial U}{\partial x_i} = U_{,i} \qquad (1.7.2)$$

Let the components of the gradient of U in the dashed system $ox_1' x_2' x_3'$ be v_p' , so that

$$v_p' = \frac{\partial U}{\partial x_p'} \qquad (1.7.3)$$

Using the law of transformation (1.3.10) and inverse law of transformation we have

$$v_p' = \frac{\partial U}{\partial x_p'}$$

$$= \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial x_p'} \qquad \text{(by chain rule)}$$

$$= \ell_{pi} \frac{\partial U}{\partial x_i} = \ell_{pi} U_{,i}$$

Using (1.7.2), we get $\qquad v_p' = \ell_{pi} v_i \qquad (1.7.4)$

Which is a transformation rule for a tensor of order one. Hence gradient of the scalar point function U is a tensor of order one.

Property 1.8 Show that the **gradient of a vector** u_i is a tensor of order two.

Proof: The gradient of the tensor u_i is defined as

$$w_{ij} = \frac{\partial u_i}{\partial x_j} = u_{i,j} \qquad (1.7.5)$$

Let the vector u_i be transformed to the vector u'_p relative to the new system $ox'_1x'_2x'_3$. Then the transformation law for tensors of orders one (1.3.10) yields

$$u'_p = \ell_{pi}u_i \qquad (1.7.6)$$

Suppose the nine quantities w_{ij} relative to new system are transformed to w'_{pq}. Then

$$w'_{pq} = \frac{\partial u'_p}{\partial x'_q}$$

$$= \frac{\partial}{\partial x'_q}(\ell_{pi}u_i) = \ell_{pi}\frac{\partial u_i}{\partial x'_q}$$

$$= \ell_{pi}\frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial x'_q} \quad \text{(by chain rule)}$$

$$= \ell_{pi}\ell_{qj}\frac{\partial u_i}{\partial x_j} = \ell_{pi}\ell_{qj}w_{ij}$$

$$\Rightarrow \qquad w'_{pq} = \ell_{pi}\ell_{qj}w_{ij} \qquad (1.7.7)$$

This is a transformation rule for tensors of order two. Hence, w_{ij} is a tensor of order two. Consequently, the gradient of a vector u_i is a tensor of order two.

Property 1.9 Show that the **gradient of a tensor of order n,** $u_{ijk.........n\,terms}$ is a tensor of order (n+1).

Proof: Let $u_{ijk.........n\,terms}$ is a tensor of order n. The gradient of the tensor $u_{ijk.........n\,terms}$ is defined as

$$w_{\tau\, pqr..............} = \frac{\partial u_{ijk..........n\,terms}}{\partial x_\tau} = u_{ijk..........n\,terms,\tau} \qquad (1.7.8)$$

Let the tensor $u_{ijk......n\,terms}$ be transformed to the tensor $u'_{prs........n\,terms}$ relative to the new system $ox'_1x'_2x'_3$. Then the transformation law for tensors of order n (1.3.13) yields

$$u'_{pqr...........n\;\;terms} = (\ell_{pi}\ell_{qj}\ell_{rk}..............n\,terms)u_{ijk...............n\,terms} \qquad (1.7.9)$$

Suppose 3^n quantities $w_{ijk.........n\,terms}$ relative to new system are transformed to $w'_{pqr.........n\,terms}$. Then

$$w'_{pqr........n\,terms,\tau} = \frac{\partial u'_{pqr.........n\,terms}}{\partial x'_\tau}$$

$$= (\ell_{pi}\ell_{qj}\ell_{rk}........n\,terms)\frac{\partial u_{ijk.........n\,terms}}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial x'_\tau}$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}........n\,terms\,\ell_{\tau\alpha}\frac{\partial u_{ijk............n\,terms}}{\partial x_\alpha}$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}..........\,..n\,terms\,\ell_{\alpha\tau}\times u_{ijk.......nterms,\tau} \qquad (1.7.10)$$

$$\Rightarrow \qquad w'_{pqr.........n\,terms,\tau} = \ell_{pi}\ell_{qj}\ell_{rk}..........\,..n\,terms\ell_{\alpha\tau}u_{ijk..........,\tau}$$

This is a transformation rule for tensors of order (n+1). Hence, $w_{ijk.........(n+1)terms}$ is a tensor of order (n+1). Consequently, the gradient of a tensor of order n is a tensor of order (n+1).

Books Recommended:

1. **Y.C.Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey, 1965.
2. **Saad, A.S.** Elasticity-Theory and Applications, Pergamon Press, Inc. NY, 1994.
3. **Sokolnikoff, I.S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

CHAPTER-II

ANALYSIS OF TENSOR

Consider an ordered set of ***N*** real variables $x_1, x_2, x_3, ...x_i, ...x_N$; these variables will be called the **coordinates** of a point. (The suffixes *1*, *2*, *3,……,i,…..N*, which we shall call superscripts, merely serve as labels and do not possess any signification as power indices. Later we shall introduce quantities of the a_i and again the i, which we shall call a subscript, will act only as a label.) Then all the point corresponding to all values of the coordinates are said to form an ***N*-dimensional space,** denoted by V_N. Several or all of the coordinates may be restricted in range to ensure a one-one correspondence between points of the V_N, and sets of coordinates.

A curve in the V_N is defined as the assemblage of points which satisfy the ***N*** equations

$$x_i = x_i(u), \quad (i = 1,2,3,..........N)$$

where u is a parameter and $x_i(u)$ are N functions of u, which obey certain continuity conditions. In general, it will be sufficient that derivatives exist up to any order required. **A subspace** V_M of V_N is defined for $M < N$ as the collection of points which satisfy the ***N*** equations

$$x_i = x_i(u_1, u_2, ..., u_M), \quad (i = 1,2,3,..........N)$$

where there are M parameters $u_1, u_2, ..., u_M$. The $x_i(u_1, u_2, ..., u_M)$ are N functions of the $u_1, u_2, ..., u_M$ satisfying certain conditions of continuity. In addition the $M \times N$

matrix formed from the partial derivatives $\frac{\partial x_i}{\partial u_j}$ is assumed to be of rank M^*. When $M = N - 1$, the subspace is called a **hyper surface**.

Let us consider a space V_N with the coordinate system $x_1, x_2, x_3, ... x_N$. The N equations

$$\vec{x}_i = \varphi_i(x_1, x_2, ... x_N), \quad (i = 1, 2, 3, ... N) \qquad (2.1)$$

where the φ_i are single-valued continuous differentiable functions of the coordinates, define a new coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3 ...., \bar{x}_N$. Equations (2.1) are said to define **a transformation of coordinates**. It is essential that the N functions φ_i be independent. A necessary and sufficient condition is that the **Jacobian determinant** formed from the partial derivatives $\frac{\partial \bar{x}_i}{\partial x_j}$ does not vanish. Under this condition we can solve equations (2.1) for the x_i as functions of the $\bar{x}_i$ and obtain

$$x_i = \varphi_i(\bar{x}_1, \bar{x}_2, \bar{x}_3, ... \bar{x}_N) \quad (i = 1, 2, 3, ... N)$$

2.1 The Symbol δ_{ij}

We will now introduce the following two conventions:

1) Latin indices, used either as subscripts or superscripts, will take all values from 1 to N unless the contrary is specified. Thus equations (2.1) are briefly written $\bar{x}_i = \varphi_i(x_1, x_2, ... x_N)$, the convention informing us that there are N equations.
2) If a Latin index is repeated in a term, then it is understood that a summation with respect to that index over the range 1, 2, 3, …. N is implied. Thus instead of the expression $\sum_{i=1}^{N} a_i x_i$, we merely write $a_i x_i$. Now differentiation of (2.1) yields

$$d\bar{x}_i = \sum_{r=1}^{N} \frac{\partial \varphi_i}{\partial x_r} dx_r = \sum_{r=1}^{N} \frac{\partial \bar{x}_i}{\partial x_r} dx_r, \qquad (i = 1, 2, 3, ..., N)$$

which simplify, when the above conventions are used, to

$$d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_r} dx_r . \qquad (2.2)$$

The repeated index r is called a **dummy index,** as it can be repeated by any other Latin index, except 'i' in this particular case. That is, equations (2.2) can equally well be written $d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_m} dx_m$ or for that matter $d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_r} dx_r$. In order to avoid confusion, the same index must not be used more than twice in any single term. For example; $\left(\sum_{i=1}^{N} a_i x_i\right)^2$ will not be written $a_i x_i a_i x_i$, but rather $a_i x_i a_j x_j$. It will always be clear from the context, usually powers will be indicated by the use of brackets; thus $(x_N)^2$ mean the square of x_N. The reason for using superscripts and subscripts will be indicated in due course. Let us introduce the **Kronecker delta.** It is defined as

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \tag{2.1.1}$$

That is, $\delta_{11} = \delta_{22} = \delta_{33} = 1$; $\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$.The symbol δ_{ij} is known as the Kronecker δ symbol, named after the German Mathematician Leopold Kronecker (1827-1891). The following property is inherent in the definition of δ_{ij}.

1) Kronecker δ is symmetric i.e $\delta_{ij} = \delta_{ji}$ (2.1.2)
2) Summation convention $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ (2.1.3)
3) The unit matrix of order 3 is $I_3 = (\delta_{ij})$ and $\det(\delta_{ij}) = 1$ (2.1.4)
4) The orthonormality of the base unit vectors $\hat{e}_i$ can be written as

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \tag{2.1.5}$$

2.1.1 Tensor Equation:- An equation of type $\alpha_{ijk} - \beta_{ij} u_k = 0$ is called a tensor equation, for checking the correctness of a tensor equation, we have the following rule

(i) In a correctly tensor equation no suffixes shall appear more than twice in any term, otherwise the operation will not be define. For example $u'_j = \alpha_{ij} u_j v_j$ is not a tensor equation.

(ii) If a suffixes appears only once in a term then it must appear only once in the remaining term also. For example, an equation $u'_j - \ell_{ij} u_i = 0$ is not a tensor equation. Hence j appears once in the first term while it appears twice in the second term.

Property 2.1 Prove the following (**Known as substitution properties of** δ_{ij})

(i) $u_j = \delta_{ij} u_i$ (ii) $\delta_{ij} u_{jk} = u_{ik}$; $\delta_{ij} u_{ik} = u_{jk}$ (iii) $\delta_{ij} u_{ij} = u_{kk} = u_{11} + u_{22} + u_{33}$

Proof. (i) Now $\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} u_3$

$$\Rightarrow u_j + \sum_{\substack{i=j \\ i \neq j}}^{3} \delta_{ij} u_i = u_j \tag{2.1.6}$$

(ii) $\delta_{ij} u_{jk} = \sum_{j=1}^{3} \delta_{ij} u_{jk}$

$= \delta_{ii} u_{ik}$ (for $j \neq i, \delta_{ij} = 0$), here summation over i is not taken

$$= u_{ik} \tag{2.1.7}$$

(iii) $\delta_{ij} u_{ij} = \sum_i \left[\sum_j \delta_{ij} u_{ij} \right]$

$= \sum_i (1.u_{ii})$, in u_{ii} summation is not being taken

$$= \sum_i u_{ii} = u_{11} + u_{22} + u_{33} = u_{kk} \tag{2.1.8}$$

Example 2.1 Given that $a_{ij} = \alpha \delta_{ij} b_{kk} + \beta b_{ij}$,where $\beta \neq 0,\ 3\alpha + \beta \neq 0$, find b_{ij} in terms of a_{ij} .

Solution. Setting $i = j$ in the relation $a_{ij} = \alpha \delta_{ij} b_{kk} + \beta b_{ij}$ and summing accordingly, we obtain $a_{ii} = \alpha.3.b_{kk} + \beta.b_{ii}$

$= (3\alpha + \beta) b_{kk}$ $(\because b_{kk} = b_{ii})$

$$\Rightarrow \qquad b_{kk} = \frac{1}{3\alpha + \beta} a_{kk}$$

Hence, $$b_{ij} = \frac{1}{\beta}\left[a_{ij} - \alpha \delta_{ij} b_{kk}\right] = \frac{1}{\beta}\left[a_{ij} - \frac{\alpha}{3\alpha + \beta} \delta_{ij} a_{kk}\right] \tag{2.1.9}$$

Property 2.2 Prove that (i) $\ell_{pi} \ell_{qi} = \delta_{pq}$ (ii) $\ell_{pi} \ell_{pj} = \delta_{ij}$ (iii) $|\ell_{ij}| = 1$, $(\ell_{ij})^{-1} = (\ell_{ij})'$

Proof. We know the transformation law of the coordinate system (1.3.10), we have

$$x'_p = \ell_{pi} x_i \text{ and } x_i = \ell_{qi} x'_q \qquad (2.1.10)$$

Now, (i) $x'_p = \ell_{pi} x_i$

$$\Rightarrow x'_p = \ell_{pi} (\ell_{qi} x'_q) \qquad (2.1.11)$$

using the relation (2.1.6) on the L.H.S. of (2.1.11)

$$\Rightarrow \delta_{pq} x'_q = \ell_{pi} \ell_{qi} x'_q$$

$$\Rightarrow (\ell_{pi} \ell_{qi} - \delta_{pq}) x'_q = 0$$

$$\Rightarrow \ell_{pi} \ell_{qi} = \delta_{pq} \qquad (2.1.12)$$

(ii) Similarly, $x_i = \ell_{pi} x'_p$

$$\Rightarrow \quad = \ell_{pi} \ell_{pj} x_j$$

Also $x_i = \delta_{ij} x_j$ (2.1.13)

Hence, $\delta_{ij} x_j = \ell_{pi} \ell_{pj} x_j$

$$(\delta_{ij} - \ell_{pi} \ell_{pj}) x_j = 0$$

$$\Rightarrow \quad \delta_{ij} = \ell_{pi} \ell_{pj} \qquad (2.1.14)$$

(iii) Using (2.1.12) gives, in the expanded form,

$$\ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 = 1, \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 = 1, \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 1$$

$$\ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} = 0, \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{33} = 0, \ell_{31}\ell_{11} + \ell_{32}\ell_{12} + \ell_{33}\ell_{13} = 0$$

The relations (2.1.12) and (2.1.14) are referred as the orthonormal relations for ℓ_{ij}. In matrix notation, the above said relations may be represented respectively, as follows

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.1.15)$$

or $LL' = L'L = 1$

these expressions show that the matrix L

Property 2.3 Show that δ_{ij} and ℓ_{ij} are tensors, each of order two.

Proof: Let u_i be any tensor of order one,

i> by the substitution property of the Kroneceker delta tensor δ_{ij} , we have

$$u_i = \delta_{ij} u_j \tag{2.1.16}$$

Now u_i and u_j are each of tensor order one. Therefore, by quotient law, we conclude that δ_{ij} is a tensor of rank two.

ii> The transformation law for the first order tensor is

$$u'_p = \ell_{pi} u_i \tag{2.1.17}$$

where u_i is a vector and $\ell_{pi} u_i$ is a vector by contraction property. Therefore, by quotient law, the quantities ℓ_{pi} are components of a second order tensor.

Note 1: The tensor δ_{ij} is called a unit tensor or an identity tensor of order two.

2. We may call the tensor ℓ_{ij} as the transformation tensor of rank two.

2.2 The Symbol $\in_{ijk}$

Euclidean geometry investigates the properties of figures which are invariant with respect to translations and rotations in space. It may be subdivided into Algebraic methods the theory applicable to entire configurations such as the class or degree of a curve. The latter discusses by means of the calculus those properties which depend on a restricted portion of the figure. For example, the total curvature of a surface at that point. Succinctly we may say that differential geometry is the study of geometry in small. This chapter is not intended to be a complete course on the subject. However, sufficient theory is developed to indicate the scope and power of the tensor method.

The symbol $\in_{ijk}$ is known as the Levi-civita $\in$-symbol, named after the Italian mathematician Tullio Levi-civita (1873-1941). The $\in$-symbol is also referred to as the **Permutation symbol/alternating symbol** or **alternator**. In terms of mutually orthogonal unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ along the Cartesian axes, it defined as

$$\hat{e}_i.(\hat{e}_j \times \hat{e}_k) = \in_{ijk} \quad \forall i, j, k = 1,2,3 \tag{2.2.1}$$

Thus, the symbol $\in_{ijk}$ gives

$$\in_{ijk}=\begin{cases} 1 & : if\ i,j,k\ take\ values\ in\ the\ cyclic\ order \\ -1 & : if\ i,j,k\ take\ values\ in\ the\ acyclic\ order \\ 0 & : if\ any\ or\ all\ of\ i,j,k\ take\ the\ same\ value \end{cases} \quad (2.2.2)$$

These relations are 27 in number. The $\in$-symbol is useful in expressing the vector product of two vectors and scalar triple product.

(i) We have $\hat{e}_i \times \hat{e}_j = \in_{ijk} \hat{e}_k$. (2.2.3)

(ii) For two vectors a_i and b_i ,we write

$$\vec{a}\times\vec{b} = (a_i\hat{e}_i)\times(b_j\hat{e}_j) = a_ib_j(\hat{e}_i\times\hat{e}_j) = \in_{ijk} a_ib_j\hat{e}_k \quad (2.2.4)$$

(iii) $\vec{a} = a_i\hat{e}_i , \vec{b} = b_j\hat{e}_j , \vec{c} = c_k\hat{e}_k$

We have

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = (\vec{a}\times\vec{b}).\vec{c} = (\in_{ijk} a_i\, b_j\, \hat{e}_k).(c_k\, \hat{e}_k)$$

$$= \in_{ijk} a_i\, b_j\, c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (2.2.5)$$

Property 2.4 Show that $\in_{ijk}$ is a tensor of order 3.

Proof: Let $\vec{a} = a_i$ and $\vec{b} = b_i$ be any two vectors. Let

$$\vec{c} = c_i = \vec{a}\times\vec{b} .$$

Then, $c_i = \in_{ijk} a_jb_k$ (2.2.6)

Now a_jb_k is a tensor of order 2 and $\in_{ijk} a_jb_k$ (by 2.2.6) is a tensor of order one. Therefore, by quotient law, $\in_{ijk}$ is a tensor of order 3.

Example 2.2 Show that $w_{ij} = \in_{ijk} u_k$ is a skew-symmetric tensor, where u_k is a vector and $\in_{ijk}$ is an alternating tensor

Solution: Since $\in_{ijk}$ is a tensor of order 3 and u_k is a tensor of order one, so by contraction, the product $\in_{ijk} u_k$ is a tensor of order 2. Further

$$w_{ij} = \in_{ijk} u_k$$

$$= - \in_{jik} u_k$$

$$= -w_{ji} \qquad (2.2.7)$$

This shows that w_{ij} is a tensor which is skew-symmetric.

Example 2.3 Show that u_{ij} is symmetric iff $\in_{ikj} u_{ij} = 0$

Solution: We find

$$\in_{ij1} u_{ij} = \in_{231} u_{23} + \in_{321} u_{32} = u_{23} - u_{32}$$

$$\in_{ij2} u_{ij} = \in_{312} u_{31} + \in_{132} u_{13} = u_{31} - u_{13}$$

$$\in_{ij3} u_{ij} = \in_{123} u_{12} + \in_{213} u_{21} = u_{12} - u_{21} \qquad (2.2.8)$$

Thus, u_{ij} is symmetric iff

$$u_{ij} = u_{ji} \text{ or } u_{12} = u_{21}, u_{13} = u_{31}, u_{23} = u_{32} \qquad (2.2.9)$$

2.3. Isotropic Tensors

Definition: A tensor is said to be an isotropic tensor if its components **remain unchanged/invariant** however the axes are rotated.

Note. 1. An isotropic tensor possesses no directional properties. Therefore a non-zero vector (or a non-zero tensor of rank 1) can never be an isotropic tensor. Tensor of higher orders, other than one, can be isotropic tensors.

2. Zero tensors of all orders are isotropic tensors.

3. By definition, a scalar (or a tensor of rank zero) is an isotropic tensor.

4. A scalar multiple of an isotropic tensor is an isotropic tensor.

5. The sum and the differences of two isotropic tensors is an isotropic tensor.

Property 2.5 Prove that substitution tensor δ_{ij} and alternating tensor $\in_{ijk}$ are isotropic tensors

Proof: A>Let the components δ_{ij} relative to x_i-system are transformed to quantities δ'_{pq} relative to x'_i-system. Then, the tensorial transformation rule is

$$\delta'_{pq} = \ell_{pi}\ell_{qj}\delta_{ij} \tag{2.3.1}$$

Now R.H.S of (2.3.1)

$$= \ell_{pi}\left[\ell_{qj}\delta_{ij}\right] = \ell_{pi}\ell_{qi}$$

$$= \delta_{pq} = \begin{cases} 0 & if \quad p \neq q \\ 1 & if \quad p = q \end{cases} \tag{2.3.2}$$

Relation (2.3.1) and (2.3.2) show that the components δ_{ij} are transformed into itself under all co-ordinate transformations. Hence, by definition, δ_{ij} is an isotropic tensor. **B>** We know that $\in_{ijk}$ is a system of 27 numbers. Let

$$\in_{ijk} = \left[\hat{e}_i \ \hat{e}_j \ \hat{e}_k\right] = \hat{e}_i.(\hat{e}_j \times \hat{e}_k) \tag{2.3.3}$$

Be related to the x_i-axis. Then, the third order tensorial law of transformation (1.3.9) gives

$$\in'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk} \in_{ijk} \tag{2.3.4}$$

where ℓ_{pi} is defined in (1.3.9). We have already check that

$$\ell_{pi}\ell_{qj}\ell_{rk} \in_{ijk} = \begin{vmatrix} \ell_{p1} & \ell_{p2} & \ell_{p3} \\ \ell_{q1} & \ell_{q2} & \ell_{q3} \\ \ell_{r1} & \ell_{r2} & \ell_{r3} \end{vmatrix} \tag{2.3.5}$$

and

$$\left[\hat{e}'_p, \hat{e}'_q, \hat{e}'_r\right] = \begin{vmatrix} \ell_{p1} & \ell_{p2} & \ell_{p3} \\ \ell_{q1} & \ell_{q2} & \ell_{q3} \\ \ell_{r1} & \ell_{r2} & \ell_{r3} \end{vmatrix} \tag{2.3.6}$$

Using (2.3.4, 2.3.5 and 2.3.6), we get

$$\in'_{pqr} = \left[\hat{e}'_p, \hat{e}'_q, \hat{e}'_r\right] = \hat{e}'_p.(\hat{e}'_q \times \hat{e}'_r) = \begin{cases} 1 \ :if \ \ p, q, r \ are \ in \ cyclic \ order \\ -1 \ :if \ \ p, q, r \ are \ in \ anticyclic \ order \\ 0 : if \ any \ two \ or \ all \ suffices \ are \ same \end{cases} \tag{2.3.7}$$

This shows that components $\in_{ijk}$ are transformed into itself under all coordinate transformations. Thus, the third order tensor $\in_{ijk}$ is an isotropic.

Property 2.6 If u_{ij} is an isotropic tensor of second order, then show that $u_{ij} = \alpha\delta_{ij}$ for some scalar α.

Proof: As the given tensor is isotropic, we have

$$u'_{ij} = u_{ij} \tag{2.3.8}$$

for all choices of the x'_i -system. In particular, we choose

$$x'_1 = x_2, x'_2 = x_3, x'_3 = x_1 \tag{2.3.9}$$

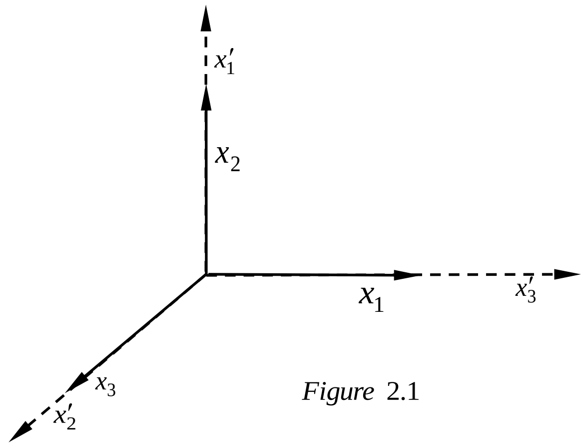


Figure 2.1

Then
$$\ell_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \tag{2.3.10}$$

and law of transformation (1.3.9), as

$$u'_{pq} = \ell_{pi}\ell_{qj}u_{ij} \tag{2.3.11}$$

Now
$$u'_{11} = \ell_{1i}\ell_{1j}u_{ij} = \ell_{1i}(\ell_{11}u_{i1} + \ell_{12}u_{i2} + \ell_{13}u_{i3})$$

$$= \ell_{1i}(0u_{i1} + \ell_{12}u_{i2} + 0u_{i3}) = \ell_{1i}\ell_{12}u_{i2}$$

$$= \ell_{12}(\ell_{11}u_{12} + \ell_{12}u_{22} + \ell_{13}u_{32}) = u_{22}$$

$\Rightarrow$
$$u'_{11} = u_{22} \tag{2.3.12}$$

Similarly,

$$u'_{22} = u_{33}, u'_{12} = u_{23}, u'_{12} = u_{23},\ u'_{23} = u_{31}, u'_{13} = u_{21}, u'_{21} = u_{32} \tag{2.3.13}$$

Now, we consider the transformation: $x'_1 = x_2, x'_2 = -x_1, x'_3 = x_3$ (2.3.14)

Then
$$\ell_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{2.3.15}$$

Using law of transformation defined in (2.3.11), we get

$$u'_{13} = u_{13} = u_{23}, u'_{23} = u_{23} = -u_{13}$$

$$\Rightarrow u'_{13} = -u_{13},\ u_{13} = 0 \text{ and } u_{23} = 0 \qquad (2.3.16)$$

using (2.3.13) and (2.3.16), we obtain

$$\ell_{ij} = \alpha\delta_{ij} \qquad \text{where } \alpha = \ell_{11} = \ell_{22} = \ell_{33} \qquad (2.3.17)$$

Note 1: If ℓ_{ijk} are components of an isotropic tensor of third order, then $\ell_{ijk} = \alpha \in_{ijk}$ for some scalar α.

Note 2: If ℓ_{ijkm} are components of a fourth-order isotropic tensor, then

$$\ell_{ijkm} = \alpha\delta_{ij}\delta_{km} + \beta\delta_{ik}\delta_{jm} + \gamma\delta_{im}\delta_{jk} \text{ for some scalars } \alpha, \beta, \gamma.$$

2.4 Contravariant tensors (vectors)

A set of N functions f_i of the N coordinates x_i are said to be the components of a **contravariant vector** if they transform according to the equation.

$$\bar{f}_i = \frac{\partial \bar{x}_i}{\partial x_j} f_j \qquad (2.4.1)$$

on change of the coordinates x_i to $\bar{x}_i$. This means that any N functions can be chosen as the components of a contravariant vector in the coordinate system x_i, and the equations (2.4.1) define the N components in the new coordinate system $\bar{x}_i$. On multiplying equations (2.4.1) by $\frac{\partial x_k}{\partial \bar{x}_i}$ and summing over the index 'i' from 1 to N, we obtain

$$\frac{\partial x_k}{\partial \bar{x}_i}\bar{f}_i = \frac{\partial x_k}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial x_j} f_j = \frac{\partial x_k}{\partial x_j} f_j = \delta_{ij} f_j = f_k \qquad (2.4.2)$$

Hence the solution of equations (2.4.1) is

$$f_k = \frac{\partial x_k}{\partial \bar{x}_i}\bar{f}_i. \qquad (2.4.3)$$

When we examine equations $d\,\bar{x}_i = \dfrac{\partial \bar{x}_i}{\partial x_r} dx_r$ (where repeated index r is called **dummy index**) we see that the differentials dx_i from the components of a contravariant vector, whose components in any other system are the differentials $d\bar{x}_i$ of the system. It follow immediately that dx_i/du is also a contravariant vector, called the **tangent vector** to the curve $x_i = x_i(u)$.

Consider now a further change of coordinates $x_i' = g_i(\bar{x}_1, \bar{x}_2, ... \bar{x}_n)$. Then the new components

$$f_i' = \frac{\partial x_i'}{\partial \bar{x}_j} \bar{f}_j = \frac{\partial x_i'}{\partial \bar{x}_j} \frac{\partial \bar{x}_j}{\partial x_k} f_k = \frac{\partial x_i'}{\partial x_k} f_k \qquad (2.4.4)$$

This equation is of the same form as (2.4.1), which shows that the transformations of contravariant vectors form a group.

2.5 Covariant vectors

A set of N functions f_i of the N coordinates x_i are said to be the components of a **covariant vector** if they transform according to the equation.

$$\bar{f}_i = \frac{\partial x_j}{\partial \bar{x}_i} f_j \qquad (2.5.1)$$

on change of the coordinates x_i to $\bar{x}_i$. Any N functions can be chosen as the components of a covariant vector in the coordinate system x_i, and the equations (2.5.1) define the N components in the new coordinate system $\bar{x}_i$. On multiplying equations (2.5.1) by $\dfrac{\partial \bar{x}_i}{\partial x_k}$ and summing over the index 'i' from 1 to N, we obtain

$$\frac{\partial \bar{x}_i}{\partial x_k} \bar{f}_i = \frac{\partial \bar{x}_i}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} f_j = \frac{\partial x_j}{\partial x_k} f_j = \delta_{jk} f_j = f_k \qquad (2.5.2)$$

Since, $\dfrac{\partial \Gamma}{\partial \bar{x}_i} = \dfrac{\partial \Gamma}{\partial x_j} \dfrac{\partial x_j}{\partial \bar{x}_i}$, it follows immediately from (2.5.1) that the quantities $\dfrac{\partial \Gamma}{\partial x_i}$ are the components of a covariant vector, whose components in any other system are the corresponding partial derivatives $\frac{\partial \Gamma}{\partial x_i}$. Such a covariant vector is called the gradient of Γ.

We now show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to transformations of the type

$$\bar{x}_i = a_{im} x_m + b_i, \tag{2.5.3}$$

where b_i are N constants which do not necessarily form the components of a contravariant vector and a_{im} are constants (not necessary forming a tensor) such that

$$a_{ir} a_{im} = \delta_{rm} \tag{2.5.4}$$

We multiply equations (2.5.3) by a_{ir} and sum over the index i from 1 to N and obtain

$$x_r = a_{ir} \bar{x}_i - a_{ir} b_i .$$

Thus,

$$\frac{\partial \bar{x}_i}{\partial x_j} = \frac{\partial x_j}{\partial \bar{x}_i} = a_{ij} \tag{2.5.5}$$

This shows that the equations (2.4.1) and (2.5.1) define the same type of entity.

Books Recommended:

4. **Y. C. Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey,1965.
5. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
6. **Barry Spain** Tensor Calculus A Concise Course, Dover Publication, INC. Mineola, New York.

CHAPTER-III

APPLICATONS OF TENSOR

3.1 EIGENVALUES AND EIGEN VACTORS

Definition: Let u_{ij} be a **second order** symmetric tensor. A scalar λ is called an **eigenvalue** of the tensor u_{ij} if there exists a non-zero vector v_i such that

$$u_{ij}v_j = \lambda v_i \qquad \forall i, j = 1, 2, 3 \tag{3.1.1}$$

The non-zero vector v_i is then called an eigenvector of tensor u_{ij} corresponding to the eigen value λ. We observe that every (**non-zero**) scalar multiple of an eigenvector is also an eigen vector.

Property 3.1 Show that it is always possible to find three mutually orthogonal eigenvectors of a second order symmetric tensor.

Proof. Let u_{ij} be a second order symmetric tensor and λ be an eigen value of u_{ij}. Let v_i be an eigenvector corresponding to λ. Then

$$u_{ij}v_j = \lambda v_i \tag{3.1.2}$$

or

$$(u_{ij} - \lambda\delta_{ij})v_j = 0 \tag{3.1.3}$$

This is a set of three homogeneous simultaneous linear equations in three unknown v_1, v_2, v_3. These three equations are

$$\left.\begin{aligned} (u_{11} - \lambda)v_1 + u_{12}v_2 + u_{13}v_3 &= 0 \\ u_{21}v_1 + (u_{22} - \lambda)v_2 + u_{23}v_3 &= 0 \\ u_{31}v_1 + u_{32}v_2 + (u_{33} - \lambda)v_3 &= 0 \end{aligned}\right\} \tag{3.1.4}$$

This set of equations possesses a non-zero solution when

$$\begin{vmatrix} u_{11}-\lambda & u_{12} & u_{13} \\ u_{21} & u_{22}-\lambda & u_{23} \\ u_{31} & u_{32} & u_{33}-\lambda \end{vmatrix} = 0 \tag{3.1.5}$$

or
$$\left| u_{ij} - \lambda\delta_{ij} \right| = 0 \tag{3.1.6}$$

expanding the determinant in (3.1.6), we find

$$\begin{aligned} (u_{11}-\lambda)&\left[(u_{22}-\lambda)(u_{33}-\lambda)-u_{32}u_{23}\right] \\ &-u_{12}\left[u_{12}(u_{33}-\lambda)-u_{31}u_{23}\right] \\ &+u_{13}\left[u_{12}u_{32}-u_{31}(u_{22}-\lambda)\right]=0 \end{aligned}$$

or
$$\begin{aligned} &-\lambda^3+(u_{11}+u_{22}+u_{33})\lambda^2 \\ &-(u_{11}u_{22}+u_{22}u_{33}+u_{33}u_{11}-u_{23}u_{32}-u_{31}u_{13}-u_{12}u_{21})\lambda \\ &+\left[u_{11}(u_{22}u_{33}-u_{23}u_{32})-u_{12}(u_{21}u_{33}-u_{31}u_{23})+u_{13}(u_{21}u_{32}-u_{31}u_{22})\right]=0 \end{aligned} \tag{3.1.7}$$

we write (3.1.7) as

$$-\lambda^3+\lambda^2 I_1-\lambda I_2+I_3=0 \tag{3.1.8}$$

where
$$I_1 = u_{11}+u_{22}+u_{33}=u_{ii}$$

$$I_2 = u_{11}u_{22}+u_{22}u_{33}+u_{33}u_{11}-u_{12}u_{21}-u_{23}u_{32}-u_{13}u_{31}=\frac{1}{2}\left[u_{ii}u_{jj}-u_{ij}u_{ji}\right]$$

$$I_3 = \left|u_{ij}\right| = \in_{ijk}\, u_{i1}u_{j2}u_{k3} \tag{3.1.9}$$

Equation (3.1.8) is a cubic equation in λ .Therefore it has three roots, say $\lambda_1,\lambda_2,\lambda_3$ which may not be distinct (real or imaginary). These roots (which are scalar) are the three eigenvalues of the symmetric tensor u_{ij} .

Further
$$\lambda_1+\lambda_2+\lambda_3=I_1 \tag{3.1.10}$$

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=I_2 \tag{3.1.11}$$

$$\lambda_1\lambda_2\lambda_3=I_3 \tag{3.1.12}$$

Each root λ_i , when substituted in equation (3.1.4), gives a set of three linear equations (homogeneous) which are not all independent. By discarding one of equations and using the condition

$$v_1^2 + v_2^2 + v_3^2 = 1 \tag{3.1.13}$$

for unit vectors, the eigenvector $\vec{v}_i$ is determined.

Property 3.2 Eigen values of a real symmetric tensor u_{ij} are real.

Proof. Let λ be eigenvalue with corresponding eigenvector v_j.

Then
$$u_{ij} v_j = \lambda v_i \tag{3.1.14}$$

Taking the complex conjugate on both sides of (3.1.14), we find

$$\bar{u}_{ij} \bar{v}_j = \bar{\lambda} \bar{v}_i$$

$$u_{ij} \bar{v}_j = \bar{\lambda} \bar{v}_i \tag{3.1.15}$$

since u_{ij} is a real tensor. Now

$$\begin{aligned} u_{ij} \bar{v}_j v_i &= (u_{ij} \bar{v}_j) v_i \\ &= (\bar{\lambda}\, \bar{v}_j) v_i \\ &= (\bar{\lambda}\, \bar{v}_i) v_i \end{aligned} \tag{3.1.16}$$

Taking complex conjugate of (3.1.16) both side

$$\begin{aligned} \overline{u_{ij} \bar{v}_j v_i} &= \bar{u}_{ij} v_j \bar{v}_i \\ &= u_{ij} v_j \bar{v}_i \quad \text{(by changing the role of } i \text{ and } j\text{)} \\ &= u_{ji} v_i \bar{v}_j \\ &= u_{ij} \bar{v}_j v_i \end{aligned} \tag{3.1.17}$$

This shows that quantity $u_{ij} \bar{v}_j v_i$ is real. Hence $\bar{\lambda} \bar{v}_i v_i$ is real. Since $\bar{v}_i v_i$ is always real, it follows that $\bar{\lambda}$ is real.

Property 3.3 Eigen vector corresponding to two distinct eigen values of the sysmmetric tensor u_{ij} are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of u_{ij}. Let A_i and B_i be the corresponding non-zero eigenvectors. Then

$$u_{ij} A_j = \lambda_1 A_i \, , u_{ij} B_j = \lambda_1 B_i \tag{3.1.18}$$

We obtain

$$u_{ij}A_jB_i = \lambda_1 A_i B_i\,,\; u_{ij}B_jA_i = \lambda_2 A_i B_i \qquad (3.1.19)$$

Interchanging the role of i and j

$$u_{ij}A_jB_i = u_{ji}A_iB_j = u_{ij}B_jA_i \qquad (3.1.20)$$

From (3.1.19) and (3.1.20), we get

$$\lambda_1 A_i B_i = \lambda_2 A_i B_i$$

$$(\lambda_1 - \lambda_2)A_i B_i = 0$$

$$\Rightarrow \qquad A_i B_i = 0 \qquad (\because\ \lambda_1 \neq \lambda_2) \qquad (3.1.21)$$

Hence, eigenvectors A_i and B_i are mutually orthogonal. This completes the proof.

Note: Now we consider various possibilities about eigenvalues λ_1, λ_2, λ_3.

Case 1: if $\lambda_1 \neq \lambda_2 \neq \lambda_3$, i.e., when all eigenvalues are different and real. Then, by property 3.3, three eigenvectors corresponding to λ_i are mutually orthogonal. Hence the results holds.

Case 2: if $\lambda_1 \neq \lambda_2 = \lambda_3$. Let $\vec{v}_{1i}$ be the eigenvector of the tensor u_{ij} corresponding to the eigenvalue λ_1 and $\vec{v}_{2i}$ be the eigenvector corresponding to λ_2. Then

$$\vec{v}_{1i} \cdot \vec{v}_{2i} = 0 \qquad (3.1.22)$$

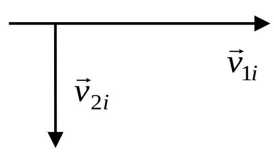


figure 3.1

Let $\vec{p}_i$ be a vector orthogonal to both $\vec{v}_{1i}$ and $\vec{v}_{2i}$. Then

$$\vec{p}_i \cdot \vec{v}_{1i} = \vec{p}_i \cdot \vec{v}_{2i} = 0 \qquad (3.1.23)$$

and $$u_{ij}\vec{v}_{1j} = \lambda_1 \vec{v}_{1i}\,,\; u_{ij}\vec{v}_{2j} = \lambda_2 \vec{v}_{2i} \qquad (3.1.24)$$

Let $$u_{ij}p_j = q_j = \text{a tensor of order 1} \qquad (3.1.25)$$

We shall show that p_i and q_i are parallel.

Now $$q_i\vec{v}_{1i} = u_{ij}p_j\vec{v}_{1i}$$

$$= u_{ji}p_i\vec{v}_{1j} \quad \text{(By interchanging the role of } i \text{ and } j\text{)}$$

$$= \lambda_1 p_i\vec{v}_{1j} = 0 \tag{3.1.26}$$

Similarly, $$q_i\vec{v}_{2i} = 0 \tag{3.1.27}$$

Thus, q_i is orthogonal to both orthogonal eigenvectors $\vec{v}_{1i}$ and $\vec{v}_{2i}$. Thus q_i must be parallel to p_i. So, we write

$$u_{ij}p_i = q_i = \alpha\, p_i \tag{3.1.28}$$

for some scalar α.

Relation (3.1.28) shows that α must be an eigenvalue and p_i must be the corresponding eigenvector of u_{ij}.

$$\vec{v}_{3i} = \frac{p_i}{|p_i|} \tag{3.1.29}$$

Since u_{ij} has only three eigenvalues λ_1, $\lambda_2 = \lambda_3$, so α must be equal to $\lambda_2 = \lambda_3$. Thus $\vec{v}_{3i}$ is an eigenvector which is orthogonal to both $\vec{v}_{1i}$ and $\vec{v}_{2i}$, where $\vec{v}_{1i} \perp \vec{v}_{2i}$. Thus, there exists three mutually orthogonal eigenvectors.

Further, let $\vec{w}_i$ be any vector which lies in the plane containing the two eigenvectors $\vec{v}_{2i}$ and $\vec{v}_{3i}$ corresponding to the repeated eigenvalues. Then

$\vec{w}_i = k_1\vec{v}_{2i} + k_2\vec{v}_{3i}$ for some scalars k_1 and k_2 and

$$\vec{w}_i \cdot \vec{v}_{1i} = k_1\vec{v}_{2i}\cdot\vec{v}_{1i} + k_2\vec{v}_{3i}\cdot\vec{v}_{1i} = 0 \tag{3.1.30}$$

and $$u_{ij}\vec{w}_i = u_{ij}(k_1\vec{v}_{2i} + k_2\vec{v}_{3i})$$

$$= k_1u_{ij}\vec{v}_{2i} + k_2u_{ij}\vec{v}_{3i}$$

$$= k_1\lambda_2\vec{v}_{2i} + k_2\lambda_3\vec{v}_{3i} \qquad (\lambda_2 = \lambda_3)$$

$$= \lambda_2(k_1\vec{v}_{2i} + k_2\vec{v}_{3i}) = \lambda_2 w_i \tag{3.1.31}$$

Thus w_i is orthogonal to $\vec{v}_{1i}$ and w_i is an eigenvector corresponding to λ_2. Hence, any two orthogonal vectors those lie on the plane normal to $\vec{v}_{1i}$ can be chosen as the other two eigenvectors of u_{ij}.

Case 3: if $\lambda_1 = \lambda_2 = \lambda_3$

In this case, the cubic equation in λ becomes

$$(\lambda - \lambda_1)^3 = 0 \qquad (3.1.32)$$

or

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_1 - \lambda & 0 \\ 0 & 0 & \lambda_1 - \lambda \end{vmatrix} = 0 \qquad (3.1.33)$$

Comparing it with equation (3.1.6), we have

$$u_{ij} = 0 \quad \text{for } i \neq j$$

and

$$u_{11} = u_{22} = u_{33} = \lambda_1$$

Thus,

$$u_{ij} = \lambda_1 \delta_{ij} \qquad (3.1.34)$$

Let $\vec{v}_i$ be any non-zero vector. Then

$$u_{ij}\vec{v}_j = \lambda_1 \delta_{ij} \vec{v}_j$$

$$= \lambda_1 \vec{v}_i \qquad (3.1.35)$$

This shows that $\vec{v}_i$ is an eigenvector corresponding to λ_1. Thus, every non-zero vector in space is an eigenvector which corresponds to the same eigenvalue λ_1. Of these vectors, we can certainly choose (at least) there vectors $\vec{v}_{1i}, \vec{v}_{2i}, \vec{v}_{3i}$ that are mutually orthogonal. Thus, in every case, there exists (at least) three mutually orthogonal eigenvectors of u_{ij}.

Example 1.Consider a second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

It is clear, the tensor u_{ij} is not symmetric. We shall find eigenvalues and eigenvectors of u_{ij}.

Solution. The characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

or $\quad (1-\lambda)[(2-\lambda)(3-\lambda)-2]-1[2-2(2-\lambda)]=0$

or $\quad (1-\lambda)(2-\lambda)(3-\lambda)=0$

Hence, eigenvalues are $\lambda_1 = 1, \lambda_2 = 2,\ \lambda_3 = 3$, all are different. (3.1.36)

We find that an unit eigenvector corresponding to $\lambda = 1$ is $\hat{v}_{1i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, the unit vector corresponding to $\lambda = 2$ is $\hat{v}_{2i} = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$, the unit vector corresponding to $\lambda = 3$ is $\hat{v}_{3i} = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$. We note that $\hat{v}_{1i} \cdot \hat{v}_{2i} \neq 0, \hat{v}_{2i} \cdot \hat{v}_{3i} \neq 0, \hat{v}_{1i} \cdot \hat{v}_{3i} \neq 0$. This happens due to non-symmetry of the tensor u_{ij}.

Example 2. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Solution. We note that the tensor is symmetric. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or $$\lambda(1-\lambda)(4-\lambda)=0$$

Hence, eigenvalues are $\lambda_1 = 0, \lambda_2 = 1,\ \lambda_3 = 4$, all are different. (3.1.37)

Let $\hat{v}_{i1}$ be the unit eigenvector corresponding to eigenvalue $\lambda_1 = 0$. Then, the system of homogeneous equations is

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_{1i} \\ \hat{v}_{2i} \\ \hat{v}_{3i} \end{bmatrix} = 0 \tag{3.1.38}$$

This gives $\hat{v}_{1i} + \hat{v}_{2i} = 0$, $\hat{v}_{1i} + \hat{v}_{2i} = 0$, $\hat{v}_{3i} = 0$

We find $\hat{v}_{1i} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right)$,

Similarly, $\hat{v}_{2i} = (0,0,1)$ and $\hat{v}_{3i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$ are eigen vectors corresponding to $\lambda_2 = 1$ and $\lambda_3 = 4$, respectively, Moreover, these vector are mutually orthogonal.

Practice 1. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 2. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 3 & -2 & 0 \\ 0 & 5 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 3. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 1 & -5 & 2 \\ 1 & -3 & 1 \\ -1 & 2 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 4. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 0 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 4. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & -3 \\ 1 & 2 & -5 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 5. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 3 & 5 & 0 \\ 1 & -1 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 6. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & 5 & 0 \\ 1 & -4 & 1 \\ 1 & 6 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Books Recommended:

1. **Y. C. Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey,1965.
2. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
3. **Barry Spain** Tensor Calculus A Concise Course, Dover Publication, INC. Mineola, New York.
4. **Shanti Narayan** Text Book of Cartesian Tensors, S. Chand & Co.,1950.

CHAPTER-IV

ANALYSIS OF STRAIN

4.1 INTRODUCTION

Rigid Body: A rigid body is an ideal body such that the ***distance between every pair of its points remains unchanged under the action of external forces***. The possible displacementsin a ***rigid body***are translation and rotation. These displacements are called rigid displacements. In translation, each point of the rigid body moves in a fixed direction. In rotation about a line, every point of the body (rigid) moves in a circular path about the line in a plane perpendicular to the line.

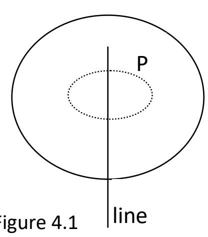


Figure 4.1

In a rigid body motion, there is a uniform motion throughout the body.

Elastic Body: A body is called elastic if it possesses the property of recovering its original shape and size when the forces causing deformation are removed.

Continuous Body: In a continuous body, the atomistic structure of matter can be disregarded and the body is replaced a continuous mathematical region of the space whose geometrical points are identified with material points of the body.

The mechanics of such continuous elastic bodies is called mechanics of continuous. This branch covers a vast range of problem of elasticity, hydromechanics, aerodynamics, plasticity and electrodynamics, seismology, etc.

Deformation of Elastic Bodies: The change in the relative position of points in a continuous is called deformation, and the body itself is then called a strained body. The study of deformation of an elastic body is known a s the analysis of strain. The deformation of the body is due to relative movements or distortions within the body.

4.2 TRANSFORMATION OF AN ELASTIC BODY

We consider the undeformed and deformed both positions of an elastic body. Let $ox_1x_2x_3$ be mutually orthogonal Cartesian coordinates fixed in space. Let a continuous body B, referred to system $ox_1x_2x_3$, occupies the region R in the undeformed state. In the deformed state, the points of the body B will occupy some region say R'.

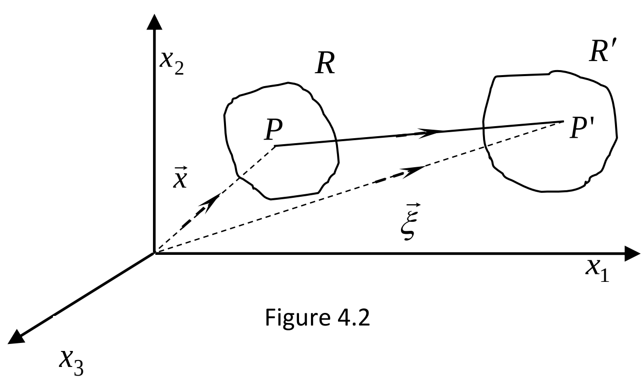


Figure 4.2

Let $P(x_1, x_2, x_3)$ be the coordinate of a material point P of the elastic body in the initial or unstained state. In the transformation or deformed state, let this material point occupies the geometric point $P'(\xi_1, \xi_2, \xi_3)$. We shall be concerned only with continuous deformation of the body from region R into the region R' and we assume that the deformation is given by the equation

$$\begin{aligned} \xi_1 &= \xi_1(x_1, x_2, x_3) \\ \xi_2 &= \xi_2(x_1, x_2, x_3) \\ \xi_3 &= \xi_3(x_1, x_2, x_3) \end{aligned} \tag{4.2.1}$$

The vector $\overrightarrow{PP'}$ is called the displacement vector of the point P and is denoted by u_i.

Thus,

$$u_i = \xi_i - x_i : i = 1, 2, 3 \tag{4.2.2}$$

or

$$\xi_i = u_i + x_i : i = 1, 2, 3 \tag{4.2.3}$$

Equation (4.2.1) expresses the coordinates of the points of the body in the transformed state in terms of their coordinates in the initial undeformed state. This type of description of deformation is known as the Lagrangian method of describing the transformation of a coordinate medium.

Another method, known as Euler's method expresses the coordinates in the undeformed state in terms of the coordinates in the deformed state.

The transformation (4.2.1) is invertible when

$$J \neq 0$$

Then, we may write

$$x_i = x_i(\xi_1, \xi_2, \xi_3) : i = 1, 2, 3 \tag{4.2.4}$$

In this case, the transformation from the region R into region R' is one to one. Each of the above description of deformation of the body has its own advantages. It is however; more convenient in the study of the mechanics of solids to use Lagrangian approach because the undeformed state of the body often possesses certain symmetries which make it convenient to use a simple system of coordinates.

A part of the transformation defined by equation (4.2.1) may represent rigid body motion. (i.e.translations and rotations) of the body as a whole. This part of the deformation leaves unchanged the length of every vector joining a pair of points within the body and is of no interest in the analysis of strain. The remaining part of the transformation (4.2.1) will be called **pure deformation**. Now, we shall learn how to distinguish between pure deformation and rigid body motions when the latter are present in the transformation equation (4.2.1)

4.3. LINEAR TRANSFORMATION OR AFFINE TRANSFORAMTION

Definition: The transformation

$$\xi_i = \xi_i(x_1, x_2, x_3)$$

is called a linear transformation or affine transformation when the function ξ_i are **linear functions**of the coordinates $x_1,\ x_2,\ x_3$. In order to distinguish between rigid motion and pure deformation, we consider the simple case in which the transformation (4.2.1) is linear.

We assume that the general form of the linear transformation (4.2.1) is of the type

$$\left.\begin{aligned}\xi_1 &= \alpha_{10} + (\alpha_{11}+1)x_1 + \alpha_{12}x_2 + \alpha_{13}x_3,\\ \xi_2 &= \alpha_{20} + \alpha_{21}x_1 + (1+\alpha_{22})x_2 + \alpha_{23}x_3,\\ \xi_3 &= \alpha_{30} + \alpha_{31}x_1 + \alpha_{32}x_2 + (1+\alpha_{33})x_3,\end{aligned}\right\} \tag{4.3.1}$$

or

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \ ;\ i, j = 1,2,3 \tag{4.3.2}$$

where the coefficients α_{ij} are constants and are well known.

Equation (4.3.2) can written in the matrix form as

$$\begin{bmatrix}\xi_1-\alpha_{10}\\ \xi_2-\alpha_{20}\\ \xi_3-\alpha_{30}\end{bmatrix} = \begin{bmatrix}1+\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{4.3.3}$$

or

$$\begin{bmatrix}u_1-\alpha_{10}\\ u_2-\alpha_{20}\\ u_3-\alpha_{30}\end{bmatrix} = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{4.3.4}$$

We can look upon the matrix $(\alpha_{ij} + \delta_{ij})$ as an operator acting on the vector $\vec{x} = x_i$ to give the vector α_{i0}.

If the matrix $(\alpha_{ij} + \delta_{ij})$ is non-singular, then we obtain

$$\left(\alpha_{ij} + \delta_{ij}\right)^{-1}\begin{bmatrix}\xi_1-\alpha_{10}\\ \xi_2-\alpha_{20}\\ \xi_3-\alpha_{30}\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{4.3.5}$$

which is also linear as inverse of a linear transformation is linear. In fact, matrix algebra was developed basically to express linear transformations in a concise and lucid manner.

Example1. Sum of two linear transformations is a linear transformation.

Solution. Let

and $$\left.\begin{aligned} \xi_i &= \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \\ \varsigma_i &= \beta_{i0} + (\beta_{ij} + \delta_{ij})x_j \end{aligned}\right\} ; \quad i, j = 1,2,3 \qquad (4.3.6)$$

are two linear transformation and suppose $\zeta_i = \xi_i + \varsigma_i$.

Now,

$$\begin{aligned} \zeta_i &= \xi_i + \varsigma_i \\ &= (\alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j) + (\beta_{i0} + (\beta_{ij} + \delta_{ij})x_j) \qquad \because (\delta_{ij}x_j = x_i) \\ &= (\alpha_{i0} + \beta_{i0}) + 2\{(\alpha_{ij} + \beta_{ij})/2 + \delta_{ij}\}x_j \end{aligned}$$

$$\zeta_i = \vartheta_{i0} + (\vartheta_{ij} + \delta_{ij})x_j \qquad (4.3.7)$$

where $\vartheta_{ij} = \alpha_{ij} + \beta_{ij}$; $i, j = 1,2,3$ relation (4.3.7) is a linear transformation by definition of linear transformation as defined in relation (4.3.2). Hence sum or difference of linear transformation is linear transformation.

Practice1. Show that product of two linear transformation is a linear transformation which is not commutative

Example2. Under a linear transformation, a plane is transformed into a plane.

Solution. Let

$$lx + my + mz + c = 0 \qquad (4.3.8)$$

be an equation of plane which is not passes through (0,0,0) in the undeformed state and (l, m, n) are direction ratios of the plane. Let

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (4.3.9)$$

Be the linear transformation of points. Let its inverse be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \qquad (4.3.10)$$

Then the equation of the plane is transformed to

$$l(L_1\xi_1 + M_1\xi_2 + N_1\xi_3) + m(L_2\xi_1 + M_2\xi_2 + N_2\xi_3) + n(L_3\xi_1 + M_3\xi_2 + N_3\xi_3) + c = 0 \quad (4.3.11)$$

or $(lL_1 + mL_2 + nL_3)\xi_1 + (lM_1 + mM_2 + nM_3)\xi_2 + (lN_1 + mN_2 + nN_3)\xi_3 + c = 0$

$$\alpha\xi_1 + \beta\xi_2 + \gamma\xi_3 + c = 0 \qquad (4.3.12)$$

Relation (4.3.12) is again an equation of a plane in terms of new coordinates (ξ_1, ξ_2, ξ_3) . Hence the result.

Practice2.A linear transformation carries line segments into line segments. Thus, it is the linear transformation that allows us to assume that a line segment is transformed to a line segment and not to a curve.

4.4. SMALL/ INFINITESIMAL LINEAR DEFORMATIONS

Definition: A linear transformation of the type $\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j;\ i, j = 1,2,3$ is said to be a small linear transformation of the coefficients α_{ij} are so small that their products can be neglected in comparison with the linear terms.

Note 1: The product of two small linear transformations is small linear transformation which is **commutative** and the product transformation is obtained by superposition of the original transformations and the result is independent of the order in which the transformations are performed.

Note 2: In the study of fine deformation (as compared to the infinitesimal affine deformation), the principle of superposition of effects and the independent of the order of transformations are no longer valid.

If a body is subjected to large linear transformation, a straight line element seldom remains straight. A curved element is more likely to result. The linear transformation then expresses the transformation of elements P_1P_2 to the tangent $P_1'T_1'$ to the curve at P_1' for the curve itself.

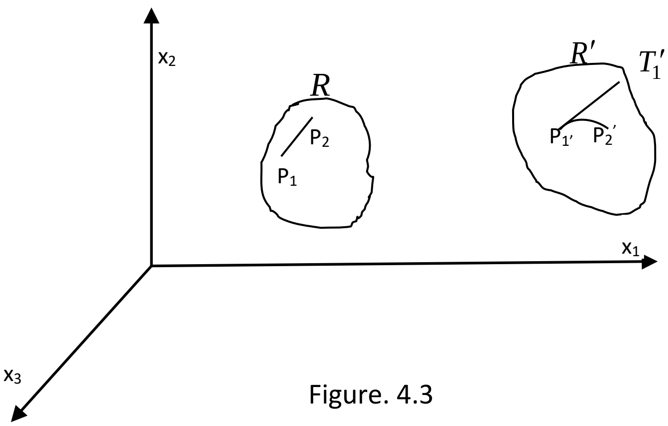


Figure. 4.3

For this reason, a linear transformation is sometimes called linear tangent transformation. It is obvious that the smaller the element P_1P_2, the better approximation of $P_1'P_2'$ by its tangent $P_1'T_1'$.

4.5 HOMOGENEOUS DEFORMATION

Suppose that a body B, occupying the region R in the undeformed state, is transformed to the region R' under the linear transformation.

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \qquad (4.5.1)$$

referred to orthogonal Cartesian system $ox_1x_2x_3$. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the unit base vectors directed along the coordinate axes x_1, x_2, x_3.

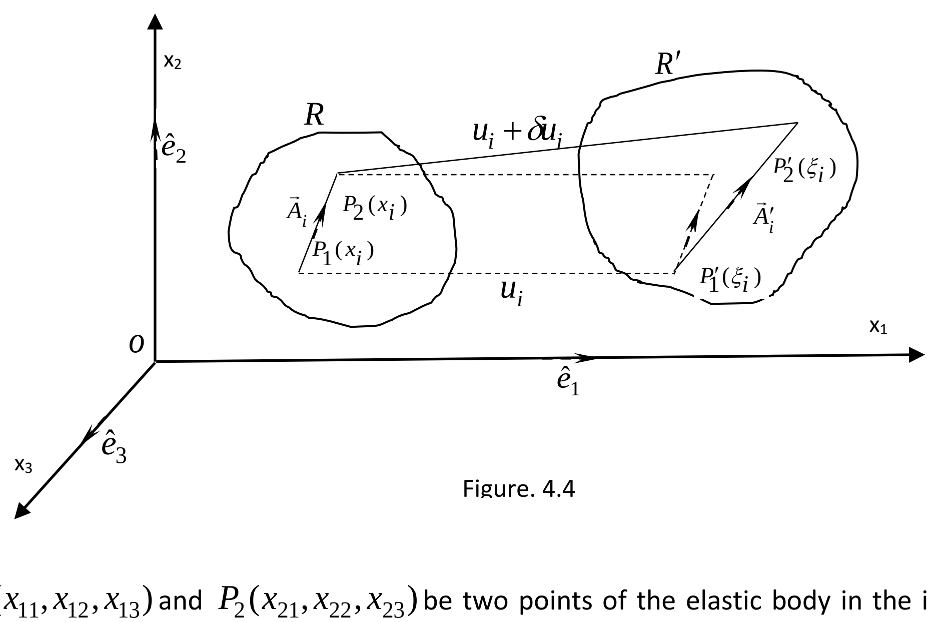


Figure. 4.4

Let $P_1(x_{11}, x_{12}, x_{13})$ and $P_2(x_{21}, x_{22}, x_{23})$ be two points of the elastic body in the initial state. Let the positions of these points in the deformed state, due to linear transformation (4.3.2), be

$P_1'(\xi_{11}, \xi_{12}, \xi_{13})$ and $P_2'(\xi_{21}, \xi_{22}, \xi_{23})$. Since transformation (4.3.2) is linear, so the line segment $\overline{P_1P_2}$ is transformed into a line segment $\overline{P_1'P_2'}$.

Let the vector $\overrightarrow{P_1P_2}$ has component A_i and vector $\overrightarrow{P_1'P_2'}$ has components A_i'. Then

$$\overrightarrow{P_1P_2} = A_i\hat{e}_i, \qquad A_i = x_{2i} - x_{1i} \quad (4.5.2)$$

and

$$\overrightarrow{P_1'P_2'} = A_i'\hat{e}_i, \qquad A_i' = \xi_{2i} - \xi_{1i} \tag{4.5.3}$$

Let
$$\delta A_i = A_i' - A_i \tag{4.5.4}$$

be change in vector A_i. The vectors A_i and A_i', in general, differ in direction and magnitude. From equations (4.5.1), (4.5.2) and (4.5.3), we write

$$\begin{aligned}
A_i' &= \xi_{2i} - \xi_{1i} \\
&= \left[\alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_{2j}\right] - \left[\alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_{1j}\right] \\
&= (x_{2i} - x_{1i}) + \alpha_{ij}(x_{2j} - x_{2i}) \\
&= A_i + \alpha_{ij}A_j \\
A_i' - A_i &= \alpha_{ij}A_j \\
\delta A_i &= \alpha_{ij}A_j
\end{aligned} \tag{4.5.5}$$

Thus, the linear transformation (4.3.2) changes the vector A_i into vector A_i' where

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} 1+\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{4.5.6}$$

or

$$\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{4.5.7}$$

Thus, the linear transformation (4.3.2) or (4.5.6) or (4.5.7) are all equivalent. From equation (4.5.6), it is clear that two vectors A_i and B_i whose components are equal transform into two vectors A_i' and B_i' whose components are again equal. Also two parallel vectors transform into parallel vectors transformation into parallel vectors.

Hence, two equal and similarly oriented rectilinear polygons located in different part of the region R will be transformed into equal and similarly oriented polygons in the transformed region R' under the linear transformation (4.5.1).

Thus, the different parts of the body B, when the latter is subjected to the linear transformation (4.5.1), experience the same deformation independent of the position of the part of the body.

For this reason, the linear deformation (4.5.1) is called a homogeneous deformation.

Theorem: Prove that the necessary and sufficient condition for an infinitesimal affine transformation

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j$$

to represent a rigid body motion is that the matrix α_{ij} is skew-symmetric

Proof: With reference to an orthogonal system $ox_1x_2x_3$ fixed in space, let the line segment $\overline{P_1P_2}$ of the body in the undeformed state be transferred to the line segment $\overline{P_1'P_2'}$ in the deformed state due to infinitesimal affine transformation

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j \tag{4.5.8}$$

In which α_{ij} are known as constants. Let A_i be vector $\overline{P_1P_2}$ and A_i' be the vector $\overline{P_1'P_2'}$

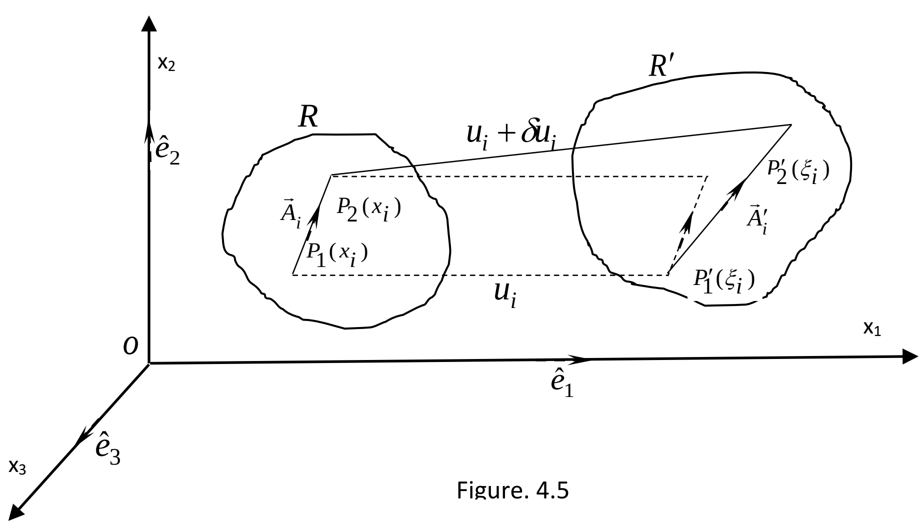


Figure. 4.5

Then

$$A_i = x_i - x_{io},\; A_i' = \xi_i - \xi_{i0} \tag{4.5.9}$$

Let

$$\delta A_i = A_i' - A_i \tag{4.5.10}$$

From (4.5.9) and (4.5.10), we find

$$A_i' = \xi_i - \xi_{i0}$$

$$= (\alpha_{i0} + \alpha_{ij}x_j + x_i) - (\alpha_{i0} + \alpha_{ij}x_{jo} + x_{i0})$$

$$= (x_i - x_{i0}) + \alpha_{ij}(x_j - x_{j0})$$

$$= A_i + \alpha_{ij}A_j$$

This gives

$$\delta A_i = A'_{i'} - A_i = \alpha_{ij}A_j \, . \qquad (4.5.11)$$

Let A denotes the length of the vector. Then

$$A = |A_i| = \sqrt{A_i A_i} = \sqrt{A_1^{\ 2} + A_2^{\ 2} + A_3^{\ 2}} \qquad (4.5.12)$$

Let δA denotes the change in length A due to deformation. Then

$$\delta A = |A_i'| - |A_i| \qquad (4.5.13)$$

It is obvious that $\delta A \neq |\delta A_i|$,but

$$\delta A = \sqrt{(A_i + \delta A_i)(A_i + \delta A_i)} - \sqrt{A_i A_i}$$

This imply

$$(A + \delta A)^2 = (A_i + \delta A_i)(A_i + \delta A_i)$$

Or

$$(\delta A)^2 + 2A\delta A = (\delta A_i)(\delta A_i) + 2A_i(\delta A_i) \qquad (4.5.14)$$

Since the linear transformation (4.5.8) or (4.5.11) is small,the term $(\delta A)^2$ and $(\delta A_i)(\delta A_i)$ are to be neglected in (4.5.14). Therefore,after neglecting these terms in(4.5.14), we write

$$2A\delta A = 2A_i \delta A_i \, ,$$

or

$$A\delta A = A_i \delta A_i = A_1\delta A_1 + A_2\delta A_2 + A_3\delta A_3 \qquad (4.5.15)$$

Using (4.5.11), equation (4.5.15) becomes

$$A\delta A = A_i(\alpha_{ij}A_j)$$

$$=\alpha_{ij}A_iA_j$$

$$=\alpha_{11}A_1^{\ 2}+\alpha_{22}A_2^{\ 2}+\alpha_{33}A_3^{\ 3}+(\alpha_{12}+\alpha_{21})A_1A_2+(\alpha_{13}+\alpha_{31})A_3A_1+(\alpha_{23}+\alpha_{32})A_2A_3 \quad (4.5.16)$$

Case 1: suppose that the infinitesimal linear transformation (4.5.9) represent a rigid body motion.Then, the length of the vector A_i before deformation and after deformation remains unchanged.

That is

$$\delta A=0 \qquad (4.5.17)$$

For all vectors A_i

Using (4.5.16), we then get

$$\alpha_{11}A^2{}_1+\alpha_{22}A_2^{\ 2}+\alpha_{33}A_3^{\ 2}+(\alpha_{12}+\alpha_{21})A_1A_2+(\alpha_{23}+\alpha_{32})A_2A_3+(\alpha_{13}+\alpha_{31})A_3A_1 \quad (4.5.18)$$

For all vectors A_i .This is possible only when

$$\alpha_{11}=\alpha_{22}=\alpha_{33}=0,$$

$$\alpha_{12}+\alpha_{21}=\alpha_{13}+\alpha_{31}=\alpha_{23}+\alpha_{32}=0 \quad ,$$

i.e., $\alpha_{ij}=-\alpha_{ji}$, for all i& j (4.5.19)

i.e. , the matrix α_{ij} is skew- symmetric.

Case 2:suppose α_{ij} is skew-symmetric. Then, equation (4.5.16) shows that

$$A\delta A=0 \qquad (4.5.20)$$

For all vectors A_i . This implies

$$\delta A=0 \qquad (4.5.21)$$

For all vectors A_i

This shows that the transformation (4.5.8) represents a rigid body linear small transformation.

This completes the proof of the theorem.

Remarks :when the quantities α_{ij} are skew –symmetric , then the linear infinitesimal transformation.

$$\delta A_i = \alpha_{ij} A_j$$

Equation (4.5.19) takes the form

$$\delta A_1 = -\alpha_{21} A_2 + \alpha_{13} A_3$$

$$\delta A_2 = \alpha_{21} A_1 - \alpha_{32} A_3$$

$$\delta A_2 = -\alpha_{13} A_1 + \alpha_{32} A_2 \qquad (4.5.22)$$

Let $w_1 = \alpha_{32} = -\alpha_{23}$

$$w_2 = \alpha_{13} = -\alpha_{31}$$

$$w_3 = \alpha_{21} = -\alpha_{12} \qquad (4.5.23)$$

Then, the transformation (4.5.22) can be written as the vectors product

$$\overline{\delta A} = \overline{w} \times \overline{A}, \qquad (4.5.24)$$

Where $\overline{w} = w_i$ is the infinitesimal rotation vector. Further

$$\delta A_i = A_i' - A_i$$

$$= (\xi_i - \xi_i^0) - (x_i - x_i^0)$$

$$== \delta x_i - \delta x_i^0 \qquad (4.5.25)$$

This yield

$$\delta x_i = \delta x_i^0 + \delta A_i,$$

$\delta x_i = \delta x_i^0 + \delta A_i$,

or

$$\delta x_i = \delta x_i^0 + (\overline{w} + \overline{A}) \qquad (4.5.26)$$

Here, the quantities

$$\delta x_i^0 = \xi_i^0 - x_i^0$$

arethe components of the displacement vector representing the translation of the point P^0 and the remaining terms of (4.5.26) represent rotation of the body about the point P^0 .

4.6 PURE DEFORMATION AND COMPONENTS OF STRAIN TENSOR

We consider the infinitesimal linear transformation

$$\delta A_i = \alpha_{ij} A_j \quad (4.6.1)$$

Let

$$w_{ij} = 1/2\,(\alpha_{ij} - \alpha_{ji}) \quad (4.6.2)$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) \quad (4.6.3)$$

Then the matrix w_{ij} is anti-symmetric while e_{ij} is symmetric.

Moreover,

$$\alpha_{ij} = e_{ij} + w_{ij} \quad (4.6.4)$$

and this decomposition of α_{ij} as a sum of s symmetric and skew-symmetric matrices is unique.

From (4.6.1) and (4.6.4), we write

$$\delta A_i = e_{ij} A_j + w_{ij} A_j \quad (4.6.5)$$

This shows that the transformation of the components of a vector A_i given by

$$\delta A_i = w_{ij} A_j \quad (4.6.6)$$

represent rigid body motion with the component of rotation vector w_i given by

$$w_1 = w_{32}, w_2 = w_{13}, w_3 = w_{21} \quad (4.6.7)$$

and the transformation

$$\delta A_i = e_{ij} A_j \,, \quad (4.6.8) \text{ with}$$

$$e_{ij} = e_{ji} \,, \quad (4.6.9)$$

represents a pure deformation.

STRAIN COMPONENTS: The symmetric coefficients, e_{ij}, in the pure deformation

$$\delta A_i = e_{ij} A_j$$

are called the strain components.

Note (1): These components of straincharacterizepure deformation of the elastic body. Since A_j and δA_i are vectors (each is a tensor of order 1),therefore,by quotient law, the strains components e_{ij} form a tensor of order 2.

Note 2: For most materials / structures, the strains are of the order 10^{-3}, such strains certainly deserve to be called small.

Note 3: The strain components e_{11}, e_{22}, e_{33} are called normal strain components while $e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}$ are called shear strain components,

Example: For the deformation defined by the linear transformation

$$\xi_1 = x_1 + x_2, \xi_2 = x_1 - 2x_2, \xi_3 = x_1 + x_2 - x_3,$$

Find the inverse transformation of rotation and strain tensor, and axis of rotation.

Solution: The given transformation is express as

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.6.10}$$

and its inverse transformation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{4.6.11}$$

giving

$$x_1 = \frac{1}{3}(2\xi_1 + \xi_2),$$

$$x_2 = \frac{1}{3}(\xi_1 - \xi_2)$$

$$x_3 = \xi_1 - \xi_3 \tag{4.6.12}$$

comparing (4.6.10) with

$$\xi_i = (\alpha_{ij} + \delta_{ij})x_j \qquad (4.6.13)$$

We find

$$\alpha_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \qquad (4.6.14)$$

Then

$$w_{ij} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad (4.6.15)$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$$

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & -3 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix} \qquad (4.6.16)$$

and

$$\alpha_{ij} = w_{ij} + e_{ij} \qquad (4.6.17)$$

The axis of rotation is

$$\overline{w} = w_i \hat{e}_i$$

where

$$w_1 = w_{32} = \frac{1}{2},$$

$$w_2 = w_{13} = -\frac{1}{2},$$

$$w_3 = w_{21} = 0 \qquad (4.6.18)$$

4.7 GEOMETRICAL INTERPRETATION OF THE COMPONENTS OF STRAIN

Normal strain component e_{11}:

Let e_{ij} be the components of strains the pure infinitesimal linear deformation of a vector A_i is given by

$$\delta A_i = e_{ij} A_j \qquad (4.7.1)$$

with $e_{ij} = e_{ji}$.

Let edenotes the extension (or change) in length per unit length of the vector A_i with magnitude A.Then, by definition,

$$e = \frac{\delta A}{A} \qquad (4.7.2)$$

We note that e is positive or negative upon whether the material line element A_i experiences an extension or a contraction. Also, $e = 0$, if and only if the vector $\overline{A}$ retains its length during a deformation.This number e is referred to as the normal strain of the vector A_i.Since the deformation is linear and infinitesimal, we have (proved earlier)

$$A\delta A = A_i \delta A_i \ (4.7.3)$$

Or
$$\frac{\delta A}{A} = \frac{A_i \delta A_i}{A^2}$$

Now from (4.7.1) and (4.7.3), we write

$$e = \frac{\delta A}{A} = \frac{A_i \delta A_i}{A^2}.$$

This implies

$$e = \frac{1}{A^2}\left[e_{11}A_2{}^2 + e_{22}A_2{}^2 + e_{33}A_3{}^2 + 2e_{12}A_1A_2 + 2e_{13}A_1A_3 + 2e_{23}A_2A_3\right] \qquad (4.7.4)$$

Since $e_{ij} = e_{ji}$

In particular,we consider the case in which the vector A_i in the underformed state is parallel to the x_1-axis. Then

$$A_1 = A, A_2 = A_3 = 0 \qquad (4.7.5)$$

Using (4.7.5), equation (4.7.4) gives

$$e = e_{11}. \qquad (4.7.6)$$

Thus, the component e_{11} of the strain tensor, to a good approximation to the extension or change in length of a material line segment (or fiber of the material) originally placed parallel to the x_1-axis in the undeformed state.

Similarly, normal strains e_{22} and e_{33} are to be interpreted.

Illustration: let $e_{ij} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then all unit vectors parallel to the x_1-axis will be extended by an amount e_{11}. In this case, one has a homogeneous deformation of material in the direction of the x_1-axis. A cube of material whose material whose edges before deformation are L unit along will become (after deformation due to e_{ij}) a rectangular parallelepiped whose dimension in the direction of the x_2 - and x_3 - axes are unchanged.

Remark: The vector

$$\overline{A} = A_i = (A, 0, 0)$$

is changed to (due to deformation)

$$\overline{A'} = (A + \delta A_1)\hat{e}_1 + \delta A\hat{e}_2 + \delta A_3\hat{e}_3$$

in which

$$\delta A_i = e_{ij} A_j = e_{i1} A_1$$

gives

Thus $$\overline{A'} = (A + e_{11}A,\ e_{12}A,\ e_{13}A)$$

this indicates that vector $A_i = (A,0,0)$ upon deformation,in general, changes its orientation also.This length of the vector due to deformation becomes $(1+e_{11})A$.

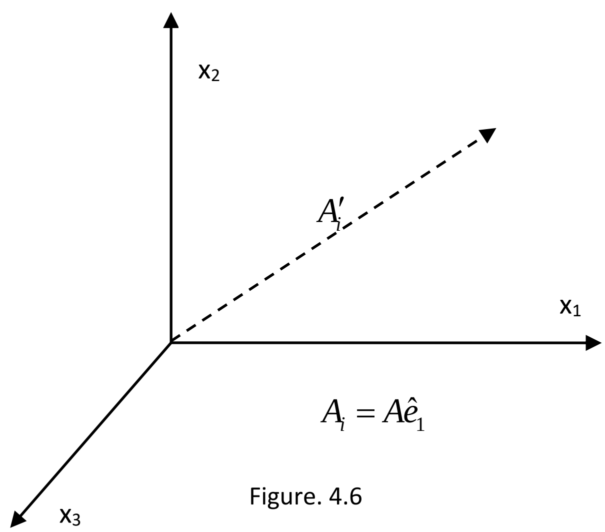


Figure. 4.6

Question: From the relation $\delta A_i = e_{ij} A_j$, find δA and δA_i for a vector lying initially along x-axis (i.e., $\overline{A} = A\hat{e}_1$) and justify the fact that $\frac{\delta A}{A} = e_{11}$. Does δA_i lie along the x-axis?

Answer: It is given that $A_i = (A,0,0)$. The given relation

$$\delta A_i = e_{ij} A_j \tag{4.7.7}$$

Gives

$$\delta A_1 = e_{11}A, \delta A_2 = e_{12}A, \delta A_3 = e_{13}A \tag{4.7.8}$$

Thus, in general, the vector δA_i does not lie along the x-axis.

Further

$$(A+\delta A) = \sqrt{\left[A(1+e_{11})^2 + (e_{12}A)^2 + (e_{13}A)^2\right]}$$

$$= A\sqrt{1+2e_{11}+e_{11}{}^2+e_{12}{}^2+e_{13}{}^2}\ . \tag{4.7.9}$$

Neglecting square terms as deformation is small, equation (4.7.9) gives

$$(A+\delta A)^2 = A^2(1+2e_{11}),$$

$$A^2 + 2A\delta A = A^2 + 2A^2 e_{11,}$$

$$2A\delta A = 2A^2 e_{11}$$

$$\frac{\delta A}{A} = e_{11}\,. \qquad (4.7.10)$$

This shows that e_{11} gives the extension of a vector (A, 0, 0) per unit length due to deformation.

Remarks: the strain components e_{ij} refer to the chosen set of coordinate axes. If the axes changed, the strain component e_{ij} will, in general, changes as per tensor transformation laws.

Geometrical interpretation of shearing Stress e_{23} :

The shearing strain component e_{23} may be interpreted by considering intersecting vectors initially parallel to two coordinate axes - x_2 -and x_3 -axis

Now, we consider in the undeformed state two vectors.

$$\overline{A} = A_2 \hat{e}_2\,,$$

$$\overline{B} = B_3 \hat{e}_3 \qquad (4.7.11)$$

directed along x_2 -and x_3 -axis, respectively.

The relations of small linear deformation are

$$\delta A_i = e_{ij} A_j\,,$$

$$\delta B_i = e_{ij} B_3\,, \qquad (4.7.12)$$

Further, the vectors A_i and B_i due to deformation become (figure 4.7)

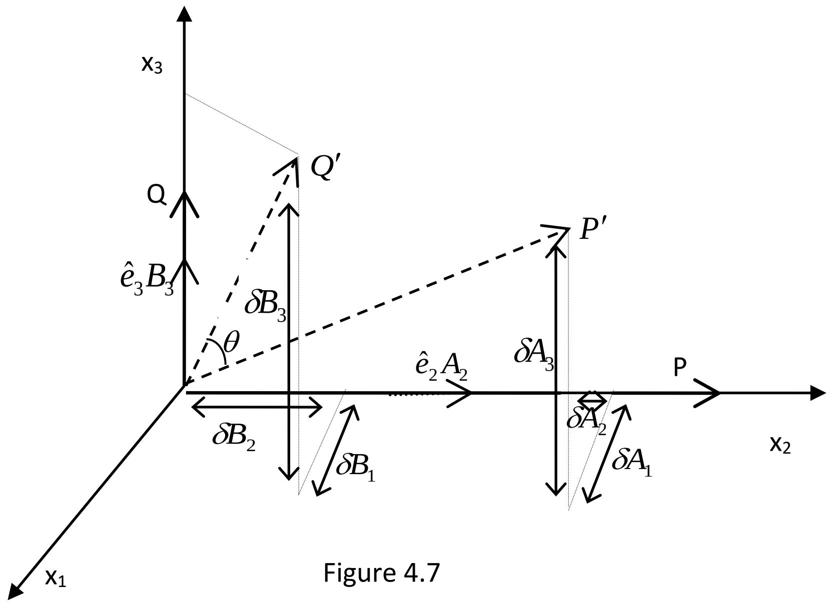


Figure 4.7

$$\overline{A'} = \delta A_1 \hat{e}_1 + (A_2 + \delta A_2)\hat{e}_2 + \delta A_3 \hat{e}_3$$
$$\overline{B'} = \delta B_1 \hat{e}_1 + \delta B_2 \hat{e}_2 + (B_3 + \delta B_3)\hat{e}_3 \quad (4.7.13)$$

Let θ be the angle between $\overline{A'}$ and $\overline{B'}$. Then

$$\cos\theta = \frac{\overline{A'}.\overline{B'}}{A'B'} = \frac{\delta A_1 \delta B_1 + (A_2 + \delta A_2)\delta B_2 + \delta A_3(B_3 + \delta B_3)}{\sqrt{(\delta A_1)^2 + (A_2 + \delta A_2)^2 + (\delta A_3)^2}\sqrt{(\delta B_1)^{12} + (\delta B_1)^2 + (B_3 \delta B_3)^2}} \quad (4.7.14)$$

Since , the deformation is small , we may neglect the product of the changes in the components of the vector A_i and B_i .Neglecting these product , equation (4.7.11) gives

$$\cos\theta = (A_2\delta B_2 + B_3\delta A_3)(A_2 + \delta A_2)^{-1}(B_3 + \delta B_3)^{-1}$$

$$= \frac{A_2\delta B_2 + B_3\delta A_3}{A_2 B_3}\left(1 + \frac{\delta A_2}{A_2}\right)^{-1}\left(1 + \frac{1 + \delta B_3}{B_3}\right)^{-1}$$

$$= \left(\frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2}\right)\left(1 - \frac{\delta A_2}{A_2}\right)\left(1 - \frac{\delta B_3}{B_3}\right),$$

Neglecting other terms,this gives

$$\cos\theta = \frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2} \quad (4.7.15)$$

Neglecting the product terms involving changes in the components of the vectors A_i and B_i.

Since in formula (4.7.15), all increments in the components of initial vectors on assuming (without loss of generality)

$$\delta A_1 = \delta A_2 \equiv 0\,,$$

And
$$\delta B_1 = \delta B_3 \equiv 0\,,$$

can be represented as shown in the figure below (it shows that vector $A_i^{'}$ and $B_i^{'}$ lie in the x_2x_3 -plane). We call that equation (4.7.13) now may be taken as

$$\overline{A'} = A_2\hat{e}_2 + \delta A_3\hat{e}_3\,,$$

$$\overline{B'} = \delta B_2\hat{e}_2 + B_3\hat{e}_3 \qquad (4.7.16)$$

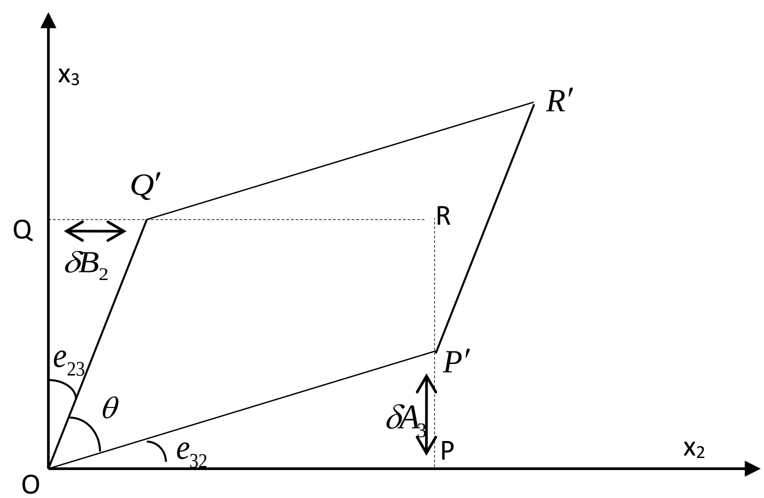


Figure 4.8

Form equation (4.7.11) and 4.7.12), we obtain

$$\delta A_3 = e_{32}A_2\,,$$

$$\delta B_2 = e_{23}B_3 \qquad (4.7.17)$$

This gives

$$e_{32} = \frac{\delta A_3}{A_2} = \tan \angle P'OP \qquad (4.7.18)$$

$$e_{23} = \frac{\delta B_2}{B_3} = TAN \angle Q'OQ \qquad (4.7.19)$$

since strain $e_{23} = e_{32}$ are small, so

$$\angle P'OP = \angle Q'OQ \cong e_{23},$$

And here

$$2e_{23} \cong 90^o - \theta = \frac{\pi}{2} - \theta \qquad (4.7.20)$$

Thus, a positive value of $2e_{23}$ represents the decrease in the right angle between the vectors A_i and B_i due to small linear deformation which were initially directed along the positive x_2 and x_3-axes.

The quantity / strain component e_{23} is called the shearing strain.

A similar interpretation can be made for the shear strain components of material arcs.

Remarks 1: By rotating the parallelogram $R'OP'Q'$ throw an angle e_{23} about the origin (in the x_2x_3 -plane), we obtain the following configurations (figure 4.9)

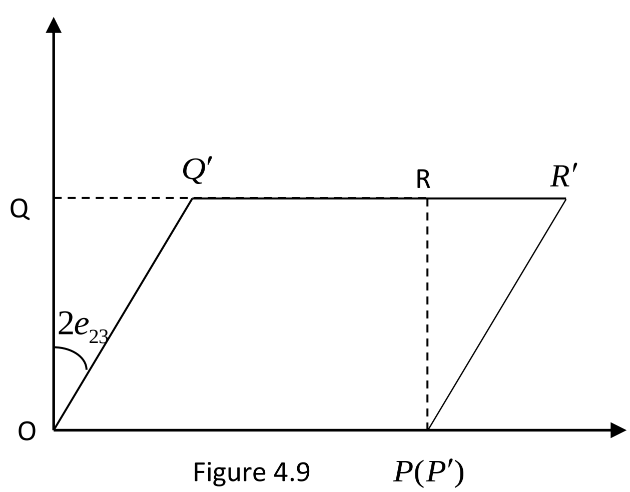


Figure 4.9

Thisfigure shows a slide or a shear of planar elements parallel to the x_1x_2 – plane.

Remarks 2: Figure shows that areas of rectangle OQRP and the parallelogram $OQ'R'P'$ are equal as they have the same height and same base in the x_2x_3-plane.

Remarks 3: For the strain tensor $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & e_{32} & 0 \end{bmatrix}$,

A cubical element is deformed into a parallelepiped and the volumes of the cube and parallelepiped remain the same.Such a small linear deformation is called a pure shear.

4.8 NORMAL AND TANGENTIAL DISPLACEMENTS

Consider a point P (x_1, x_2, x_3) of the material. Let it be moved to Q under a small linear transformation. Let the components of the displacement vector $\overline{PQ}$ be u_1, u_2, u_3. In the plane OPQ, let $\overline{PN} = \bar{n}$ be the projection of $\overline{PQ}$ on the line OPN and let $\overline{PT} = \bar{t}$ be the tangential of $\overline{PQ}$ in the plane of OPQ or PQN.

Definition: vectors $\bar{n}$ and $\bar{t}$ are, respectively, called the normal and the tangentialcomponents of the displacement of P.

Note: The magnitude n of normal displacement $\bar{n}$ is given by the dot product of vectors $\overline{OP} = (x_1, x_2, x_3)$ and $\overline{PQ} = (u_1, u_2, u_3)$.

the magnitude t of tangential vector $\bar{t}$ is given the vector product of vectors $\overline{OP}$ and $\overline{PQ}$ (this does not give the direction of $\bar{t}$).

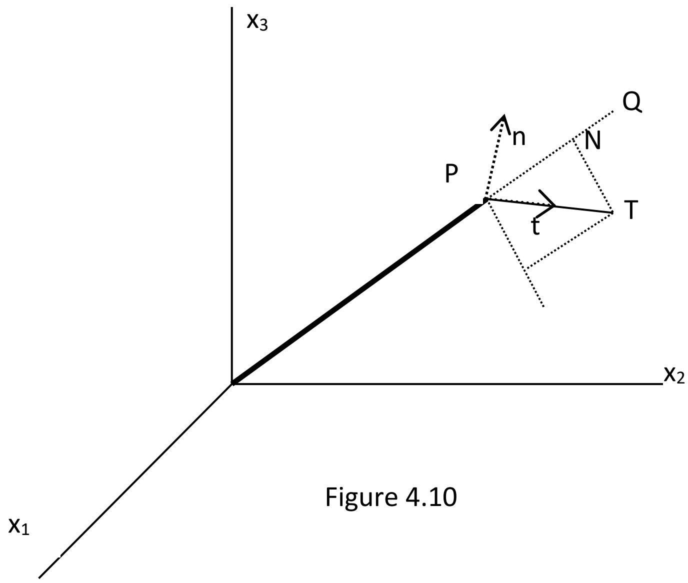


Figure 4.10

Thus

$$n = \cos\angle NPQ = \frac{\overline{OP}.\overline{PQ}}{\left|\overline{OP}\right|},$$

$$t = PQ\sin\angle NPQ = \frac{(OP)(PQ)\sin(NPQ)}{OP} = \frac{\left|\overline{OP}\times\overline{PQ}\right|}{\left|\overline{OP}\right|},$$

And

$$n^2 + t^2 = u_1^2 + u_2^2 + u_3^2 .$$

Books Recommended:

1. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

2. **Shanti Narayan** Text Book of Cartesian Tensors, S. Chand & Co.,1950.

CHAPTER-V

STRAIN QUADRIC OF CAUCHY

5.1 Strain Quadric of Cauchy

Let $P^0\left(x_1^0, x_2^0, x_3^0\right)$ be any fixed point of a continuous medium with reference axis

$0\,x_1x_2x_3$ fixed in space. We introduce a local system of axis with origin at point P^0 and with axes parallel to the fixed axes (figure 5.1)

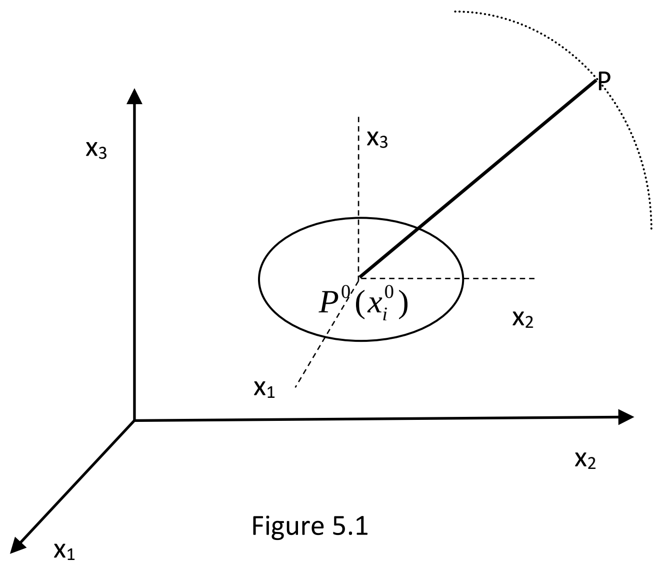


Figure 5.1

with reference to these axes, consider the equation

$$e_{ij}x_ix_j = \pm k^2 \tag{5.1.1}$$

where k is a real constant and is the strain tensor at P^0. This equation represents a quadric of Cauchy. The sign + or – in equation (5.1.1) be chosen so that the quadric surface (5.1.1) becomes a real one. The nature of this quadratic surface depends on the value of the strain e_{ij}.

If $\left|e_{ij}\right| \neq 0$, the quadratic is either an ellipsoid or a hyperboloid.

If $\left|e_{ij}\right| = 0$, the quadratic surface degenerates into a cylinder of the elliptic or hyperbolic type or else into two parallel planes symmetrically situated with respect to the quadric surface.

This strain quadric is completely determined once the strain components e_{ij} at point $P^{(0)}$ are known. Let $\overline{P^0P}$ be the radius vector A_i of magnitude A to any point $P(x_1, x_2, x_3)$, referred to local axis, on the strain quadric surface (5.1.1). Let e be the extension of the vector A_i due to some linear deformation characterized by

$$\delta A_i = e_{ij} A_j \, , \qquad (5.1.2)$$

Then, by definition,

$$e = \frac{\delta A}{A} = \frac{A\delta A}{A^2} = \frac{A_i \delta A_i}{A^2}$$

This gives

$$e = \frac{e_{ij} A_i A_j}{A^2} \qquad (5.1.3)$$

using (5.1.2)

Since $\overline{P^0P} = A_i$ and the coordinate of point P, on the surface (5.1.1), relative to P^0 are (x_1, x_2, x_3) , it follows that

$$A_i = x_i \qquad (5.1.4)$$

From equation (5.1.1), (5.1.2) and (5.1.4); we obtain

$$eA^2 = e_{ij} A_i A_j = e_{ij} x_i x_j = \pm k^2$$

Or $$e = \pm \frac{k^2}{A^2} \qquad (5.1.5)$$

Result (1): Relation (5.1.5) shows that the extension or elongation of any radius vector A_i of the strain quadric of Cauchy, given by equation (5.1.1), is inversely proportional to the length 'A' of any radius vector this deformation the elongation of any radius vector of the strain quadric at the point $P^0(x_i^0)$.

Result (2): we know that the length 'A' of the radius vector A_i of strain quadric (5.1.1) at the point $P^0(x_i^{\,0})$ has maximum and minimum values along the axes of the quadric. In general, axes of the strain quadric (5.1.1) differs from the coordinates axes through $P^0(x_i^0)$. Therefore, the maximum and minimum extensions or elongation of the radius vectors of strain quadric (5.1.1) will be along its axes.

Result (3): Another interesting property of the strain quadric (5.1.1) is that normal v_i to this surface at the end point P of the vector $\overline{P^0P} = A_i$ is parallel to the displacement vector δA_i.

To prove this property, let us write equation (5.1.1) in the form

$$G = e_{ij} x_j x_i \pm k^2 = 0 \qquad (5.1.6)$$

Then the direction of the normal $\hat{v}$ to the strain quadric (5.1.6) is given by the gradient of the scalar function G. The components of the gradient are

$$\frac{\partial G}{\partial x_k} = e_{ij} \delta_{ik} x_j + e_{ij} x_i \delta_{kj}$$

$$= e_{kj} x_j + e_{ik} x_i$$

$$= 2 e_{kj} x_j$$

Or

$$\frac{\partial G}{\partial x_k} = 2 \delta A_k \qquad (5.1.7)$$

This shows that vector $\frac{\partial G}{\partial x_k}$ and vector δA_k are parallel. Hence, the vector $\overline{\delta A}$ is directed along the normal at P to the strain quadric of Cauchy.

5.2 STRAIN COMPONENTS AT A POINT IN A ROTATION OF COORDINATE AXES

Let new axes $0x_1' x_2' x_3'$ be obtained from the old reference system $0x_1 x_2 x_3$ by a rotation. Let the directions of the new axes x_i' be the specified relative to the old system x_i by the following table of direction cosines in which ℓ_{pi} is the cosine of the angle between the $x_{p'}$ -and x_i axis.

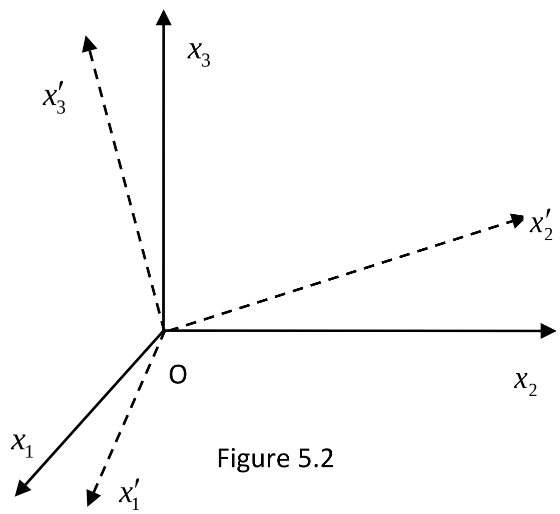


Figure 5.2

That is $\qquad \ell_{pi} = \cos(x_p', x_i)$.

Thus

	x_1	x_2	x_3
x_1'	ℓ_{11}	ℓ_{12}	ℓ_{13}
x_2'	ℓ_{21}	ℓ_{22}	ℓ_{23}
x_3'	ℓ_{31}	ℓ_{32}	ℓ_{33}

Then the transformation law for coordinates is

$$x_i = \ell_{pi} x_p' \tag{5.2.1}$$

Or

$$x_p' = \ell_{pi} x_i \tag{5.2.2}$$

The well –known orthogonality relations are

$$\ell_{pi}\ell_{qi} = \delta_{pq} \tag{5.2.3}$$

$$\ell_{pi}\ell_{pj} = \delta_{ij} \tag{5.2.4}$$

with reference to new x_p' -system, a new set of strain components e_{pq}' is determined at the point O while e_{ij} are the components of strain at O relative to old axes $Ox_1x_2x_3$.

Let

$$e_{ij}x_ix_j = \pm k^2 \tag{5.2.5}$$

be the equation of the strain quadric surface relative to old axis. The equation of quadric surface with reference to new prime system becomes

$$e'_{pq}x'_p x'_q = \pm k^2 \qquad (5.2.6)$$

As we know that quadric form is invariant w. r. t. an orthogonal transformation of coordinates. Further, equation (5.2.2) to (5.2.6) together yield

$$e'_{pq}x'_p x'_q = e_{ij}x_i x_j$$

$$= e_{ij}\left(\ell_{pi}x'_p\right)\left(\ell_{qj}x'_q\right)$$

$$= \left(e_{ij}\ell_{pi}\ell_{qj}\right)x'_p x'_q$$

Or

$$(e'_{pq} - \ell_{pi}\ell_{qj}e_{ij})x'_p x'_q = 0 \qquad (5.2.7)$$

Since equation (5.2.7) is satisfied for arbitrary vector x'_p, we must have

$$e'_{pq} = \ell_{pi}\ell_{qj}e_{ij} \qquad (5.2.8)$$

Equation (5.2.8) is the law of transformation for second order tensors. We, therefore, conclude that the components of strain form a second order tensor.

Similarly, it can be verified that

$$e_{ij} = \ell_{pi}\ell_{qj}e'_{pq} \qquad (5.2.9)$$

Question: Assuming that e_{ij} is a tensor of order 2, show that quadratic form $e_{ij}x_i x_j$ is an invariant.

Solution: We have

$$e_{ij} = \ell_{pi}\ell_{qj}e'_{pq}$$

So,

$$e_{ij}x_i x_j = \ell_{pi}\ell_{qj}e'_{pq}x_i x_j$$

$$= e'_{pq}\left(\ell_{pi}x_i\right)\left(\ell_{qj}x_j\right)$$

$$= e'_{pq}x'_p x'_q . \qquad (5.2.10)$$

Hence the result

5.3 PRINCIPAL STRAINS AND INVERIANTS

From a material point $P^0\left(x_i^0\right)$, there emerge infinitely many material arcs/ filaments, and each of these arcs generally changes in length and orientation under a deformation. We seek now the lines through $P^0\left(x_i^0\right)$ whose orientation is left unchanged by the small linear deformation given by

$$\delta A_i = e_{ij} A_j \tag{5.3.1}$$

where the strain components e_{ij} are small and constant. In this situation, vectors A_i and δA_i are parallel and, therefore,

$$\delta A_i = e A_i \tag{5.3.2}$$

for some constant e.

Equation (5.3.2) shows that the constant e represents the extension.

$$\left(e = \frac{|\delta A_i|}{|A_i|} = \frac{\delta A}{A} \right)$$

of vector A_i. From equation (4.11.1) and (4.11.2), we write

$$e_{ij} A_j = e A_i$$

$$= e\delta_{ij} A_j \tag{5.3.3}$$

This implies

$$\left(e_{ij} - e\delta_{ij}\right) A_j = 0 \tag{5.3.4}$$

We know that e_{ij} is a real symmetric tensor of order 2. The equation (5.3.3) shows that the scalar e is an eigen value of the real symmetric tensor e_{ij} with corresponding eigenvector A_i. Therefore, we conclude that there are precisely three mutually orthogonal direction are not changed on account of deformation and these direction coincide with the three eigenvectors of the strain tensor e_{ij} .These directions are known as principle direction of strain. Equation (5.3.4) gives us a system of three homogeneous equations in the unknown A_1, A_2, A_3. This system possesses a non-trivial solution if and only if the determination of the coefficients of the A_1, A_2, A_3 is equal to zero, i.e.,

$$\begin{vmatrix} e_{11}-e & e_{12} & e_{13} \\ e_{21} & e_{22}-e & e_{23} \\ e_{31} & e_{32} & e_{33}-e \end{vmatrix} = 0 \qquad (5.3.5)$$

which is cubic equation in e.

Let e_1, e_2, e_3 be the three roots of equation (5.3.5), these are known as principal strains. Evidently, the principal strains are the eigenvalues of the second order real symmetric strain tensor e_{ij}. Consequently, these principal strains are real (not necessarily distinct). Physically, the principal strains e_1, e_2, e_3 (all different) are the extensions of the vectors, say $\overline{A_i}$, in the principal / invariant of strain. So, vectors A_i, δA_i, $A+\delta A_i$ are collinear. At the point P^0 consider the strain quadric

$$e_{ij}x_i x_j = \pm k^2 \qquad (5.3.6)$$

For every principal direction of strain A_i, we know that δA_i is normal to the quadric surface (5.3.6). Therefore, the principal directions of strain are also normal to the strain quadric of Cauchy. Here, principal direction of strain must be the three principal axes of the strain quadric of Cauchy. If some of the principal strains e_i are equal, then the associated directions become indeterminate but one can always select three directions that all mutually orthogonal If the $e_1 \neq e_2 = e_3$, then the quadric surface of Cauchy is a surface revolution and our principal direction, say $\tilde{A}_1$, will be directed along the axis of revolution.

In this case, any two mutually perpendicular vectors lying in the plane normal to $\tilde{A}_1$ may be taken as the other two principal directions of strain.

If $e_1 = e_2 = e_3$, then strain quadric of Cauchy becomes a sphere and any three orthogonal directions may be chosen as the principal directions of strain.

Result: If the principal directions of strain are taken as the coordinate axes, then

$$e_{11} = e_1, e_{22} = e_2, e_{33} = e_3$$

And

$$e_{12} = e_{13} = e_{23} = 0,$$

As a vector initially along an axis remains in the same direction after deformation (so change in right angles are zero). In this case, the strain quadric Cauchy has the equation.

$$e_1x_1^2 + e_2x_2^2 + e_3^2x_3^2 = \pm k^2 \tag{5.3.7}$$

Result 2: Expanding the cubic equation (5.3.5), we write

$$-e^3 + v_1e^2 - v_2e + v_3 = 0$$

where
$$v_1 = e_{11} + e_{22} + e_{33}$$
$$= e_{ii} = tr(E), \tag{5.3.8}$$

$$v_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{23}^2 - e_{13}^2 - e_{12}^2$$
$$= tr(E^2) = \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ji}), \tag{5.3.9}$$

$$v_3 = \varepsilon_{ijk}e_{1i}e_{2j}e_{3k}$$
$$= |e_{ij}| = tr(E^3) \tag{5.3.10}$$

Also e_1, e_2, e_3 are roots of a cubic equation (5.3.8), so

$$\left.\begin{aligned} v_1 &= e_1 + e_2 + e_3 \\ v_2 &= e_1e_2 + e_2e_3 + e_3e_1 \\ v_3 &= e_1e_2e_3 \end{aligned}\right\} \tag{5.3.11}$$

We know that eigenvalues of a second order real symmetric tensor are independent of the choice of the coordinate system.

It follows that v_1, v_2, v_3 are given by (5.3.10) three invariants of the strain tensor e_{ij} with respect to an orthogonal transformation of coordinates.

Geometric meaning of the first strain invariant $\vartheta = e_{ii}$

The quantity $\vartheta = e_{ii}$ has a simple geometric meaning. Consider a volume element in the form of rectangle parallelepiped whose edges of length l_1, l_2, l_3 are parallel to the direction of strain. Due to small linear transformation /deformation, this volume element becomes again rectangle parallelepiped with edges of length $l_1(1+e_1), l_2(1+e_2), l_3(1+e_3)$, where e_1, e_2, e_3 are principal strains. Hence, the change δV in the volume V of the element is

$$\delta V = l_1l_2l_3(1+e_1)(1+e_2)(1+e_3) - l_1l_2l_3$$
$$= l_1l_2l_3(1+e_1+e_2+e_3) - l_1l_2l_3, \qquad \text{ignoring small strains } e_i.$$

$$= l_1 l_2 l_3 (e_1 + e_2 + e_3)$$

This implies

$$\frac{\delta V}{V} = e_1 + e_2 + e_3 = \vartheta$$

Thus the first strain invariant ϑ represents the change in volume per unit initial volume due to strain produced in the medium. The quantity ϑ is called the cubical dilatation or simply the dilatation.

Note: If $e_1 > e_2 > e_3$ then e_3 is called the minor principal strain, e_2 is called the intermediate principal strain, and e_1 is called the major principal strain.

Question: For small linear deformation, the strains e_{ij} are given by

$$(e_{ij}) = \alpha \begin{bmatrix} x_2 & \frac{(x_1 + x_2)}{2} & x_3 \\ \frac{(x_1 + x_2)}{2} & x_1 & x_3 \\ x_3 & x_3 & 2(x_1 + x_2) \end{bmatrix}, \qquad \alpha = \text{constant}$$

Find the strain invariants, principal strain and principal direction of strain at the point P(1,1,0).

Solution: The strain matrix at the point P(1,1,0) becomes

$$(e_{ij}) = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 4\alpha \end{bmatrix},$$

whose characteristics equation becomes

$$e(e - 2\alpha)(e - 4\alpha) = 0.$$

Hence, the principal strains are

$$e_1 = 0, e_2 = 2\alpha, e_3 = 4\alpha.$$

The three scalar invariants are

$$v_1 = e_1 + e_2 + e_3 = 6\alpha, v_2 = 8\alpha^2, v_3 = 0$$

The three principal unit directions are found to be

$$A_i^1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \qquad A_i^2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \ A_i^3 = (0,0,1)$$

Exercise: The strain field at a point P(x, y, z) in an elastic body is given by

$$e_{ij} = \begin{bmatrix} 20 & 3 & 2 \\ 3 & -10 & 5 \\ 2 & 5 & -8 \end{bmatrix} \times 10^{-6}.$$

Determine the strain invariant and the principal strains.

Question: Find the principal directions of strain by finding the extremal value of the extension ϑ. OR, Find the direction in which the extension ϑ is stationary.

Solution: Let ϑ be the extension of a vector A_i due to small linear deformation

$$\delta A_i = e_{ij} A_j \qquad (5.3.12)$$

Then

$$\vartheta = \frac{\delta A}{A} \qquad (5.3.13)$$

We know that for an infinitesimal linear deformation (5.3.12), we have

$$A\delta A = A_i \delta A_i \qquad (5.3.14)$$

Thus $$\vartheta = \frac{A\delta A}{A^2} = \frac{A_i \delta A_i}{A^2} = \frac{e_{ij} A_i A_j}{A^2} \qquad (5.3.15)$$

Let $$\frac{A_i}{A} = a_i \qquad (5.3.16)$$

Then $$a_i a_i = 1 \qquad (5.3.17)$$

And equation (5.3.15) then gives

$$e(a_1, a_2, a_3) = e_{ij} a_i a_j \qquad (5.3.18)$$

Thus the extension e_i is a function of a_1, a_2, a_3 which are not independent because of relation (5.3.17). The extreme/stationary (or max/min) values of the extension e are to be found by making use of Lagrange's method of multipliers. For this purpose, we consider the auxiliary function

$$F(a_1, a_2, a_3) = e_{ij} a_i a_j - \lambda(a_i a_i - 1) \qquad (5.3.19)$$

where λ is a constant.

In order to find the values of a_i, a_2, a_3 for which the function (5.3.18) may have a maximum or minimum, we solve the equations.

$$\frac{\partial F}{\partial a_k} = 0, \quad \text{k=1, 2, 3.} \qquad (5.3.20)$$

Thus, the stationary values of e are given by

$$e_{ij}\left(\delta_{ik} a_j + a_i \delta_{jk}\right) - \lambda 2 a_i \delta_{ik} = 0$$

Or $$e_{kj} a_j + e_{ik} a_i - 2\lambda a_k = 0$$

Or $$2 e_{ki} a_i - 2\lambda a_k = 0$$

Or $$e_{ki} a_i = \lambda a_k . \qquad (5.3.21)$$

This shows that λ is an eigenvalue of the strain tensor e_{ij} and a_i is the corresponding eigenvector. Therefore, equation in (5.3.21) determines the principal strains and the stationary/extreme values are precisely the principal strains.

Thus, the extension e assumes the stationary values along the principal direction of strain and the stationary/extreme values are precisely the principal strains.

Remarks: Let M be the square matrix with eigenvectors of the strain tensor e_{ij} as columns. That is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Then $$e_{ij} A_{1j} = e_1 A_{1i}$$

$$e_{ij} A_{2j} = e_2 A_{2i}$$

$$e_{ij} A_{3j} = e_3 A_{3i}$$

The matrix M is called the modal matrix of strain tensor e_{ij}.

Let

$$E = \left(e_{ij}\right), D = dia\left(e_1, e_2, e_3\right).$$

Then, we find

$$EM=MD$$

Or $$M^{-1}EM=D.$$

This shows that the matrices E and D are similar.

We know that two similar matrices have the same eigenvalues. Therefore, the characteristic equation associated with $M^{-1}EM$ is the same as the one associated with E. Consequently, eigenvalues of E and D are identical.

Question: Show that, in general, at any point of the elastic body there exists (at least) three mutually perpendicular principal directions of the strain due to an infinitesimal linear deformation.

Solution: Let e_1, e_2, e_3 be the three principal strains of the strain tensor e_{ij}. Then, they are the roots of the cubic equation

$$(e-e_1)(e-e_2)(e-e_3)=0$$

And $$e_1+e_2+e_3=e_{11}+e_{22}+e_{33}=e_{ii},$$

$$e_1e_2+e_2e_3+e_3e_1=\frac{1}{2}\left(e_{ii}e_{jj}-e_{ij}e_{ji}\right),$$

$$e_1e_2e_3=\left|e_{ij}\right|=\in_{ijk}\ e_{1i}e_{2j}e_{3k}.$$

We further assume that coordinate axes coincide with the principal directions of strain. Then, the strain components are given by

$$e_{11}=e_1, e_{22}=e_2, e_{33}=e_3,$$

$$e_{12}=e_{13}=e_{23}=0,$$

and the strain quadric of Cauchy becomes

$$e_1x_1^2+e_2x_2^2+e_3x_3^2=\pm k^2. \qquad (5.3.22)$$

Now, we consider the following three possible cases for principal strains.

Case: 1 When $e_1 \neq e_2 \neq e_3$. In this case, it is obvious that there exists three mutually orthogonal eigenvectors of the second order real symmetric strain tensor e_{ij}. These eigenvectors are precisely the three principal directions that are mutually orthogonal.

Case: 2 When $e_1 \neq e_2 = e_3$.

Let A_{1i} and A_{2i} be the corresponding principal orthogonal directions corresponding to strains (distinct) e_1 and e_2, respectively. Then

$$e_{ij}A_{1j} = e_1 A_{1i}$$

$$e_{ij}A_{2j} = e_2 A_{2i} \qquad (5.3.23)$$

Let p_i be a vector orthogonal to both A_{1i} and A_{2i}. Then

$$p_i A_{1i} = p_i A_{2i} = 0 \qquad (5.3.24)$$

Let
$$e_{ij} p_i = q_j \qquad (5.3.25)$$

Then
$$q_j A_{1j} = (e_{ij} p_i) A_{1j} = (e_{ij} A_{1j}) p_i = e_1 A_{1i} p_i = 0 \qquad (5.3.26a)$$

similarly
$$q_j A_{2j} = 0 \qquad (5.3.26b)$$

This shows that the vector q_j is orthogonal to both A_{1j} and A_{2j}. Hence, the vectors q_i and p_i must be parallel. Let

$$q_i = \alpha\, p_i \qquad (5.3.27)$$

for some scalar α. From equation (5.3.25) and (5.3.27), we write

$$e_{ij} p_j = q_i = \alpha p_i \qquad (5.3.28)$$

which shows that the scalar α is an eigenvalue /principal strain tensor e_{ij} with corresponding principal direction p_i. Since e_{ij} has only three principal strains e_1, e_2, α and two of these are equal, so α must be equal to $e_2 = e_3$. We denote the normalized form of p_i by A_{3i}. This shows the existence of three mutually orthogonal principal directions in this case. Further, let v_i be any vector normal to A_{1i}. Then v_i lies in the plane containing principal directions A_{2i} and A_{3i}. Let

$$v_i = k_1 A_{2i} + k_2 A_{3i} \quad \text{for some constant } k_1 \text{ and } k_2 \qquad (5.3.29)$$

Now
$$e_{ij} v_j = e_{ij}(k_1 A_{2j} + k_2 A_{3j})$$

$$= k_1(e_{ij} A_{2j}) + k_2(e_{ij} A_{3j})$$

$$= k_1(e_2 A_{2i}) + k_2(e_3 A_{3i})$$

$$= e_2(k_1 A_{2i} + k_2 A_{3i}) \; (\because e_2 = e_3)$$

$$= e_2 v_i$$

This shows that the direction v_i is also a principal directions strain e_2. Thus, in this case, any two orthogonal (mutually) vectors lying on the plane normal to A_{1i} can be chosen as the other two principal directions. In this case, the strain quadric surface is a surface of revolution.

Case3: when $e_1 = e_2 = e_3$, then the strain quadric of Cauchy is a sphere with equation

$$e_1\left(x_1^2 + x_2^2 + x_3^2\right) = \pm k^2$$

Or
$$x_1^2 + x_2^2 + x_3^2 = \pm \frac{k^2}{e_1}$$

and any three mutually orthogonal directions can be taken as the coordinate axes which are coincident with principal directions of strain. Hence, the result.

5.4 GENERAL INFINITESIMAL DEFORMATION

Now we consider the general functional transformation and relation to the linear deformation. Consider an arbitrary material point $P^0\left(x_i^0\right)$ in a continuous medium. let the same material point assume after deformation the point $Q^0\left(\xi_i^0\right)$. Then

$$\xi_i^0 = x_i^0 + u_1\left(x_1^0, x_2^0, x_3^0\right) \qquad (5.4.1)$$

where u_i are the components of the displacement vector $\overline{P^0Q^0}$. We assume that as well as their partial derivatives is a continuous function. The nature of the deformation in the neighborhood of the point P^0 can be determined by considering the change in the vector $\overline{P^0}P = A_i$; in undeformed state.

Let $Q(\xi_1, \xi_2, \xi_3)$ be the deformed position of P. then the displacement u_i at the point P is

$$u_i(x_1, x_2, x_3) = \xi_i - x_i \qquad (5.4.2)$$

The vector
$$A_i = x_i - x_i^0 \qquad (5.4.3)$$

Has now deformed to the vector

$$\xi_i - \xi_i^0 = A_i' \text{ (say)} \qquad (5.4.4)$$

Therefore, $$\delta A_i = A_i' - A_i$$

$$= \left(\xi_i - \xi_i^0\right) - \left(x_i - x_i^0\right)$$

$$= \left(\xi_i - x_i\right) - \left(\xi_i^0 - x_i^0\right)$$

$$= u_i\left(x_1, x_2, x_3\right) - u_i\left(x_1^0, x_2^0, x_3^0\right)$$

$$= u_i\left(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3\right) - u_i\left(x_1^0, x_2^0, x_3^0\right)$$

$$= \left(\frac{\partial u_i}{\partial x_j}\right) A_j \qquad (5.4.5)$$

plus the higher order terms of Taylor's series. The subscript o indicates that the derivatives are to be evaluated at the point P^0. If the region in the neighborhood of P^0 is chosen sufficiently small, i.e. if the vector A_i is sufficiently small, then the product terms like A_i, A_j may be ignored. Ignoring the product terms and dropping the subscript 0 in (5.4.5), we write

$$\delta A_i = u_{i,j} A_j \qquad (5.4.6)$$

where the symbol $u_{i,j}$ has been used for $\frac{\partial u_i}{\partial x_j}$. Result (5.4.6) holds for small vectors A_i. If we further assume that the displacements u_i as well as their partial derivatives are so small that their products can be neglected, then the transformation (which is linear) given by (5.4.4) becomes infinitesimal in the neighborhood of the point P^0 under consideration and

$$\delta A_i = \alpha_{ij} A_j \qquad (5.4.7)$$

with $$\alpha_{ij} = u_{i,j} \qquad (5.4.8)$$

Hence, all results discussed earlier are immediately applicable. The transformation (5.4.6) can be spited into deformation and rigid body motion as

$$\delta A_i = u_{i,j} A_j = \left(\frac{u_{i,j} + u_{j,i}}{2} + \frac{u_{i,j} - u_{j,i}}{2}\right) A_j$$

$$= e_{ij} A_j + w_{ij} A_j \qquad (5.4.9)$$

Where $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (5.4.10)$$

$$w_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{5.4.11}$$

The transformation

$$\delta A_i = e_{ij} A_j \tag{5.4.12}$$

represents pure deformation and

$$\delta A_i = w_{ij} A_j \tag{5.4.13}$$

represents rotation. In general, the transformation (5.4.9) is no longer homogeneous as both strain components e_{ij} and components of rotation w_{ij} are function of the coordinates. We find

$$v = e_{ij} = \frac{\partial u_i}{\partial x_i} = u_{i,i} = div\bar{u} \tag{5.4.14}$$

That is, the cubic dilatation is the divergence of the displacement vector $\bar{u}$ and it differs, in general, from point of the body. The rotation vector w_i is given by

$$w_1 = w_{32}, w_2 = w_{13}, w_3 = w_{21}. \tag{5.4.15}$$

Question: For the small linear deformation given by

$\bar{u} = \alpha\, x_1\, x_2\left(\hat{e}_1 + \hat{e}_2\right) + 2\alpha\left(x_1 + x_2\right)x_3\hat{e}_3$, α =constant.

Find the strain tensor, the rotation and the rotation vector.

Solution: We have

$$u_1 = \alpha x_1 x_2, u_2 = \alpha x_1 x_2, u_3 = 2\alpha\left(x_1 + x_2\right)x_3.$$

Then strains are given by

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \alpha x_2, e_{22} = \frac{\partial u_2}{\partial x_2} = \alpha x_1, e_{33} = \frac{\partial u_3}{\partial x_3} = 2\alpha\left(x_1 + x_2\right)$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\alpha}{2}\left(x_1 + x_2\right)$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \alpha x_3, e_{23} = \alpha x_3$$

We know that

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \qquad (5.4.16)$$

We find

$$w_{11} = w_{22} = w_{33} = 0$$

$$w_{12} = \frac{\alpha}{2}[x_1 - x_2] = -w_{21},\ w_{13} = -\alpha x_3 = -w_{31},\quad w_{23} = -\alpha x_3 = -w_{32}$$

Therefore

$$(w_{ij}) = \alpha \begin{bmatrix} 0 & \frac{(x_1 - x_2)}{2} & -x_3 \\ -\frac{(x_1 - x_2)}{2} & 0 & -x_3 \\ x_3 & x_3 & 0 \end{bmatrix} \qquad (5.4.17)$$

The rotation vector $\overline{w} = w_i$ is given by $w_i = \in_{ijk} u_{kj}$. We find

$$w_1 = w_{32} = \alpha x_3, w_2 = w_{13} = -\alpha x_3, w_3 = w_{21} = \frac{\alpha}{2}(x_2 - x_1)$$

So $$\overline{w} = \alpha x_3(\hat{e}_1 - \hat{e}_2) + \frac{\alpha}{2}(x_2 - x_1)\hat{e}_3 \qquad (5.4.18)$$

Exercise 1: For small deformation defined by the following displacement, find the strain tensor, rotation tensor and rotation vector.

(i) $u_1 = -\alpha x_2 x_3, u_2 = \alpha x_1 x_2, u_3 = 0$

(ii) $u_1 = \alpha^2 (x_1 - x_3)^2, u_2 = \alpha^2 (x_2 + x_3)^2, u_3 = -\alpha x_1 x_2, \alpha =$ constant (5.4.19)

Exercise 2: the displacement components are given by

$u = -yz, v = xz, w = \phi(x, y)$ calculate the strain components. (5.4.20)

Exercise 3: Given the displacements

$$u = 3x^2 y, v = y^2 + 6xz, w = 6z^2 + 2yz$$

Calculate the strain components at the point (1, 0, 2). What is the extension of a line element (parallel to the x- axis) at this point? (5.4.21)

Exercise 4: Find the strain components and rotation components for the small displacement components given below

(a) Uniform dilation- u=ex, v=ey, w=ez

(b) Simple extension- u=ex, v=w=0

(c) Shearing strain- u=2sy, v=w=0

(d) Plane strain- u=u(x, y), v=v(x, y), w=0 (5.4.22)

5.5 SAINT-VENANT'S EQUATIONS OF COMPATIBILITY

By definition, the strain components e_{ij} in terms of displacement components u_i are given by

$$e_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}] \tag{5.5.1}$$

Equation (5.5.1) is used to find the components of strain if the components of displacement are given. However, if the components of strain, e_{ij} ,are given then equation (5.4.1) is a set of six partial differential equations in the three unknown u_1, u_2, u_3 .Therefore, the system (5.5.1) will not have single valued solution for u_i unless given strains e_{ij} satisfy certain conditions which are known as the conditions of compatibility or equations of compatibility.

Equations of compatibility

we have

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{5.5.2}$$

so,

$$e_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right) \tag{5.5.3}$$

Interchanging i with k and j with l in equation (5.4.3), we write

$$e_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right) \tag{5.5.4}$$

adding (5.5.3) and (5.5.4), we get

$$e_{ij,kl} + e_{kl,ij} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}\right) \tag{5.5.5}$$

Interchanging i and l in (5.5.5), we get

$$e_{lj,ki} + e_{ki,lj} = \frac{1}{2}\left(u_{l,jki} + u_{j,lki} + u_{k,lij} + u_{i,ljk}\right) \tag{5.5.6}$$

From (5.5.5) and (5.5.6), we obtain

$$e_{ij,kl} + e_{kl,ij} = e_{lj,ki} + e_{ki,lj}$$

Or $$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \tag{5.5.7}$$

These equations are known as equations of compatibility.

These equations are necessary conditions for the existence of a single valued continuous displacement field. These are 81 equations in number. Because of symmetry in indices i, j and k, l; some of these equations are identically satisfied and some are repetitions. Only 6 out of 81 equations are essential. These equations were first obtained by **Saint-Venant's in 1860.**

A strain tensor e_{ij} that satisfies these conditions is referred to as a possible strain tensor.

Show that the conditions of compatibility are sufficient for the existence of a single valued continuous displacement field.

Let $P^0(x_i^0)$ be some point of a simply connected region at which the displacements u_i^0 and rotations w_{ij}^0 are known. The displacements u_i of an arbitrary point $P'(x_i')$ can be obtained in terms of the known functions e_{ij} by mean of a line integral along a continuous curve C joining the point P^0 and P'.

$$u_j(x_1', x_2', x_3') = u_j^0(x_1^0, x_2^0, x_3^0) + \int_{P^0}^{P'} du_j \tag{5.5.8}$$

If the process of deformation does not create cracks or holes, i.e., if the body remains continuous, the displacements u_j' should be independent of the path of integration. That is, u_j' should have the same value regardless of whether the integration is along curve C or any other curve. We write

$$du_j = \frac{\partial u_j}{\partial x_k} dx_k = u_{j,k} dx_k = (e_{jk} + w_{jk}) dx_k \tag{5.5.9}$$

Therefore

$$u_j' = u_j^0 + \int_{P^0}^{P'} e_{jk} dx_k + \int_{p^0}^{P'} w_{jk} dx_k \,, \ P(x_k) \text{ being point the joining curve.} \tag{5.5.10}$$

Integrating by parts the second integral, we write

$$\int_{p^0}^{P'} w_{jk}dx_k = \int_{P^0}^{P'} w_{jk}d\left(x_k - x'_k\right)$$ the point $P'\left(x'_k\right)$ being fixed so $dx'_k = 0$

$$= \left\{(x_k - x_k')w_{jk}^0 \right\}_{P^o}^{P'} - \int_{P^0}^{P'}\left(x_k - x_k'\right)w_{jk,l}dx_l \qquad (5.5.11)$$

From equations (5.5.10) and (5.5.11), we write

$$u_j\left(x_1', x_2', x_3'\right) = u_j^0 + \left(x_k' - x_k^0\right)w_{jk}^0 + \int_{P^o}^{P'} e_{jk}dx_k + \int_{P^o}^{P'}\left(x_k' - x_k\right)w_{jk,l}dx_l$$

$$= u_j^0 + \left(x_k' - x_k^0\right)w_{jk}^0 + \int_{P^0}^{P'}[e_{jl} + \left(x_k' - x_k\right)w_{jk,l}]dx_l \qquad (5.5.12)$$

where the dummy index k of e_{jk} has been changed to l.

but $$w_{jk,l} = \frac{1}{2}\frac{\partial}{\partial x_l}[u_{j,k} - u_{k,j}]$$

$$= \frac{1}{2}[u_{j,kl} - u_{k,jl}]$$

$$= \frac{1}{2}[u_{j,kl} + u_{l,jk}] - \frac{1}{2}[u_{l,jk} - u_{k,jl}]$$

$$= e_{jl,k} - e_{lk,j} \qquad (5.5.13)$$

using (5.5.13), equation (5.5.12) becomes

$$u_j(x_1', x_2', x_3') = u_j^0 + \left(x_k' - x_k^0\right)w_{jk}^0 + \int_{P^0}^{P'}[e_{jl} + \{x_k' - x_k\}\{e_{jl,k} - e_{kl,j}\}]dx_l$$

$$= u_j^0 + \left(x_k' - x_k^0\right)w_{jk}^0 + \int_{P^0}^{P'}U_{jl}dx_l \qquad (5.5.14)$$

where for convenience we have set

$$U_{jl} = e_{jl} + \left(x_k' - x_k\right)\left(e_{jl,k} - e_{kl,j}\right) \qquad (5.5.15)$$

which is known function as e_{ij} are known. The first two terms in the side of equation (5.5.14) are independent of the path of integration. From the theory of line integrals, the third term becomes independent of the path of integration when the integrands $U_{jl}dx_1$ must be exact differentials. Therefore, if the displacements $u_i(x_1', x_2', x_3')$ are to be independent of the path of integration, we must have

$$\frac{\partial U_{jl}}{\partial x_i} = \frac{\partial U_{ji}}{\partial x_l} \qquad \text{for } i, j, l = 1,2,3 \qquad (5.5.16)$$

Now

$$U_{jl,i} = e_{jl,i} + (x_k' - x_k)(e_{jl,ki} - e_{kl,ji}) - \delta_{ki}(e_{jl,k} - e_{kl,j})$$
$$= e_{jl,i} - e_{kl,i} + e_{li,j} + (x_k' - x_k)(e_{jl,ki} - e_{kl,ji}) \qquad (5.5.17)$$

and

$$U_{jl,i} = e_{ji,l} + (x_k' - x_k)(e_{ji,kl} - e_{ki,jl}) - \delta_{kl}(e_{ji,k} - e_{ki,j})$$
$$= e_{ji,l} - e_{ji,l} + e_{li,j} + (x_k' - x_k)(e_{ji,kl} - e_{ki,jl}) \qquad (5.5.18)$$

Therefore, equations (5.5.16) and (5.5.17), (5.5.18) yields

$$(x_k' - x_k)[e_{jl,ki} - e_{kl,ji} - e_{ji,kl} + e_{ki,kl}] = 0$$

Since this is true for an arbitrary choice of $x_k' - x_k$ (as P' is arbitrary), it follows that

$$e_{ji,kl} + e_{kl,ji} - e_{ik,jl} - e_{jl,ki} = 0 \qquad (5.5.19)$$

This is true as these are the compatibility relations. Hence, the displacement (5.5.8) independent of the path of integration. Thus, the compatibility conditions (5.5.7) are sufficient also.

Remarks1: The compatibility conditions (5.4.7) are necessary and sufficient for the existence of a single valued continuous displacement field when the strain components are prescribed.

In details form, these 6 conditions are

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{-\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3}\right)$$

$$\frac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{-\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{-\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2}\right)$$

$$\frac{2\partial^2 e_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 e_{11}}{\partial x_2{}^2} + \frac{\partial^2 e_{22}}{\partial x_1{}^2}$$

$$\frac{2\partial^2 e_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 e_{22}}{\partial x_3{}^2} + \frac{\partial^2 e_{33}}{\partial x_3{}^2}$$

$$\frac{2\partial^2 e_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 e_{33}}{\partial x_1{}^2} + \frac{\partial^2 e_{11}}{\partial x_3{}^2}. \qquad (5.5.20)$$

These are the necessary and sufficient conditions for the components e_{ij} to give single valued displacements u_i for a simply connected region.

Definition: A region space is said to be simply connected if an arbitrary closed curve lying in the region can be shrunk to a point, by continuous deformation, without passing outside of the boundaries.

Remarks2: The specification of the strains e_{ij} only does not determine the displacements u_i uniquely because the strains e_{ij} characterize only the pure deformation of an elastic neighborhood of the point x_i.

The displacements u_i may involve rigid body motions which do not affect e_{ij}.

Example1: (i) Find the compatibility condition for the strain tensor e_{ij} if e_{11}, e_{22}, e_{33} are independent of x_3 and $e_{31} = e_{32} = e_{33} = 0$.

(ii) Find the condition under which the following are possible strain components.

$$e_{11} = k\left(x_1{}^2 - x_2{}^2\right), e_{12} = k' x_1 x_2, e_{22} = k x_1 x_2,$$

$$e_{31} = e_{32} = e_{33} = 0, k \,\&\, k' \text{ are constants}$$

(iii) When e_{ij} given above are possible strain components, find the corresponding displacements, given that $u_3 = 0$

Solution: (i) We verify that all the compatibility conditions except one are obviously satisfied. The only compatibility to be satisfied by e_{ij} is

$$e_{11,22} + e_{22,11} = 2e_{12,21}. \quad (5.5.21)$$

(ii) Five conditions are trivially satisfied. The remaining condition (5.5.20) is satisfied iff

$$k' = k \text{ as } \quad e_{11,22} = -2k, e_{12,12} = k', e_{22,11} = 0$$

(iii) We find

$$e_{11} = u_{1,1} = k\left(x_1^2 - x_2^2\right), u_{2,2} = kx_1x_2, u_{1,2} + u_{2,1} = -2kx_1x_2, \ (\therefore k' = -k)$$

$$u_{2,3} = u_{1,3} = 0$$

This shows that the displacement components u_1 and u_2 are independent of x_3.

We find (exercise)

$$u_1 = \frac{1}{6}(2x_1^{\ 3} - 6x_1x_2^{\ 2} + x_2^{\ 3}) - cx_2 + c_1$$

$$u_2 = \frac{1}{2}kx_1x_2^{\ 2} + cx_1 + c_2 \text{ where } c_1, c_2 \text{ and c constants.}$$

Example: Show that the following are not possible strain components

$$e_{11} = k\left(x_1^{\ 2} + x_2^{\ 2}\right), e_{22} = k\left(x_2^{\ 2} + x_3^{\ 2}\right), e_{33} = 0$$

$$e_{12} = k' x_1x_2x_3, e_{13} = e_{21} = 0, k \,\&\, k' \text{ being constants.}$$

Solution: The given components e_{ij} are possible strain components if each of the six compatibility conditions are satisfied. On substitution, we find

$$2k = 2k' x_3$$

This can't be satisfied for $x_3 \neq 0$. For $x_3 = 0$, this gives k=0 and then all e_{ij} vanish. Hence, the given e_{ij} are not possible strain components.

Exercise1: Consider a linear strain field associated with a simply connected region R such that $e_{11} = Ax_2^{\ 2}, e_{22} = Ax_1^{\ 2}, e_{12}Bx_1x_2, e_{13} = e_{23} = e_{33}0$.find the relationship between constant A and B such that it is possible to obtain a single- valued continuous displacement field which corresponds to the given strain field.

Exercise2: Show by differentiation of the strain displacement relation that the compatibility conditions are necessary condition for the existence of continuous single-valued displacements.

Exercise3: Is the following state of strain possible? (c=constant)

$$e_{11} = c\left(x_1^{\ 2} + x_2^{\ 2}\right)x_3, e_{22} = cx_2^{\ 2}x_3, e_{12} = 2cx_1x_2x_3, e_{31} = e_{32} = e_{33} = 0$$

Exercise4: Show that the equations of compatibility represent a set of necessary and sufficient conditions for the existence single-valued displacements. Drive the equations of compatibility for plane strain.

Exercise 5: If $e_{11} = e_{22} = e_{33} = 0, e_{13} = \phi,_2$ and $e_{23} = \phi_{,1}$; where ϕ is a function of x_1 and x_2, show that ϕ must satisfy the equation

$$\nabla^2\phi = \text{constant}$$

Exercise 6: If e_{13} and e_{23} are the only non-zero strain components and e_{13}, e_{23} are independent of x_3, show that the compatibility condition may be reduced to the following condition

$$e_{13,2} - e_{23,1} = \text{constant.}$$

Exercise 7: Find which of the following values of e_{ij} are possible linear strains

(i) $e_{11} = \alpha\left(x_1^{\ 2} + x_2^{\ 2}\right), e_{22} = \alpha x_2^{\ 2}, e_{12} = 2\alpha x_1x_2, e_{31} = e_{32} = e_{33} = 0, \alpha =$ constant.

(ii) $$e_{ij} = \begin{bmatrix} x_1 + x_2 & x_1 & x_2 \\ x_1 & x_2 + x_3 & x_3 \\ x_2 & x_3 & x_1 + x_3 \end{bmatrix}$$

Compute the displacements in the case (i).

5.6 FINITE DEFORMATIONS

All the results reported in the preceding sections of this chapter were that of the classical theory of infinitesimal strains. Infinitesimal transformations permit the application of the derivatives of superposition of effects. Finite deformations are those deformations in which the displacements u_i together with their derivatives are no longer small. Consider an aggregate of particles in a continuous medium. We shell use the same reference frame for the location of particles in the deformed and undeformed states.

Let the coordinates of a particle lying on a curve C_0, before deformation, be denoted by (a_1, a_2, a_3) and let the coordinates of the same particle after deformation (now lying same curve C) be (x_1, x_2, x_3). Then the elements of arc of the curve C_0 and C are given, respectively, by

$$ds_o^{\ 2} = da_i da_i \tag{5.6.1}$$

and
$$ds^2 = dx_i dx_i \tag{5.6.2}$$

we consider first the Eulerian description of the strain and write

$$a_i = a_i(x_1, x_2, x_3) \tag{5.6.3}$$

then
$$da_i = a_{i,j} dx_j = a_{i,k} dx_k \tag{5.6.4}$$

substituting from (5.6.3) into (5.6.1), we write

$$ds_0^{\ 2} = a_{i,j} a_{i,k} dx_j dx_k \tag{5.6.5}$$

using the substitution tensor, equation (5.6.2) can be rewritten as

$$ds^2 = \delta_{jk} dx_j dx_k \tag{5.6.6}$$

We know that the measure of the strain is the difference $ds^2 - ds_0^{\ 2}$

from equations (5.6.5) and (5.6.6), we get

$$ds^2 - ds_0^{\ 2} = (\delta_{jk} - a_{i,j} a_{i,k}) dx_j dx_k$$

$$= 2\eta_{jk} dx_j dx_k \tag{5.6.7}$$

where

$$2\eta_{jk} = \delta_{jk} - a_{i,j} a_{i,k} \tag{5.6.8}$$

We now write the strain components η_{jk} in term of displacement components u_i , where

$$u_i = x_i - a_i \tag{5.6.9}$$

this gives

$$a_i = x_i - u_i$$

Hence

$$a_{i,j} = \delta_{ij} - u_{i,j} \tag{5.6.10}$$

$$a_{i,k} = \delta_{ik} - u_{i,k} \tag{5.6.11}$$

Equations (5.6.8), (5.6.10) and (5.6.11) yield

$$2\eta_{jk} = \delta_{jk} - (\delta_{ij} - u_{i,j})(\delta_{ik} - u_{i,k})$$

$$= \delta_{jk} - [\delta_{jk} - u_{k,j} - u_{j,k} + u_{i,j} u_{i,k}]$$

$$= (u_{j,k} + u_{k,j}) - u_{i,j} u_{i,k} \tag{5.6.12}$$

The quantities η_{jk} are called the Eulerian strain components.

If, on the other hand, Lagrangian coordinates are used, and equations of transformation are of the form

$$x_i = x_i(a_1, a_2, a_3) \tag{5.6.13}$$

then

$$dx_i = x_{i,j} da_i = x_{i,k} da_k \tag{5.6.14}$$

and

$$ds^2 = x_{i,j} x_{i,k} da_i da_k \tag{5.6.15}$$

while

$$ds_0{}^2 = \delta_{ij} da_j da_k \tag{5.6.16}$$

The Lagrangian components of strain $\in_{jk}$ are defined by

$$ds^2 - ds_0{}^2 = 2 \in_{jk} da_j da_k \tag{5.6.17}$$

Since

$$x_i = a_i + u_i \tag{5.6.18}$$

Therefore,

$$x_{i,j} = \delta_{ij} + u_{i,j}$$

$$x_{i,k} = \delta_{ik} + u_{i,k}$$

Now

$$ds^2 - ds_0{}^2 = (x_{i,j} x_{i,k} - \delta_{jk}) da_j da_k$$

$$= [(\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) - \delta_{jk}] da_j da_k$$

$$= (u_{j,k} + u_{k,j} + u_{i,j} u_{i,k}) da_j da_k \tag{5.6.19}$$

Equation (5.6.17) and (5.6.19) give

$$2 \in_{jk} = u_{j,k} + u_{k,j} + u_{i,j} u_{i,k} \tag{5.6.20}$$

It is mentioned here that the differentiation in (5.6.12) is carried out with respect to the variable x_i , while in (5.6.19) the 'a_i' are regarded as the independent as the independent variables. To make the difference explicitly clear, we write out the typical expressions η_{jk} and $\in_{jk}$ in unabridged notation,

$$\eta_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \quad (5.6.21)$$

$$2\eta_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \quad (5.6.22)$$

$$\in_{xx} = \frac{\partial u}{\partial a} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial b}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right] \quad (5.6.23)$$

$$2 \in_{xy} = \left(\frac{\partial u}{\partial a} + \frac{\partial v}{\partial a}\right) + \left(\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right) \quad (5.6.24)$$

When the strain components are large, it is no longer possible to give simple geometrical interpretations of the strain $\in_{jk}$ and η_{jk} .

Now we consider some particular cases.

Case1: Consider a line element with

$$ds_0 = da_1, da_2 = 0, da_3 = 0 \quad (5.6.25)$$

Define the extension E_1 of this element by

$$E_1 = \frac{ds - ds_0}{ds_0}$$

then

$$ds = (1 + E_1)ds_o \quad (5.6.26)$$

and consequently

$$ds^2 - ds_0^{\ 2} = 2 \in_{jk} da_j da_k$$

$$= 2 \in_{11} da_j^{\ 2} \quad (5.6.27)$$

Equation (5.6.25) to (5.6.27) yield

$$(1+E_1)^2-1=2\in_{11}$$

Or $$E_1=\sqrt{1+2\in_{11}}-1 \qquad (5.6.28)$$

When the strain $\in_{11}$ is small,(5.6.28) reduced to

$$E_1 \cong \in_{11}$$

As was shown in discussion of strain infinitesimal strains.

Case II: Consider next two line elements

$$ds_0=da_2, da_1=0, da_3=0 \qquad (5.6.29)$$

and

$$d\overline{s_0}=d\overline{a_3}, d\overline{a_1}=d\overline{a_2}=0 \qquad (5.6.30)$$

These two elements lie initially along the a_2-and a_3-axes.

Let θ denote the angle between the corresponding deformed dx_i and $d\overline{x_i}$, of length ds and $d\bar{s}$ respectively. Then

$$dsd\bar{s}\cos\theta = dx_i d\overline{x_i} = x_{i,\alpha}\overline{x_{i,\beta}}da_\alpha d\overline{a_\beta} = x_{i,2}\overline{x_{i,3}}da_2 d\overline{a_3}$$

$$=2\in da_2 d\overline{a_3} \qquad (5.6.31)$$

Let $$\alpha_{23}=\frac{\pi}{2}-\theta \qquad (5.6.32)$$

Denotes the change in the right angle between the line elements in the initial state. Then, we have

$$\sin\alpha_{23}=2\in_{23}\left(\frac{da_2}{ds}\right)\left(\frac{d\overline{a_3}}{d\bar{s}}\right) \qquad (5.6.33)$$

$$=\frac{2\in_{23}}{\sqrt{1+2\in_{22}}\sqrt{1+2\in_{33}}} \qquad (5.6.34)$$

using relations (5.6.26) and (5.6.28).

Again, if the strains $\in_{ij}$ are so small that their products can be neglected, then

$$\alpha_{23}\cong 2\in_{23} \qquad (5.6.35)$$

As proved earlier for infinitesimal strains.

Remarks: If the displacements and their derivatives are small, then it is immaterial whether the derivatives are calculated at the position of a point before or after deformation. In this case, we may neglect the nonlinear terms in the partial derivatives in (5.6.12) and (5.6.20) and reduce both sets of formulas to

$$2\eta_{jk} = u_{j,k} + u_{k,j} = 2\in_{jk}$$

Which were obtained for an infinitesimal transformation, It should be emphasized of finite homogeneous strain are not in general commutative and that the simple superposition of effects is no longer applicable to finite deformation.

Books Recommended:

4. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

5. **S. Timoshenko and N. Goodier,** Theory of Elasticity, McGraw Hill, New York, 1970.

CHAPTER-VI

ANALYSIS OF STRESS

6.1 INTRODUCTION

Deformation and motion of an elastic body are generally caused by external forces such as surface loads or internal forces such as earthquakes, nuclear explosions etc. When an elastic body is subjected to such force, its behaviour depends on magnitude of forces, upon their direction and upon the inherent strength of the material of which the body is made. Such forces give rise to interaction between neighbouring portions in the interior parts of the elastic solid. The concept of stress vector on a surface and state of stress at a point of the medium shall be discussed.

An approach to the solutions of problems in elastic solid mechanics is to examine deformation initially and then consider stresses and applied loads. Another approach is to establish relationship between applied loads and internal stresses first and then to consider deformations. Regardless of the approach selected, it is necessary to derive the components relations individually.

6.2 BODY FORCES AND SURFACE FORCES

Consider a continuous medium. We refer the points of this medium to a rectangular Cartesian coordinate system. Let τ represents the region occupied by the body in deformed state. A deformable body may be acted upon by two different types of external forces.

(i) **Body forces:** These forces are those forces which act on every volume element of the body and hence on the entire volume of the body. Forexample, gravitational force and magnetic forces are body forces. Let ρ denotes the density of a volume element$\Delta\tau$ of the bodyτ. Let g be the gravitational force/acceleration. Then the force acting on mass $\rho\Delta\tau$ contained in volume $\Delta\tau$ is g $\rho\Delta\tau$.

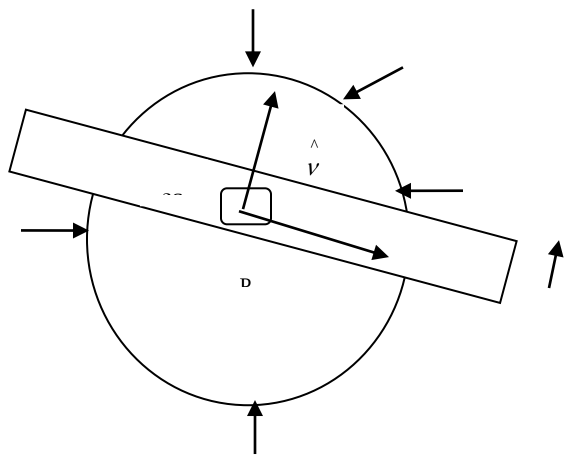
(ii) **Surface forces:** These forces act on every surface element of the body .Such forces are also called contact forces. Loads applied over the external surface or bounding surface are examples of surface forces. Hydrostatic pressure acting on the surface of a body submerged in a liquid /water is a surface force.

(iii) **Internal forces:** Internal forces such as earthquakes, nuclear explosions arise from the mutual interaction between various parts of the elastic body.

Now we consider an elastic body in its unreformed state with no forces acting on it. Let a system of forces applied on it. Due to these forces, the body is deformed and a system of internal forces is set up to oppose this deformation. These internal forces give rise to stress within the body. It is therefore necessary to consider how external forces are transmitted through the medium.

6.3 STRESS VECTOR ON A PLANE AT A POINT

Let us consider an elastic body in equilibrium under the action of a system of external forces.



Let us pass a fictitious plane π through a point $P(x_1, x_2, x_3,)$ in the interior of this body. The body can be considered as consisting of two parts, say, A and B and these parts are in welded contacts at the interface.Part A of the body is in equilibrium under forces(external) and the effect of part B on the plane π. We assume that this effect is continuously distributed over the surface of intersection around the point P, let us consider a small surface δS(on the place π) and let $\hat{\nu}$ be an outward unit normal unit vector (for the part A of the body).The effect of part B on this small surface element can be reduced to a force and a vector couple $\bar{C}$. Now let usshrink in size towards zero in amanner such that the point P always remains aside and remains the normal vector.

$$\lim_{\delta S \to 0} \frac{\overline{Q}}{\delta S} = \overline{T}(x_1, x_2, x_3),$$

$$\lim_{\delta S \to 0} \frac{\overline{C}}{\delta S} = 0,$$

Now $\overline{T}$ is a surface force per unit area. The force $\overline{T}$ is called the stress vector or traction on the plane π at P.

Note 1: Forces acting over the surface of a body are never idealized point forces; they are, in reality, forces per unit area applied over some finite area. These external forces per unit area are also called tractions.

Note 2:Cauchy's stress postulate

If we consider another oriented plane containing same point $P(x_i)$, then the stress vector is likely to have a different direction. For this purpose, Cauchy made the following postulated known as **Cauchy's stress postulate**

"**The stress vector** $\underset{\sim}{T}$ **depends on the orientation of the plane upon which it acts**".

Let $\hat{\nu}$ be the unit normal to the plane π through the point P.This normal characterizes the orientation of the plane upon which the stress vector acts. For this reason, we write the stress vector as $\underset{\sim}{\overset{\hat{\nu}}{T}}$, indicating the dependence on the orientation $\hat{\nu}$.

Cauchy's Reciprocal Relation

When the plane π is in the interior of the elastic body, the normal $\hat{\nu}$ has two possible directions that are opposite to each other and we choose one of these directions.

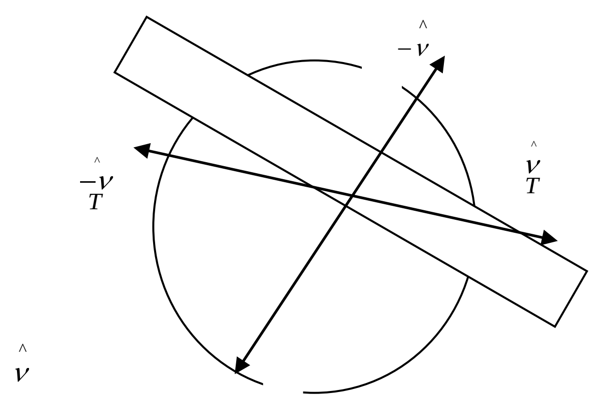


Figure 6.2

For a chosen $\hat{\nu}$,the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is interpreted as the internal surface force per unit area acting on plane π due to the action of part B of the material/body which $\hat{\nu}$ is directed upon the part A across the plane π.

Consequently, $\overset{-\hat{\nu}}{\underset{\sim}{T}}$ is the internal surface force per unit area acting on π due to the action of part A for which $\hat{\nu}$ is the outward drawn unit normal. By Newton's third law of motion, vectors $\overset{-\hat{\nu}}{\underset{\sim}{T}}$ and $-\overset{\hat{\nu}}{\underset{\sim}{T}}$ balance each other as the body is in equilibrium.

$$\therefore \qquad \overset{-\hat{\nu}}{\underset{\sim}{T}} = -\overset{\hat{\nu}}{\underset{\sim}{T}}$$

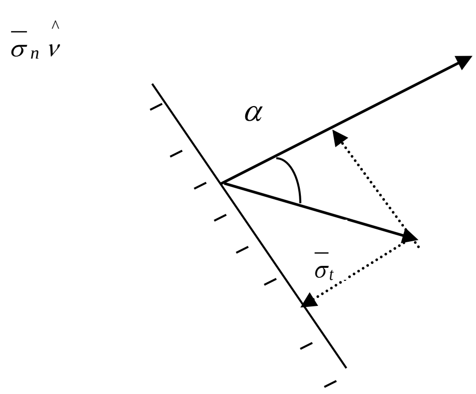
which is known as Cauchy's Reciprocal Relation.

Homogenous State of Stress

If π and π' are any two parallel planes through any two points P and P' of a continuous elastic body, and if the stress vector on π at P is equal to the stress on π' at P', then the state of stress in the body is said to be a homogeneous state of stress.

6.4 NORMAL AND TANGENTIAL STRESSES

In general, the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is inclined to the plane on which it acts and need not be in the direction of unit normal. The projection of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ on the normal $\hat{\nu}$ is called **the normal stress**. It is denoted by σ or σ_n . The projection of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ on the plane π, in the plane of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ and $\hat{\nu}$, is called **the tangential or shearing stress**. It is denoted by τ or σ_t .



$\overset{\wedge}{\nu}$
$\underline{T}$

Figure 6.3

Thus, $$\sigma = \sigma_n = \overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu}$$

$$\tau = \sigma_t = \overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{t} \qquad (6.4.1)$$

$$\left| \overset{\hat{\nu}}{\underset{\sim}{T}} \right|^2 = \sigma_n^2 + \sigma_t^2 \qquad (6.4.2)$$

where $\overset{\hat{\nu}}{\underset{\sim}{T}}$ unit vector normal to $\hat{\nu}$ and lies in the place π.

A stress in the direction of the outward normal is considered positive (i.e.σ> 0) and is called a **tensile stress**. A stress in the opposite direction is considered negative (σ< 0) and is called a **compressible stress.**

If σ = 0, $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is perpendicularto $\hat{\nu}$ .The stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is called a **pure shear stress** or a **pure tangential stress.**

If τ = 0, then $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is parallel to $\hat{\nu}$ .The stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is then called **pure normal stress**.When $\overset{\hat{\nu}}{\underset{\sim}{T}}$ acts opposite to the normal $\hat{\nu}$,then the pure normal stress is called **pressure** (σ <0, τ= 0).

From (6.4.1), we can write $\overset{\hat{\nu}}{\underset{\sim}{T}} = \sigma \hat{\nu} + \tau \hat{t}$ (6.4.3)

$$\tau = \sqrt{\left| \overset{\hat{\nu}}{\underset{\sim}{T}} \right|^2 - \sigma^2}$$

(6.4.4)

Note: $\tau = \sigma_t = |\overset{\nu}{T}| Sin\alpha$ (6.4.5)

$$|\sigma| = \left| \overset{\hat{\nu}}{T} \times \hat{\nu} \right| \quad \text{as} \quad |\hat{\nu}| = 1$$

Thisτ in magnitude is given by the magnitude of vector product of $\underset{\sim}{\overset{\hat{\nu}}{T}}$ and $\hat{\nu}$.

6.5 STRESS COMPONENTS

Let $P(x_i)$ be any point of the elastic medium whose coordinates are (x_1 , x_2 , x_3) relative to rectangular Cartesian system $ox_1x_2\, x_3$.

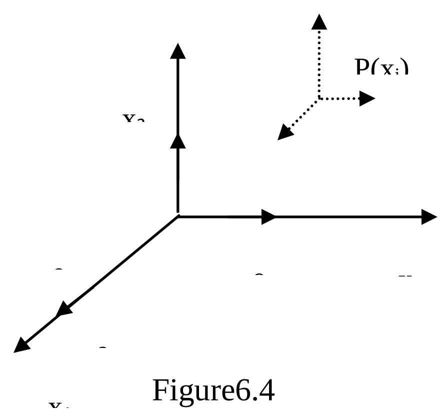


Figure6.4

Let $\overset{\hat{1}}{T}$ denote the stress vector on theplane, with normal along x_1 – axis, at the point P.

Let the stress vector $\overset{\hat{1}}{T}$ has components τ_{11},τ_{12}, τ_{13}, i.e.

$$\underset{\sim}{\overset{\hat{1}}{T}} = \tau_{11}\hat{e}_1 + \tau_{12}\hat{e}_2 + \tau_{13}\hat{e}_3 = \tau_{1j}\hat{e}_j \qquad 6.5.1)$$

Let $\overset{\hat{2}}{T}$ denote the stress vector on the plane, with normal along x_2 – axis, at the point P.

$$\overset{\hat{2}}{\underset{\sim}{T}} = \tau_{21}\hat{e}_1 + \tau_{22}\hat{e}_2 + \tau_{23}\hat{e}_3 = \tau_{2j}\hat{e}_j \quad (6.5.2)$$

Similarly $$\overset{\hat{3}}{\underset{\sim}{T}} = \tau_{31}\hat{e}_1 + \tau_{32}\hat{e}_2 + \tau_{33}\hat{e}_3 = \tau_{3j}\hat{e}_j \quad (6.5.3)$$

Equations (6.5.1) to (6.5.3) can be condensed in the following form

$$\overset{\hat{i}}{\underset{\sim}{T}} = \tau_{ij}\hat{e}_j \quad (6.5.4)$$

$$\overset{\hat{i}}{\underset{\sim}{T}}.\hat{e}_k = (\tau_{ij}\hat{e}_j).\hat{e}_k = \tau_{ij}\delta_{jk} = \tau_{ik} \quad (6.5.5)$$

Thus, for given i & j, the quantity τ_{ij} represent the jth components of the stress vector $\overset{\hat{i}}{\underset{\sim}{T}}$ acting on a plane having $\hat{e}_i$ as the unit normal. Here, the first suffix i indicates the direction of the normal to the plane through P and the second suffix j indicates the direction of the stress component. In all, we have 9 components τ_{ij} at the point $P(x_i)$ in the $ox_1x_2\,x_3$ system. These quantities are called stress — components. The matrix

$$(\tau_{ij}) = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \quad (6.5.6)$$

whose rows are the components of the three stress vectors, is called the matrix of the state of stress at P. The dimensions of stress components are force/(length)^2=$ML^{-1}T^{-2}$.

The stress components $\tau_{11}, \tau_{22}, \tau_{33}$ are called **normal stresses** and other components $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}$ are called as **shearing stresses(** $\overset{1}{\underset{\sim}{T}}\hat{e}_1 = e_{11}, \overset{1}{\underset{\sim}{T}}.\hat{e}_2 = e_{12}$ *etc.*). In CGS system, the stress is measured in **dyne per square centimetre.**In English system, it measured in **pounds per square inch or tons per square inch.**

DYADIC REPRESENTATION OF STRESS

It may be helpful to consider the stress tensor as a vector - like quantity having a magnitude and associated direction (s), specified by unit vector. The dyadic is such a representation. We write the **stress tensor or stress dyadic** as

$$\tau = \tau_{ij}\hat{e}_i\hat{e}_j = \tau_{11}\hat{e}_1\hat{e}_1 + \tau_{12}\hat{e}_1\hat{e}_2 + \tau_{13}\hat{e}_1\hat{e}_3 + \tau_{21}\hat{e}_2\hat{e}_1 + \tau_{22}\hat{e}_2\hat{e}_2$$
$$+ \tau_{23}\hat{e}_2\hat{e}_3 + +\tau_{31}\hat{e}_3\hat{e}_1 + \tau_{32}\hat{e}_3\hat{e}_2 + \tau_{33}\hat{e}_3\hat{e}_3 \qquad (6.5.7)$$

where the juxtaposed double vectors are called **dyads.**

The stress vector $\overset{i}{\underset{\sim}{T}}$ acting on a plane having normal along $\hat{e}_i$ is evaluated as follows:

$$\overset{i}{\underset{\sim}{T}} = \overline{\sigma}\hat{e}_i = (\tau_{jk}\hat{e}_j\hat{e}_k).\hat{e}_i = \tau_{jk}\hat{e}_j\delta_{ki} = \tau_{ji}\hat{e}_j = \tau_{ij}\hat{e}_j \qquad (6.5.8)$$

6.6 STATE OF STRESS AT A POINT-THE STRESS TENSOR

We shall show that the state of stress at any point of an elastic medium on an oblique plane is completely characterized by the stress components at P.

ANALYSIS OF STRESS

Let $\overset{\hat{\nu}}{\underset{\sim}{T}}$ be the stress vector acting on an oblique plane at the material point P, the unit normal to this plane being $\hat{\nu} = \nu_i$.

Through the point P, we draw three planar elements parallel to the coordinate planes. A fourth plane ABC at a distance h from the point P and parallel to the given oblique plane at P is also drawn. Now, the tetrahedron PABC contains the elastic material.

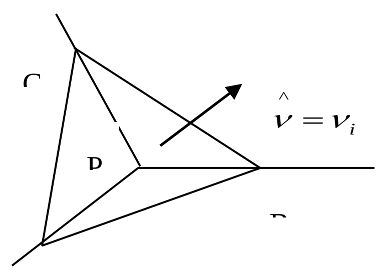


Figure6.5

Let τ_{ij} be the components of stress at the point P regarding the signs (negative or positive) of scalar quantities τ_{ij}, we adopt the following convention.

If one draws an exterior normal (outside the medium) to a given face of the tetrahedron PABC ,then the positive values of components τ_{ij} are associated with forces acting in the positive directions of the coordinate axes. On the other hand, if the exterior normal to a given face is pointing in a direction opposite to that of the coordinate axes, then the positive values of τ_{ij} are associated with forces directed oppositely to the positive directions of the coordinate axes.

Let σ be the area of the face ABC of the tetrahedron in figure. Let σ_1, σ_2, $\sigma_3$be the areas of the plane faces PBC, PCA and PAB (having normal's along $x_1 -$, $x_2 - \& \, x_3 -$axes) respectively.

Then $$\sigma_i = \sigma \cos(x_i, \hat{\nu}) = \sigma \nu_i \tag{6.6.1}$$

The volume of the tetrahedron is

$$v = \frac{1}{3} h\sigma \tag{6.6.2}$$

Assuming the continuity of the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}} = \overset{\nu}{T}_i$, the x_i component of the stress force acting on the face ABC of the tetrahedron PABC (made of elastic material) is $(\overset{\nu}{T}_i + \varepsilon_i)\sigma$

provided $$\lim_{h \to 0} \varepsilon_i = 0 \tag{6.6.3}$$

Here ε_i are inserted because the stress force acts at points of the oblique plane ABC and not on the given oblique plane through P. Under the assumption of continuing of stress field, quantities ε_1' are infinitesimals. We note that the plane element PBC is a part of the boundary surface of the material contained in the tetrahedron. As such, the unit outward normal to PBC is $-\hat{e}_i$. Therefore, the xi component of force due to stress acting on the face PBC of area σ_i is

$$(\tau_{1i} + \varepsilon_{1i})\sigma_1 \tag{6.6.4a}$$

where $\lim_{h \to 0} \varepsilon_{1i} = 0$

Similarly forces on the face PCA and PAB are

$$(\tau_{2i} + \varepsilon_{2i})\sigma_2, \ (\tau_{3i} + \varepsilon_{3i})\sigma_3$$

with $\lim_{h\to 0}\varepsilon_{2i} = \lim_{h\to 0}\varepsilon_{3i} = 0$

(6.6.4b)

On combining (6.6.4a) and (6.6.4b) , we write

$$(-\tau_{ji} + \varepsilon_{ji})\sigma_j \tag{6.6.5}$$

as the x_i -- component of stress force acting on the face of area provided $\lim_{h\to 0}\varepsilon_{ji} = 0$

In equation (6.6.5), the stress components are taken with the negative sign as the exterior normal to a face of area σj is in the negative direction of the xj axis.Let F_i be the body force per unit volume at the point P. Then the $x_i$component of the body force acting on the volume of tetrahedron PABC is

$$\frac{1}{3}h\sigma(F_i + \varepsilon_i') \tag{6.6.6}$$

where ε_i''s are infinitesimal and

$$\lim_{h\to 0}\varepsilon_i' = 0$$

Since the tetrahedral element PABC of the elastic body is in equilibrium, therefore, the resultant force acting on the material contained in PABC must be zero. Thus

$$(\overset{\nu}{T}_i + \varepsilon_i)\sigma + (-\tau_{ji} + \varepsilon_{ji})\sigma_j + \frac{1}{3}h\sigma(F_i + \varepsilon')h = 0$$

Using (6.6.1), above equation (after cancellation of σ) becomes

$$(\overset{\nu}{T}_i + \varepsilon_i) + (-\tau_{ji} + \varepsilon_{ji})\nu_j + \frac{1}{3}h\sigma(F_i + \varepsilon')h = 0 \tag{6.6.7}$$

As we take the $\lim h \to 0$ in (6.6.7), the oblique face ABC tends to the given oblique plane at P. Therefore, this limit gives

$$\overset{\nu}{T}_i - \tau_{ji}\nu_j = 0$$

or

$$\overset{\nu}{T}_i = \tau_{ji}\nu_j \tag{6.6.8}$$

This relation connecting the stress vector $\overset{\wedge}{\underset{\sim}{\overset{\nu}{T}}}$ and the stress components τ_{ij} is known as **Cauchy's law or formula.**

It is convenient to express the equation (6.6.8) in the matrix notation. This has the form

$$\begin{bmatrix} \overset{\nu}{T}_1 \\ \overset{\nu}{T}_2 \\ \overset{\nu}{T}_3 \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \qquad (6.6.8a)$$

As $\overset{\wedge}{\underset{\sim}{\overset{\nu}{T}}}$ and ν_i are vectors. Equation (6.6.8) shows, by quotient law for tensors, that **new components form a second order tensor.**

This stress tensor is called the **CAUCHY'S STRESS TENSOR.**

We note that, through a given point, there exist infinitely many surface plane elements. On every one of these elements we can define a stress vector. The totality of all these stress vectors is called the state of stress at the point. The relation (6.6.8) enables us to find the stress vector on any surface element at a point by knowing the stress tensor at that point. As such, the state of stress at a point is completely determined by the stress tensor at the point.

Note: In the above, we have assumed that stress can be defined everywhere in a body and secondly that the stress field is continuous. These are the basic assumptions of continuum mechanics. Without these assumptions, we can do very little. However, in the further development of the theory, certain mathematical discontinuities will be permitted / allowed.

6.7 BASIC BALANCE LAWS

(A) Balance of Linear Momentum:

So far, we have discussed the state of stress at a point. If it is **desired to move from one point to another,** the stress components will change. Therefore, it is necessary to investigate the equations / conditions which control the way in which they change.

While the strain tensor e_{ij} has to satisfy six compatibility conditions, the components of stress tensor must satisfy three linear **partial differential equations of the first order.** The principle of balance of linear momentum gives us these differential equations. This law, consistent with the Newton's second law of motion, states that **the time rate of change of linear momentum is equal to the resultant force on the elastic body.**

Consider a continuous medium in equilibrium with volume τ and bounded by a closed surface σ. Let F_i be the components of the **body force per unit volume** and $\overset{\nu}{T_i}$ be the component of the surface force in the x_i direction. For equilibrium of the medium, the resultant force acting on the matter within τ must vanish i.e.

$$\int_\tau F_i d\tau + \int_\sigma \overset{\nu}{T_i}\, d\sigma = 0 \qquad \text{for i = 1,2,3} \qquad (6.7.1)$$

We know the following Cauchy's formula

$$\overset{\nu}{T_i} = \tau_{ji}\nu_j \qquad \text{for i = 1,2,3} \qquad (6.7.2)$$

whereτ_{ij} is the stress tensor and ν_j is the unit normal to the surface. Using (6.7.2) into equation (6.7.l), we obtain

$$\int_\tau F_i d\tau + \int_\sigma \tau_{ji}\nu_j d\sigma = 0 \qquad \text{for i = 1,2,3} \qquad (6.7.3)$$

We assume that stresses τ_{ij} and their first order partial derivatives are also continuous and single valued in the region τ . Under these assumptions, Gauss-divergence theorem can be applied to the surface integral in (3) and we find

$$\int_\tau \tau_{ji,j} d\tau = \int_\sigma \tau_{ji}\nu_j d\sigma \qquad (6.7.4)$$

From equations (6.7.3) and (6.7.4), we write

$$\int_\tau (\tau_{ji} + F_i) d\tau = 0 \qquad (6.7.5)$$

for each i = l, 2 , 3. Since the region τ of integration is arbitrary (every part ofthe medium is in equilibrium) and the integrand is continuous, so, we must have

$$\tau_{ji.j} + F_i = 0 \qquad (6.7.6)$$

for each i = 1,2,3 .and at every interior point of the continuous elastic body. These equations are

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + F_1 = 0,$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + F_2 = 0, \qquad (6.7.7)$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + F_3 = 0,$$

These equations are referred to as **Cauchy's equations of equilibrium**. These equations are also called **stress equilibrium equations**. These equations are associated with undeformed Cartesian coordinates. These equations were obtained by Cauchy in 1827.

Note 1: In the case of motion of an elastic body, these equations (due to balance of linear momentum) take the form

$$\tau_{ji.j} + F_i = \rho \ddot{u}_i \qquad (6.7.8)$$

where $\ddot{u}_i$ is the acceleration vector and p is the density (mass per unit volume) of the body.

Note 2: When body force F_i is absent (or negligible), equations of equilibrium reduce to

$$\tau_{ji.j} = 0 \qquad (6.7.9)$$

Example: Show that for zero body force, the state of stress for an elastic body given by

$\tau_{11} = x^2 + y + 3z^2$, $\tau_{22} = 2x + y^2 + 2z$, $\tau_{33} = -2x + y + z^2$

$\tau_{12} = \tau_{21} = -xy + z^3$, $\tau_{13} = \tau_{31} = y^2 - xz$, $\tau_{23} = \tau_{32} = x^2 - yz$ is possible.

Example: Determine the body forces for which the following stress field describes a state of equilibrium

$\tau_{11} = -2x^2 - 3y^2 - 5z$, $\tau_{22} = -2y^2 + 7$, $\tau_{33} = 4x + y + 3z - 5$

$\tau_{12} = \tau_{21} = z + 4xy - 6$, $\tau_{13} = \tau_{31} = -3x + 2y + 1$, $\tau_{23} = \tau_{32} = 0$

Example: Determine whether the following stress field is admissible in an elastic body when body forces are negligible.

$$[\tau_{ij}] = \begin{bmatrix} yz+4 & z^2+2x & 5y+z \\ . & xz+3y & 8x^3 \\ . & . & 2xyz \end{bmatrix}$$

(B) Balance of Angular momentum

The principle of balance of angular momentum for an elastic solid is **"The time rate of change of angular momentum about the origin is equal to the resultant moment about of origin of body and surface forces."** This law assures the symmetry of the stress tensor τ_{ij}.

Let a continuous elastic body in equilibrium occupies the region τ bounded by surface σ. Let F_i be the body force acting at a point $P(x_i)$ of the body, Let the position vector of the point P relative to the origin be $\bar{r} = x_i \hat{e}_i$ Then , the moment of force F is $\bar{r} \times \bar{F} = \varepsilon_{ijk} x_j F_k$, where ε_{ijk} is the alternating tensor.

As the elastic body is in equilibrium, the resultant moment due to body and surface forces must be zero. So

$$\int_\tau \varepsilon_{ijk} x_j F_k \, d\tau + \int_\sigma \varepsilon_{ijk} x_j \overset{\nu}{T}_k \, d\sigma = 0 \text{ for each i = 1,2,3} \qquad (6.7.9)$$

Since, the body is in equilibrium, so the Cauchy's equilibrium equationsgive

$$F_k = -\tau_{lk,l} \qquad (6.7.10)$$

The stress vector $\overset{\nu}{T}_k$ in terms of stress components is given by $\overset{\nu}{T}_k = \tau_{lk} \nu_l$ (6.7.11)

The Gauss divergence theorem gives us

$$\int_\sigma \varepsilon_{ijk} x_j \tau_{lk} \nu_l \, d\sigma = \int_\tau \left[\varepsilon_{ijk} x_j \tau_{lk}\right]_{,l} d\tau$$

$$= \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \delta_{jl} \tau_{lk}] d\tau$$

$$= \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] d\tau \qquad (6.7.12)$$

From equations (6.7.9), (6.7.10) and (6.7.12); we write

$$\int_\tau \varepsilon_{ijk} x_j (-\tau_{lk,l}) d\tau + \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] d\tau = 0 \qquad (6.7.13)$$

This gives

$$\int_\tau \varepsilon_{ijk} x_j \tau_{jk} d\tau = 0 \qquad (6.7.14)$$

fori = 1, 2 , 3. Since the integrand is continuous and the volume is arbitrary, so

$$\varepsilon_{ijk} \tau_{jk} = 0 \qquad (6.7.15)$$

fori = 1, 2 , 3 and at each point of the elastic body. Expanding (6.7.5) , we write

$$\varepsilon_{123}\tau_{23} + \varepsilon_{132}\tau_{32} = 0$$

$$\Rightarrow \tau_{23} - \tau_{32} = 0$$

$$\varepsilon_{213}\tau_{13} + \varepsilon_{231}\tau_{31} = 0$$

$$\Rightarrow \tau_{23} - \tau_{32} = 0$$

$$\varepsilon_{312}\tau_{12} + \varepsilon_{321}\tau_{21} = 0$$

$$\Rightarrow \tau_{12} - \tau_{21} = 0$$

i.e. $\Rightarrow \tau_{ij} = \tau_{ji}$ $\quad for\ i \neq j$ at every point of the medium. (6.7.16)

This proves the symmetry of stress tensor. This law is also referred to **as Cauchy's second law**. It is due to **Cauchy** in 1827.

Note 1: On account of this symmetry, the state of stress at every point is specified by six instead of nine functions of position.

Note 2: In summary, the six components of the state of the stress must satisfy three partial differential equations $\tau_{ij,j} + F_i = 0$ within the body and the three relations ($\overset{\nu}{T}_i = \tau_{ji,j}\nu_j$) on the bounding surface. The equations $\overset{\nu}{T}_i = \tau_{ji,j}\nu_j$ are called the boundary conditions.

Note 3: Because of symmetry of the stress tensor, the equilibrium equations may be written as $\tau_{ij,j} + F_i = 0$

Note 4: Since $\overset{i}{T}_j = \tau_{ji}$, equations of equilibrium (using symmetry of τ_{ij}) may also be expressed as $\overset{i}{T}_{j,j} = -F_i$ or $div\,\underset{\sim}{\overset{i}{T}} = -F_i$

Note 5: Because of the symmetry of τ_{ij} , the boundary conditions can be expressed as

$\overset{\nu}{T}_i = \tau_{ij}\nu_j$

Remark: It is obvious that the three equations of equilibrium do not suffice for the determination of the six functions that specify the stress field. This may be expressed by the statement that the stress field is statistically indeterminate. To determine the stress field, the equations of equilibrium must be supplemented by other relations that can't be obtained from static considerations.

6.8 TRANSFORMATION OF COORDINATES

We have defined earlier the components of stress with respect to Cartesian system $ox_1x_2x_3$. Let $ox_1'x_2'x_3'$ be any other Cartesian system with thesame origin but oriented differently. Let these coordinates be connected by the linear relations

$$x_p' = \ell_{pi}x_i \qquad (6.8.1)$$

where ℓ_{pi} are the direction cosines of the x_p' - axis with respect to the x_1 - axis.

i.e
$$\ell_{pi} = \cos(x_p', x_i) \qquad (6.8.2)$$

Let τ'_{pq} be the components of stress in the new reference system (Figure 6.6)

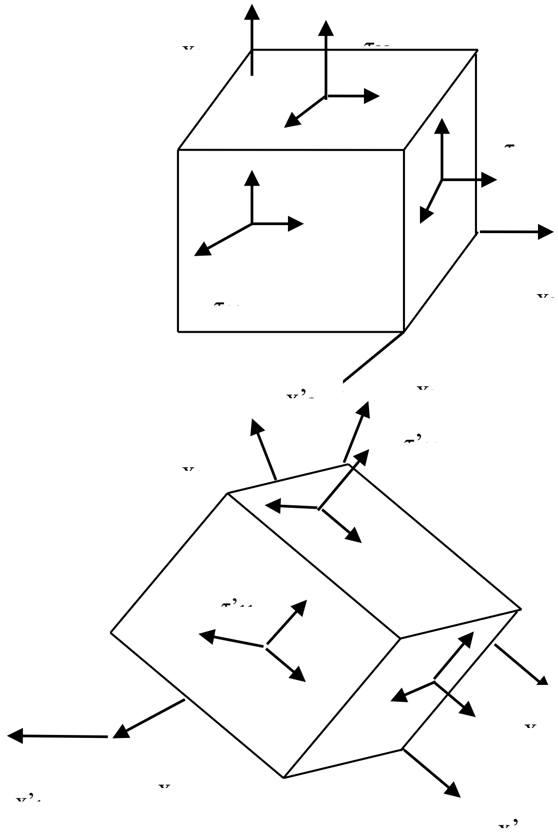


Figure6.6& 6.7

Figure6.7, Transformation of stress components under rotation of co-ordinates system.

Theorem: let the surface element $\Delta\sigma$ and $\Delta\sigma'$, with unit normal $\hat{\nu}$ and $\hat{\nu}'$, pass through the point P. Show that the component of the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ acting on $\Delta\sigma$ in the direction of $\hat{\nu}'$ is equal to the component of the stress vector $\overset{\hat{\nu}'}{\underset{\sim}{T}}$ acting on $\Delta\sigma'$ in the direction of $\hat{\nu}$

Proof: In this theorem, it is required to show that

Thus, $$\overset{\hat{\nu}'}{\underset{\sim}{T}} . \hat{\nu}' = \overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu} \qquad (6.8.3)$$

The Cauchy's formulagives us

$$\underset{\sim}{\overset{\hat{\nu}}{T}} = \tau_{ji}\nu_j \quad (6.8.4)$$

and

$$\underset{\sim}{\overset{\hat{\nu}'}{T}} = \tau_{ji}\nu'_j \quad (6.8.5)$$

due to symmetry of stress tensors as with

$$\hat{\nu} = \nu_j, \ \hat{\nu}' = \nu'_j$$

Now $\underset{\sim}{\overset{\hat{\nu}'}{T}} . \hat{\nu} = \overset{\hat{\nu}'}{T_i} . \hat{\nu}_i$

$$= (\tau_{ij}\nu'_j)\nu_i$$

$$= (\tau_{ji}\nu'_j)\nu_i$$

$$= \overset{\hat{\nu}}{T_i}\, \nu'_i \quad (6.8.6)$$

This completes the proof of the theorem.

Article: Use the formula (6.8.3) to derive the formulas of transformation of the components of the stress tensor τ_{ij}.

Solution: Since the stress components τ'_{pq} is the projection on the x'_p — axis of the stress vector acting on a surface element normal to the x'_p — axis (by definition), we can write

$$\tau'_{pq} = \overset{P}{T_q} = \overset{\hat{\nu}'}{T} . \hat{\nu} \quad (6.8.7)$$

where

$\hat{\nu}'$ is parallel to the x'_p-axis (6.8.8)

$\hat{\nu}$ is parallel to the x'_q - axis

Equations (6.8.6) and (6.8.7) imply

$$\tau'_{pq} = \tau_{ij}\nu'_i\nu_j \quad (6.8.9)$$

Since

$$\nu_i' = \cos(x_p', x_i) = \ell_{pi} \tag{6.8.10}$$

$$\nu_i = \cos(x_q', x_i) = \ell_{qi}$$

Equation (6.8.9) becomes

$$\tau_{pq}' = \tau_{ij}\nu_i'\nu_j \tag{6.8.11}$$

Equation (6.8.11) and definition of a tensor of order 2, show that the stress components τ_{ij} transform like a Cartesian tensor of order 2. Thus, the physical concept of stress which is described by τ_{ij}agrees with the mathematical definition of a tensor of order 2 in a Euclidean space.

6.9 Theorem: Show that the quantity

$\Theta = \tau_{11} + \tau_{22} + \tau_{33}$ is invariant relative to an orthogonal transformation of Cartesian coordinates.

Proof: Let τ_{ij} be the tensor relative to the Cartesian system $ox_1x_2x_3$. Let these axes be transformed to $ox_1'x_2'x_3'$ under the orthogonal transformation

$$x_p' = \ell_{pi}x_i \tag{6.9.1}$$

where

$$\ell_{pi} = \cos(x_p', x_i) \tag{6.9.2}$$

Let τ_{pi} be the stress components relative to new axes, then these components are given by the rule for second order tensors.

$$\tau_{pp}' = \ell_{pi}\ell_{pj}\tau_{ij} \tag{6.9.3}$$

Putting q= p and taking summation over the common suffix, we write

This implies $\tau_{pp}' = a_{pi}a_{pj}\tau_{ij}$

$$= \delta_{ij}\tau_{ij} = \tau_{ij}$$

$$\tau_{11}' + \tau_{22}' + \tau_{33}' = \tau_{11} + \tau_{22} + \tau_{33} = \Theta \tag{6.9.4}$$

This proves the theorem.

Remark: This theorem shows that whatever be the orientation of three mutually orthogonal planes passing through a given point, the sum of the normal stresses is independent of the orientation of these planes.

Exercise 1: Prove that the tangential traction parallel to a line l , across a plane at right angles to a line l' , the two lines being at right angles to each other , is equal to the tangential traction, parallel to the line l' , across a plane at right angles to l.

Exercise 2: Show that the following two statements are equivalent.

(a) The components of the stress are symmetric.

(b) Let the surface elements Δσ and Δσ' with respective normal $\hat{\nu}$ and $\hat{\nu}'$ passes through a point P. Then $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu}' = \underset{\sim}{\overset{\hat{\nu}'}{T}} . \hat{\nu}$

Hint: (b) ⇒ (a)

Let $\hat{\nu} = \hat{i}$ and $\hat{\nu}' = \hat{j}$

Then $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu}' = \underset{\sim}{\overset{i}{T}} . \hat{j} = \overset{i}{T}_j = \tau_{ij}$

and $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu} = \underset{\sim}{\overset{i}{T}} . \hat{i} = \overset{j}{T}_i = \tau_{ji}$

by assumption $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu}' = \underset{\sim}{\overset{\hat{\nu}'}{T}} . \hat{\nu}$,

therefore $\tau_{ij} = \tau_{ji}$

This shows that τ_{ij} is symmetric.

Example l: The stress matrix at a point P in a material is given as

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}.$$

Find

(i) The stress vector on a plane element through P and parallel to the plane $2x_1 + x_2 - x_3 = 1$,

(ii) The magnitude of the stress vector, normal stress and the shear stress.

(iii) The angle that the stress vector makes with normal to the plane.

Solution: (i) The plane element on which the stress vector is required is parallel to the plane $2x_1 + x_2 - x_3 = 1$. Therefore, direction ratios of the normal to the required plane at P are< 2, 1,-1>. So, the d.c.'s of the unit normal $\hat{\nu} = \nu_i$ to the required plane at P are

$$\nu_1 = \frac{2}{\sqrt{6}}, \nu_2 = \frac{1}{\sqrt{6}}, \nu_3 = -\frac{1}{\sqrt{6}}$$

let $\overset{\nu}{\underset{\sim}{T}}. = \overset{\nu}{\underset{\sim}{T_i}}$ be the required stress vector. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T_1} \\ \overset{\nu}{T_2} \\ \overset{\nu}{T_3} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}$$

or $$\overset{\nu}{T_1} = \sqrt{3/2}, \overset{\nu}{T_2} = 3\sqrt{3/2}, \overset{\nu}{T_3} = \sqrt{3/2}$$

So, the required stress vector at P is

$$\overset{\nu}{T_1} = \sqrt{3/2}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3) \text{ and } |\overset{\nu}{\underset{\sim}{T}}| = \sqrt{33/2}$$

(ii) The normal stress is given by

$$\sigma = \overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}.\hat{\nu} = \sqrt{\frac{3}{2}}..\frac{1}{\sqrt{6}}(2+3-1) = \frac{1}{2} \times 4 = 2$$ the shear stress is given by

$$\tau = \sqrt{|\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}|^2 - \sigma^2} = \sqrt{33/2-4} = \frac{5}{\sqrt{2}}$$

(As $\tau \neq 0$,so the stress vector $\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}.$ need not be along the normal to the plane element)

iii)let θ be the angle between the stress vector $\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}.$ and normal $\hat{\nu}$.

Then

$$\cos\theta = \frac{\overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu}}{\left|\overset{\hat{\nu}}{\underset{\sim}{T}}\right| . \left|\hat{\nu}\right|} = 2 \Big/ \sqrt{33/2} = \sqrt{8/33}$$

This determines the required inclination.

Example 2: The stress matrix at a point P(x_i) in a material is given by

$$\left[\tau_{ij}\right] = \begin{bmatrix} x_3 x_1 & x_3^2 & 0 \\ x_3^2 & 0 & -x_2 \\ 0 & -x_2 & 0 \end{bmatrix}$$

Find the stress vector at the point Q (l, 0, -l) on the surface $x_2^2 + x_3^2 = x_1$

Solution: The stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is required on the surface element

f(x_1 , x_2 , x_3) $= x_1 - x_2^2 - x_3^2 = 0$, at the point Q(l , 0 , -l). We find $\nabla f = \hat{e}_1 + 2\hat{e}_3$ and $\left|\nabla f\right| = \sqrt{5}$ at the point Q.

Hence, the unit outward normal $\hat{\nu} = \nu_i$ to the surface f = 0 at the pointQ(1,0,-1) is

$$\hat{\nu} = \frac{\nabla f}{\left|\nabla f\right|} = \frac{1}{5}(\hat{e}_1 + 2\hat{e}_3)$$

giving

$$\nu_1 = \frac{1}{\sqrt{5}}, \nu_2 = 0, \nu_3 = \frac{2}{\sqrt{5}}$$

The stress matrix at the point Q(1, 0, -l) is

$$\left[\tau_{ij}\right] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

let $\overset{\nu}{\underset{\sim}{T}} = \overset{\nu}{\underset{\sim}{T_i}}$ be the required stress vector at the point Q. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T}_1 \\ \overset{\nu}{T}_2 \\ \overset{\nu}{T}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

or $$\overset{\nu}{T}_1 = -\sqrt{1/5}\,, \overset{\nu}{T}_2 = \sqrt{1/5}\,, \overset{\nu}{T}_3 = 0$$

So, the required stress vector at P is

$$\overset{\nu}{T}_1 = \frac{1}{\sqrt{5}}(-\hat{e}_1 + \hat{e}_2)$$

Example 3: The stress matrix at a certain point in a given material is given by

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Find the normal stress and the shear stress on the octahedral plane element through the point.

Solution: An octahedral plane is a plant whose normal makes equal angles withpositive directions of the coordinate axes.Hence, the components of the unit normal $\hat{\nu} = \nu_i$ are

$$\nu_1 = \nu_2 = \nu_3 = \frac{1}{\sqrt{3}}$$

let $\underset{\sim}{\overset{\nu}{T}} = \underset{\sim}{\overset{\nu}{T_i}}$ be the required stress vector. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T}_1 \\ \overset{\nu}{T}_2 \\ \overset{\nu}{T}_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

or $$\overset{\nu}{T}_1 = \sqrt{5/3}\,, \overset{\nu}{T}_2 = \sqrt{3}\,, \overset{\nu}{T}_3 = \sqrt{3}$$

The magnitude of this stress vector is

$$\left|\overset{\nu}{\underset{\sim}{T}}\right| = \sqrt{43/3}$$

let σ be the normal stress and τ be the shear stress.Then

$$\sigma = \overset{\nu}{\underset{\sim}{T}}.\hat{\nu} = \frac{1}{3}(5+3+3) = \frac{11}{3} \text{ and } \tau = \sqrt{\frac{43}{3} - \frac{121}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Since σ> 0, the normal stress on the octahedral plane is tensile.

Example 4: The state of stress at a point P in cartesian coordinates is given by

τ_{11}=500, τ_{12}= τ_{21}=500, τ_{13}= τ_{31}=800, τ_{22}=1000,τ_{33}= -300, τ_{23}= τ_{32}= -750

Compute the stress vector $\overline{T}$ and the normal and tangential components of stress on the plane passing through P whose outward normal unit vector is $\hat{\nu} = \frac{1}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$

Solution: The stress vectoris given by $\overset{\nu}{T}_i = \tau_{ji}\nu_j$,

We find $\overset{\nu}{T}_1 = \tau_{11}\nu_1 + \tau_{21}\nu_2 + \tau_{31}\nu_3 = 250 + 250 + 400\sqrt{2}$ =1064(approx.)

$$\overset{\nu}{T}_2 = \tau_{12}\nu_1 + \tau_{22}\nu_2 + \tau_{32}\nu_3 = 250 + 500 + \frac{750}{\sqrt{2}} = 221(approx.)$$

$$\overset{\nu}{T}_3 = \tau_{13}\nu_1 + \tau_{23}\nu_2 + \tau_{33}\nu_3 = 400 - 375 + 150\sqrt{2} = 237(approx.)$$

Books Recommended:

6. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

7. **S. Timoshenko and N. Goodier,** Theory of Elasticity, McGraw Hill, New York, 1970.

CHAPTER-I

CARTESIAN TENSOR

1.1 Introduction

The concept of a tensor has its origin in the developments of differential geometry by Gauss, Riemann and Christoffel. The emergence of Tensor calculus, as a systematic branch of Mathematics is due to Ricci and his pupil Levi-Civita. In collaboration they published the first memoir on this subject: - '***Methods de calcul differential absolu et leurs applications***' **Mathematische Annalen, Vol. 54,** (**1901**).

The investigation of relations which remain valid when we change from one coordinate system to any other is the chief aim of Tensor calculus. The laws of Physics cannot depend on the frame of reference which the physicist chooses for the purpose of description. Accordingly it is aesthetically desirable and often convenient to utilize the Tensor calculus as the mathematical background in which such laws can be formulated. In particular, Einstein found it an excellent tool for the presentation of his General Relativity theory. As a result, the Tensor calculus came into great prominence and is now invaluable in its applications to most branches of theoretical Physics; it is also indispensable in the differential geometry of hyperspace.

A physical state or a physical phenomenon of the quantity which is invariant, i.e remain unchanged, when the frame of reference within which the quantity is defined is changed that quantity is called **tensor.** In this chapter, we have to confine ourselves to Cartesian frames of reference.

As a Mathematical entity, a tensor has an existence independent of any coordinate system. Yet it may be specified in a particular coordinate system by a certain set of quantities, known as its components. Specifying the components of a tensor in one coordinate system determines the components in any other system according to some definite law of transformation.

Under a transformation of cartesian coordinate axes, a scalar quantity, such as the **density** or the **temperature**, remain unchanged. This means that a scalar is an invariant under a coordinate transformation. Scalars are called **tensors of zero rank.** All physical quantities having magnitude only are tensors of zero order. It is assumed that the reader has an elementary knowledge of determinants and matrices.

Rank/Order of tensor

1) If the value of the quantity at a point in space can be described by a single number, the quantity is a scalar or a tensor of rank/order zero. For example, '5' is a scalar or tensor of rank/order zero.
2) If three numbers are needed to describe the quantity at a point in the space, the quantity is a tensor of rank one. For example vector is a tensor of rank/order one.
3) If nine numbers are needed to describe the quantity, the quantity is a tensor of rank three. The 3×3, 1×9 and 9×1, nine numbers describe the quantity is an example of tensor of rank/order 3.
4) In general, if 3^n numbers are needed to describe the value of the quantity at a point in space, the quantity is a tensor of rank/order n. A quantity described by 12 or 10 or 8 ………… numbers, then the quantity is not a tensor of any order/rank.

OR

Tensor: A set of members/numbers 3^n represents the physical quantity in the reference coordinates, then the physical quantity is called a tensor of order n.

1.1.1 Characteristics of the tensors

1) Tensors are the quantities describing the same phenomenon regardless of the coordinate system used; they provide an important guide in the formulation of the correct form of physical law. Equations describing physical laws must be tensorially homogenous, which means that every term of the equation must be a tensor of the same rank.

2) The tensor concept provides convenient means of transformation of an equation from one system of coordinates to another.
3) An advantage of the use of Cartesian tensors is that once the properties of a tensor of a certain rank have been established, they hold for all such tensors regardless of the physical phenomena they represent.

   Note: For example, in the study of strain, stress, inertia properties of rigid bodies, the common bond is that they are all symmetric tensors of rank two.

1.2 Notation and Summation Convention

Let us begin with the matter of notation. In tensor analysis one makes extensive use of indices. A set of n variables $x_1, x_2, ..., x_n$ is usually denoted as x_i, $i = 1, 2, 3, ... n$. Consider an equation describing a plane in a three-dimensional space

$$a_1x_1 + a_2x_2 + a_3x_3 = p \qquad (1.2.1)$$

where a_i and p are constants. This equation can be written as

$$\sum_{i=1}^{3} a_i x_i = p \qquad (1.2.2)$$

However, we shall introduce the summation convention and write the equation above in the simple form $a_i x_i = p$ (1.2.3)

The convention is as follow: The repetition of an index (*whether superscript or subscript*) in a term will denote a summation with respect to that index over its range. The range of an index i is the set of n integer values 1 to n. An index that is summed over is called a *dummy index*, and one that is not summed out is called a free index.

1.3 Law of Transformation

Let $P(x_1, x_2)$ be a physical quantity in $ox_1x_2x_3$ is the Cartesian coordinate systems before deformation and $P'(x_1', x_2')$ be corresponding to $P(x_1, x_2)$ in the new coordinate system $ox'_1x'_2x'_3$ after rotating the x_3-axis about itself at an angle θ, i.e., after deformation.

From the figure given below (Figure1.1)

$$\begin{aligned} x_1 &= OM \\ &= ON - MN \\ &= ON - M'N' \end{aligned}$$

$$= x'_1 \cos\theta - x'_2 \sin\theta \tag{1.3.1}$$

$$\begin{aligned} x_2 &= PM \\ &= PN' + N'M \\ &= PN' + M'N \end{aligned}$$

$$= x'_2 \cos\theta + x'_1 \sin\theta \tag{1.3.2}$$

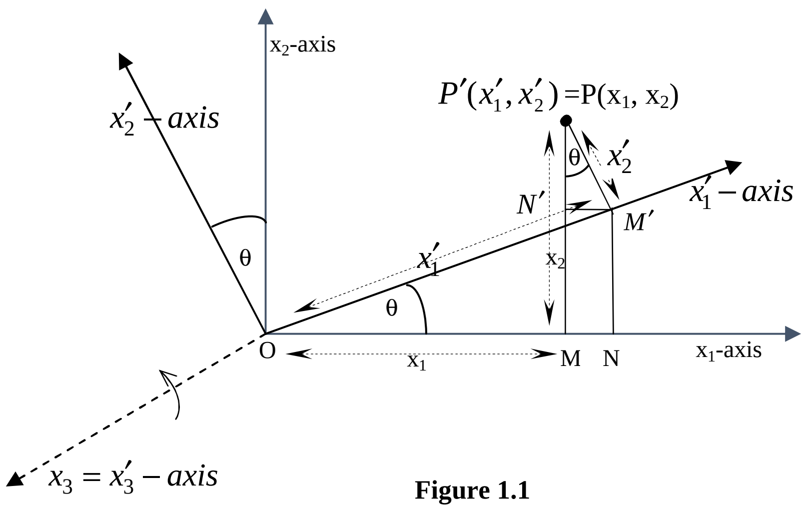


Figure 1.1

Using the relation (1.3.1) and (1.3.2) we get

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta + 0x_3 \tag{1.3.3}$$

$$x'_2 = -x_1 \sin\theta + x_2 \cos\theta + 0x_3 \tag{1.3.4}$$

$$x'_3 = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \tag{1.3.5}$$

Relation (1.3.3), (1.3.4) and (1.3.5) can be written as

$$x_1' = x_1\ell_{11} + x_2\ell_{12} + x_3\ell_{13} \tag{1.3.6}$$

$$x_2' = x_1\ell_{21} + x_2\ell_{22} + x_3\ell_{23} \tag{1.3.7}$$

$$x_3' = x_1\ell_{31} + x_2\ell_{32} + x_3\ell_{33} \tag{1.3.8}$$

where $\ell_{ij} = \cos(angle between x_i' \; and \, x_j)$; $i, j = 1,2,3$ that is (1.3.9)

$$\ell_{11} = \cos(angle between \, x_1' \; and \; x_1) = \cos\theta$$

$$\ell_{12} = \cos(angle between \, x_1' \; and \; x_2) = \cos(90-\theta) = \sin\theta$$

$$\ell_{13} = \cos(angle between \, x_1' \; and \; x_3) = \cos 90$$

$$\ell_{21} = \cos(angle between \, x_2' \; and \; x_1) = \cos(90+\theta) = -\sin\theta$$

$$\ell_{22} = \cos(angle between \, x_2' \; and \; x_2) = \cos\theta$$

$$\ell_{23} = \cos(angle between \, x_2' \; and \; x_3) = \cos 90$$

$$\ell_{31} = \cos(angle between \, x_3' \; and \; x_1) = \cos 90$$

$$\ell_{32} = \cos(angle between \, x_3' \; and \; x_2) = \cos 90$$

$$\ell_{33} = \cos(angle between \, x_3' \; and \; x_3) = \cos 0 = 1$$

Law of transformation can be written in a tensor form of order one as follow

$$x_1' = \ell_{11}x_1 + \ell_{12}x_2 + \ell_{13}x_3 = \ell_{1j}x_j \quad ; j = 1,2,3$$

$$x_i' = \ell_{ij}x_j \; ; \; i, j = 1,2,3 \tag{1.3.10}$$

$$\frac{\partial x_i'}{\partial x_j} = \ell_{ij} \text{ and } \frac{\partial x_i}{\partial x_j'} = \ell_{ji}$$

Similarly, law of transformation for a tensor of order two

$$x_{pq}' = \ell_{pi}\ell_{qj}x_{ij} \; ; \; i, j = 1,2,3; \text{ p, q are dummy variables} \tag{1.3.11}$$

law of transformation for a tensor of order three

$$x'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}x_{ijk};\ i, j, k = 1,2,3; \text{ p, q, r are dummy variables} \quad (1.3.12)$$

and law of transformation of order n

$$x'_{pqr..........n\ terms} = (\ell_{pi}\ell_{qj}\ell_{rk}............n\,terms)x_{ijk.............n\,terms} \quad (1.3.13)$$

where $i, j, k,..........n\,terms = 1,2,3,..........n$; p, q, r, ……….n terms are dummy variables

Example.1. The x'_i-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ=30^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find it's coordinate in the x'_i-system. If a point has coordinate (1, 3, 2) in the x'_i-system, find its coordinates in the x_i -system.

Solution. The figure (1.2) shows how the x'_i-system is related to the x_i-system. The direction cosines for the given transformation is represented in relation (1.3.14)

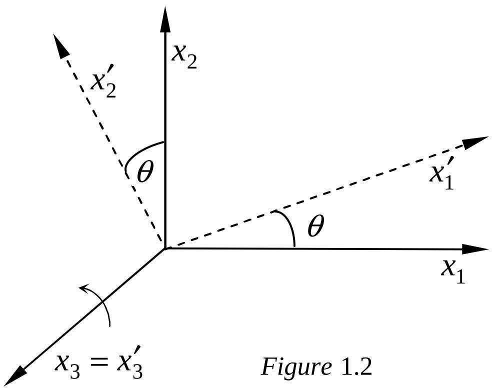


Figure 1.2

Hence, the matrix of the transformation by using (1.9) is

$$(\ell_{ij}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.3.14)$$

Using law of transformation for a tensor of order one, i.e, form (1.3.10), we get

$x'_i = \ell_{ij} x_j$; i, j=1, 2, 3

$$x'_1 = \ell_{11}x_1 + \ell_{12}x_2 + \ell_{13}x_3$$

$$\Rightarrow \quad x'_1 = 2\cos\theta + 4\sin\theta + 1\times 0 = 2\times \sqrt{3}/2 + 4\times 1/2 + 1\times 0 = (\sqrt{3}+2)$$

$$\Rightarrow \quad x'_2 = 2\sin\theta + 4\cos\theta + 1\times 0 = -2\times 1/2 + 4\times \sqrt{3}/2 + 1\times 0 = (2\sqrt{3}-1)$$

$$\Rightarrow \quad x'_3 = 2\times 0 + 4\times 0 + 1\times 1 = 1 \qquad (1.3.15)$$

Hence, (x'_1, x'_2, x'_3) = ($\sqrt{3}+2, 2\sqrt{3}-1, 1$) is in new coordinate system.

Further for the second, (1, 3, 2) are the coordinate of a point in new coordinate system, i.e. ($x'_1 = 1, x'_2 = 3, x'_3 = 2$) to finding the corresponding coordinate in to old coordinate system i.e. (x_1, x_2, x_3) . Using law of transformation (1.3.10),

we have $\quad x_i = \ell_{ji} x'_j$; i, j= 1,2,3 (1.3.16)

or $\quad x_1 = \ell_{11}x'_1 + \ell_{21}x'_2 + \ell_{31}x'_3$

$$x_2 = \ell_{12}x'_1 + \ell_{22}x'_2 + \ell_{32}x'_3$$

$$x_3 = \ell_{13}x'_1 + \ell_{23}x'_2 + \ell_{33}x'_3$$

$$\Rightarrow \quad x_1 = \cos\theta x'_1 - \sin\theta\, x'_2, \qquad x'_3 = 1\times\sqrt{3}/2 - 3\times 1/2 + 2\times 0 = (\sqrt{3}/2 - 3/2)$$

$$\Rightarrow \quad x_2 = \sin\theta x'_1 + \cos\theta\, x'_2, \qquad x'_3 = 1\times 1/2 + 3\times\sqrt{3}/2 + 2\times 0 = (1/2 - \sqrt{3}/2)$$

$$\Rightarrow \quad x_3 = \cos 90^0\, x'_1 + \sin 90^0\, x'_2 + 1, \qquad x'_3 = 1\times 0 + 3\times 0 + 2\times 1 = 2 \qquad (1.3.17)$$

Hence, (x_1, x_2, x_3) = $(\sqrt{3}/2 - 3/2, 1/2 - \sqrt{3}/2, 2)$ in old coordinate system.

Practice 1. The x_i'-system is obtained by rotating the x_i-system about the x_2-axis through an angle θ=45^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Practice 2. The x_i'-system is obtained by rotating the x_i-system about the x_1-axis through an angle θ=60^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Practice 3. The x_i'-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ= 60^0 in the sense of right handed screw. Find the transformation matrix. If a point has coordinates (2, 4, 1) in the x_i-system, find its coordinate in the x_i'-system. If a point has coordinate (1, 3, 2) in the x_i'-system, find its coordinates in the x_i-system.

Example2. The x_i'-system is obtained by rotating the x_i-system about the x_2-axis through an angle θ= 60^0 in the sense of right handed screw. Find the transformation matrix. If a tensor of rank/order two has components $[a_{ij}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

Solution. The figure (1.3) shows how the x_i'-system is related to the x_i-system. The direction cosines for the given transformation are represented in the (1.3.18) when x_2

-axis is rotated at an angle 60^0 about itself in right handed screw, where a'_{pq} are the components of the tensor of order two in new coordinate system corresponding to a_{ij} in old coordinate system.

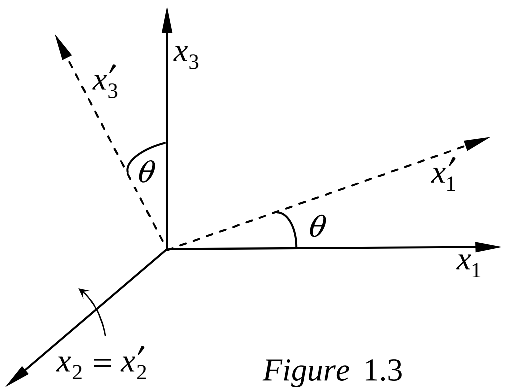


Figure 1.3

Hence, the matrix of the transformation is by using (1.3.9)

$$(\ell_{ij}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \qquad (1.3.18)$$

Using law of transformation (1.3.11) for a tensor of order two, i.e

$$x'_{pq} = \ell_{pi}\ell_{qj}x_{ij}$$

$$a'_{pq} = \ell_{pi}\ell_{qj}a_{ij}$$

$$\Rightarrow \; a'_{11} = \ell_{1i}\ell_{1j}a_{ij}$$

$$= \ell_{1i}(\ell_{11}a_{i1} + \ell_{12}a_{i2} + \ell_{13}a_{i3})$$

$$= \ell_{11}(\ell_{11}a_{11} + \ell_{12}a_{12} + \ell_{13}a_{13}) + \ell_{12}(\ell_{11}a_{21} + \ell_{12}a_{22} + \ell_{13}a_{23}) + \ell_{13}(\ell_{11}a_{31} + \ell_{12}a_{32} + \ell_{13}a_{33})$$

using value of ℓ_{ij} from (1.3.18), we have

$$a'_{11} = 1/2\,(1/2\times 1 + 0\times 0 - \sqrt{3}/2\times 1)$$
$$+\,0(1/2\times 0 + 0\times 2 + \sqrt{3}/2\times 2)$$
$$+\,\sqrt{3}/2\,(-1/2\times 2 + 0\times 0 + \sqrt{3}/2\times 1)$$

$$= \frac{1}{2}\left(\frac{1-\sqrt{3}}{2}\right) + 0 + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-2}{2}\right) = \left(\frac{4-3\sqrt{3}}{4}\right) \qquad (1.3.19)$$

Similarly, $a'_{22} = 2, a'_{33} = \dfrac{4+3\sqrt{3}}{4}$

and $\quad a'_{23} = \ell_{2i}\ell_{3j}a_{ij}$

$$= \ell_{2i}(\ell_{31}a_{i1} + \ell_{32}a_{i2} + \ell_{33}a_{i3})$$

$$= \ell_{21}(\ell_{31}a_{11} + \ell_{32}a_{12} + \ell_{33}a_{13})$$
$$+\,\ell_{22}(\ell_{31}a_{21} + \ell_{32}a_{22} + \ell_{33}a_{23})$$
$$+\,\ell_{33}(\ell_{31}a_{31} + \ell_{32}a_{32} + \ell_{33}a_{33})$$

$$= 0\times(1/2\times 1 + 0\times 0 - \sqrt{3}/2\times 1)$$
$$+\,1\times(-\sqrt{3}/2\times 0 + 0\times 2 + 1/2\times 2)$$
$$+\,0\times(-1/2\times 2 + 0\times 0 + \sqrt{3}/2\times 1)$$

$$a'_{23} = 0 + 1 + 0 = 1 \qquad (1.3.20)$$

Similarly, $a'_{31} = \dfrac{1}{4}, a'_{13} = \dfrac{5}{4}, a'_{12} = 0, a'_{21} = \sqrt{3}, a'_{32} = \dfrac{1}{2}$

Hence,

the tensor $[a_{ij}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ is transformed into $[a'_{pq}] = \begin{bmatrix} \left(\dfrac{4-3\sqrt{3}}{4}\right) & 0 & \dfrac{5}{4} \\ \sqrt{3} & 2 & 1 \\ \dfrac{1}{4} & \dfrac{1}{2} & \dfrac{4+3\sqrt{3}}{4} \end{bmatrix}$

Practice 4. The x_i'-system is obtained by rotating the x_i-system about the x_3-axis through an angle θ= 45[0] in the sense of right handed screw. Find the transformation matrix. If a tensor of rank two has components $\left[a_{ij}\right]=\begin{bmatrix}3 & -2 & 1\\ 1 & 3 & 2\\ 2 & -1 & 4\end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

Practice 5. The x_i'-system is obtained by rotating the x_i-system about the x_1-axis through an angle θ= 30[0] in the sense of right handed screw. Find the transformation matrix. If a tensor of rank two has components $\left[a_{ij}\right]=\begin{bmatrix}1 & -2 & 0\\ -1 & 3 & -2\\ 2 & -1 & 1\end{bmatrix}$ in the x_i-system, find its coordinate in the x_i'-system.

1.4 Some Properties of Tensor

Zero Tensors: A tensor whose all components in one Cartesian coordinates system are 0 is called a zero. A tensor may have any order n.

Property 1.4.1 If all component of a tensor are '0' in one coordinate system then they are '0' in all coordinate systems.

Proof. Let $u_{ijk.............n\,terms}$ and $u'_{pqr.............n\,terms}$ the component of a n[th] order tensor in two coordinates systems $ox_1x_2x_3$ and $ox_1'x_2'x_3'$.

Suppose $\quad u_{ijk............n\ terms}=0, \forall\, i, j, k.......$ (1.4.1)

We know the law of transformation of tensor of order n as

$$u'_{pqr...........nterms}=(\ell_{pi}\ell_{qj}\ell_{rk}............nterms)u_{ijk.............nterms} \qquad (1.4.2)$$

Using (1.4.10) into (1.4.11) we get

$u'_{pqr..........nterms} = 0$, $\forall\, p, q, r........$ Hence, zero tensor of any order in one coordinate system remains always zero tensor of same order in all other coordinate systems.

Property 1.4.2 If the corresponding components of two tensors of the same order are equal in one coordinate system, then they are equal in all coordinate systems.

Property 1.4.3 Equality of Tensors: Two tensors of the same order whose corresponding components are equal in a coordinate system (and hence in all coordinates) are called equal tensors.

Thus, in order to show that two tensors are equal, it is sufficient to show that their corresponding components are equal in any one of the coordinate system.

Property 1.4.4 (**Scalar multiplication of a tensor**): If components of a tensor of order n are multiplied by a scalar α, then the resulting components form a tensor of the same order n.

Proof: Let $u_{ijk.........nterms}$ be a tensor of order n in $ox_1x_2x_3$ system. Let $u'_{pqr...........n\,terms}$ be the corresponding components in the dashed ($ox'_1x'_2x'_3$) system. The transformation rule for a tensor of order n, (1.3.13) yields.

$$u'_{pqr.............nterms} = \ell_{pi}\ell_{qj}\ell_{rk}............n\,terms(u_{ijk..............n\,terms}) \qquad (1.4.3)$$

Now $$\alpha u'_{pqr.............nterms} = \ell_{pi}\ell_{qj}\ell_{rk}............n\,terms(\alpha u_{ijk..............n\,terms}) \qquad (1.4.4)$$

This shows that components $\alpha u_{ijk..............}$ form a tensor of rank n.

Property 1.4.5 (Sum and Difference of tensors) If $u_{ijk..........nterms}$ and $v_{ijk.........nterms}$ are tensors of the same rank n then their sum ($u_{ijk..........nterms} + v_{ijk.........nterms}$) is a tensor of the same order n.

Proof: Let $$w_{ijk..........nterms} = u_{ijk..........nterms} + v_{ijk.........nterms} \qquad (1.4.5)$$

and let $u'_{pqr..........n\,terms}$ and $v'_{pqr..........n\,terms}$ be the components of the given tensors of order n relative to the new system $ox'_1x'_2x'_3$. Then transformation rules for these tensors

are $$u'_{pqr............n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}...........n\,terms(u_{ijk..............n\,terms}) \quad (1.4.6)$$

and $$v'_{pqr............n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}...........n\,terms(v_{ijk..............n\,terms}) \quad (1.4.7)$$

where $$\ell_{pi} = \cos(x'_p, x_i) \quad (1.4.8)$$

let $$w'_{pqr..........n\,terms} = u'_{pqr..........n\,terms} + v'_{pqr..........n\,terms} \quad (1.4.9)$$

using relations (1.4.6 and 1.4.7) in the relation (1.4.9), we get

$$w'_{pqr..........n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}........n\,terms\,(u_{ijk..........n\,terms} + v_{ijk..............n\,terms} \quad (1.4.10)$$

$$w'_{pqr..........n\,terms} = \ell_{pi}\ell_{qj}\ell_{rk}........n\,terms\,(w_{ijk..........n\,terms}) \quad (1.4.11)$$

Thus quantities $w_{ijk..........n\,terms}$ obey the transformation rule of a tensor of order n. Therefore, they are components of a tensor of rank/order n.

Corollary: Similarly, their difference $u_{ijk..........n\,terms} - v_{ijk..............n\,terms}$ is also a tensor of rank n.

Property 1.4.6 (Tensor Multiplication)

The product of two tensors is also a tensor whose order is the sum of orders of the given tensors.

Proof: Let $u_{ijk.............m\,terms}$ and $v_{\alpha\beta\gamma.............n\,terms}$ be two tensors of order m and n respectively in the coordinate system $ox_1x_2x_3$ also $u'_{pqr............m\ terms}$ and $v'_{\sigma\tau\varsigma............n\ terms}$ are corresponding components of tensors in $ox'_1x'_2x'_3$ system.

We shall show that the product

$$w_{ijk.........m\,terms+\alpha\beta\gamma.......n\,terms} = u_{ijk.........n\,terms} \times v_{\alpha\beta\gamma.........n\,terms} \quad (1.4.5)$$

is tensor of order m+n. Using the law of transformation (1.3.13), we have

$$\left.\begin{aligned} u'_{pqr\ldots\ldots m\,terms} &= \ell_{pi}\ell_{qj}\ell_{rk}\ldots\ldots m\,terms(u_{ijk\ldots\ldots m\,terms}) \\ v'_{\sigma\tau\varsigma\ldots\ldots n\,terms} &= \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}\ldots\ldots n\,terms(v_{\alpha\beta\gamma\ldots\ldots m\,terms}) \end{aligned}\right\} \quad (1.4.6)$$

where, ℓ_{ij} is having its standard meaning as defined in relation (1.3.9).

Let $\quad w'_{pqr\ldots m\,terms+\sigma\tau\varsigma\ldots n\,terms} = u'_{pqr\ldots n\,terms} \times v'_{\sigma\tau\varsigma\ldots n\,terms}$ (1.4.7)

Using relation (1.4.6) in to (1.4.7), we get

$$w'_{pqr\ldots m\,terms+\sigma\tau\varsigma\ldots n\,terms} =$$

$$\ell_{pi}\ell_{qj}\ell_{rk}\ldots m\,terms(u_{ijk\ldots m\,terms}) \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}\ldots n\,terms(v_{\alpha\beta\gamma\ldots m\,terms})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}\ldots m\,terms \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}\ldots n\,terms(v_{\alpha\beta\gamma\ldots m\,terms} \times u_{ijk\ldots m\,terms})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}\ldots m\,terms \times \ell_{\sigma\alpha}\ell_{\tau\beta}\ell_{\varsigma\gamma}\ldots n\,terms(w_{\alpha\beta\gamma\ldots m\,terms} +_{ijk\ldots m\,terms}) \quad (1.4.8)$$

This shows that components $w_{ijk\ldots m\,terms+\alpha\beta\gamma\ldots n\,terms}$ obey the transformation rule of a tensor of order (m+n). Hence $u_{ijk\ldots n\,terms} \times v_{\alpha\beta\gamma\ldots n\,terms}$ are components of a $(m+n)^{th}$ order tensor.

Practice 6. If u_i and v_j are components of vectors, then show that $u_i v_j$ are components of a second-order tensor.

Practice 7. If u_{ij} and v_k are components of tensors of second-order and first-order, respectively, then prove that $u_{ij}v_k$ are components of a third order tensor.

Practice 8. If u_{ij} and v_{km} are components of second-order tensors, then prove that $u_{ij}v_{km}$ are components of a fourth order tensor.

Practice 9. If u_i and v_j are components of two tensors. Let $w_{ij} = u_i v_j + u_j v_i$ and $\alpha_{ij} = u_i v_j - u_j v_i$. Show that each of w_{ij} and α_{ij} is a second order tensor.

1.5 Contraction of a Tensor

The operation or process of setting two suffixes equal in a tensor and then summing over the dummy suffix is called a contraction operation or simply a contraction. *The tensor resulting from a contraction operation is called a* ***contraction*** *of the original tensor. Contraction operations are applicable to tensor of all orders higher than 1 and each* ***such operation reduces the order of a tensor by 2.***

Property 1.5 Prove that the result of applying a contraction of a tensor of order n is a tensor of order (n-2).

Proof: Let $u_{ijk.........n\,terms}$ and $u'_{pqr.........n\,terms}$ be the components of the given tensor of order n relative to two Cartesian coordinate systems $ox_1x_2x_3$ and $ox'_1x'_2x'_3$. The rule of transformation of tensor of order n (1.3.13) is

$$u'_{pqr..........n\ terms} = (\ell_{pi}\ell_{qj}\ell_{rk}............n\,terms)u_{ijk.............n\,terms} \qquad (1.5.1)$$

without loss of generality, we contract the given tensor by setting $i = j$ and summation convention. Let

$$v_{kl...........} = u_{iikl.............} \qquad (1.5.2)$$

Now $$u'_{pqr..........n\ terms} = (\ell_{pi}\ell_{qi})\ell_{rk}............n\,terms \times u_{iik.............n\,terms} \qquad (1.5.3)$$

$$= (\delta_{pq})\ell_{rk}...........n\,terms \times v_{kl.................(n-2)\,terms}$$

$$u'_{ppr................} = \ell_{rk}.........(n-2)terms \times v_{kl.............(n-2)\,terms} \quad \because \delta_{pq} = \begin{cases} 1 \ if \ \ p = q \\ 0 \ if \ \ p \neq q \end{cases}$$

$$v'_{r...............(n-2)\,terms} = \ell_{rk}.........(n-2)terms \times v_{kl.............(n-2)\,terms} \qquad (1.5.4)$$

Hence, the resulting tensor is tensor of order n-2. So contraction applying once on a tensor of order greater then 1, the order of the tensor reduces by 2. Similarly contraction applying twice on a tensor of order n the order of that tensor reduces by 4.

1.6 Quotient law of Tensors

(*Quotient law is the partial converse of the contraction law)*

Property 1.6 If there is an entity represents by the set of 9 quantities u_{ij} relative to any given system of Cartesian axes, and if $u_{ij}v_j$ is a vector for an arbitrary vector v_j, then show that u_{ij} is a second order tensor.

Proof: $$w_i = u_{ij}v_j \tag{1.6.1}$$

Suppose that u'_{pq}, u'_p and w'_p be the corresponding components in the dashed system $ox'_1x'_2x'_3$. Then by using law of transformation and inverse law of transformation (1.3.10 and 11)

Now $$u'_{pq}v'_p = w'_p \tag{1.6.2}$$

$$= \ell_{pi}w_i$$

$$= \ell_{pi}(u_{ij}v_j)$$

$$= \ell_{pi}\ell_{qj}u_{ij}v'_q$$

$\Rightarrow$ $$(u'_{pq} - \ell_{pi}\ell_{qj}u_{ij})v'_q = 0 \tag{1.6.3}$$

for an arbitrary vector v'_q. Therefore, we must have

$$u'_{pq} = \ell_{pi}\ell_{qj}u_{ij} \tag{1.6.4}$$

This rule shows that components u_{ij} obey the tensor law of transformation of a second order. Hence, u_{ij} is a tensor of order two.

Practice 10. Let α_i be an ordered triplet and β_i be a vector, referred to the x_i-axis. If $\alpha_i\beta_i$ is a scalar, show that α_i are component of a vector.

Example 3. If there is an entity representable by a set of 27 quantities u_{ijk} relative to $ox_1x_2x_3$ system and if $u_{ijk}v_{jk}$ is a tensor of order one for an arbitrary tensor v_{jk} if order 2, show that u_{ijk} is tensor of order 3.

Solution. Let $w_i = u_{ijk}v_{jk}$ (1.6.5)

It is given that v_{jk} is a tensor of order 2 and $u_{ijk}v_{jk}$ is a tensor of order one, and v'_{pq}, u'_{pqr} are corresponding to v_{jk}, u_{ijk} in new coordinate system $ox'_1x'_2x'_3$. Then by using transformation law and inverse transformation law (1.3.10 and 11) we get.

$$u'_{pqr}v'_{qr} = w'_p \tag{1.6.6}$$

$$= \ell_{pi}w_i$$

$$= \ell_{pi}u_{ijk}v_{jk} \quad \text{(by using 1.6.5)}$$

$$= \ell_{pi}u_{ijk}(\ell_{qj}\ell_{rk}v'_{qr})$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk}v'_{qr}$$

$$\Rightarrow \quad (u'_{pqr} - \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk})v'_{qr} = 0 \tag{1.6.7}$$

for an arbitrary vector v'_{qr}. Therefore, we must have

$$u'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}u_{ijk} \tag{1.6.8}$$

This rule shows that components u_{ijk} obey the tensor law of transformation of a second order. Hence, u_{ijk} is a tensor of order two.

Practice 11. If there is an entity representable by a set of 27 quantities u_{ijk} relative to $ox_1x_2x_3$ system and if $u_{ijk}v_k$ is a tensor of order two for an arbitrary tensor v_k of order one, show that u_{ijk} is tensor of order 3.

Practice 12. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_{jkl}$ is a tensor of order one for an arbitrary tensor v_{jkl} if order 3, show that u_{ijkl} is tensor of order 4.

Practice 13. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_l$ is a tensor of order three for an arbitrary tensor v_l if order one, show that u_{ijkl} is tensor of order 4.

Practice 14. If there is an entity representable by a set of 81 quantities u_{ijkl} relative to $ox_1x_2x_3$ system and if $u_{ijkl}v_{kl}$ is a tensor of order two for an arbitrary tensor v_{kl} of order 2, show that u_{ijkl} is tensor of order 4.

1.7 Symmetric & Skew symmetric tensors

1.7.1 A second order tensor u_{ij} is said to be symmetric if $u_{ij} = u_{ji} \;\; \forall i, j$. For example unit matrix of order 3×3 is symmetric tensor of order two.

1.7.2 A second order tensor u_{ij} is said to be skew-symmetric if $u_{ij} = -u_{ji} \;\; \forall i, j$. For example skew-symmetric matrix of order 3×3 is skew-symmetric tensor of order two.

Definition: (Gradient) if $u_{pqr.........n\,terms}$ is a tensor of order n in $ox_1x_2x_3$ system, then

$$v_{spqr..........(n+1)\,terms} = \frac{\partial}{\partial s} u_{pqr..........n\,terms}$$

$$= u_{pqr.........n\,terms,s} \tag{1.7.1}$$

is defined as the gradient of the tensor $u_{pqr.........n\,terms}$.

For example $u_{p,q} = \frac{\partial}{\partial x_q} u_p$ represents the gradient of vector u_p.

Property 1.7 Show that the gradient of a scalar point function is a tensor of order one.

Proof: Suppose that $U = U(x_1, x_2, x_3)$ be a scalar point function and

$$v_i = \frac{\partial U}{\partial x_i} = U_{,i} \qquad (1.7.2)$$

Let the components of the gradient of U in the dashed system $ox_1'x_2'x_3'$ be v_p', so that

$$v_p' = \frac{\partial U}{\partial x_p'} \qquad (1.7.3)$$

Using the law of transformation (1.3.10) and inverse law of transformation we have

$$v_p' = \frac{\partial U}{\partial x_p'}$$

$$= \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial x_p'} \qquad \text{(by chain rule)}$$

$$= \ell_{pi} \frac{\partial U}{\partial x_i} = \ell_{pi} U_{,i}$$

Using (1.7.2), we get $\quad v_p' = \ell_{pi} v_i \qquad (1.7.4)$

Which is a transformation rule for a tensor of order one. Hence gradient of the scalar point function U is a tensor of order one.

Property 1.8 Show that the **gradient of a vector** u_i is a tensor of order two.

Proof: The gradient of the tensor u_i is defined as

$$w_{ij} = \frac{\partial u_i}{\partial x_j} = u_{i,j} \tag{1.7.5}$$

Let the vector u_i be transformed to the vector u'_p relative to the new system $ox'_1x'_2x'_3$. Then the transformation law for tensors of orders one (1.3.10) yields

$$u'_p = \ell_{pi}u_i \tag{1.7.6}$$

Suppose the nine quantities w_{ij} relative to new system are transformed to w'_{pq}. Then

$$w'_{pq} = \frac{\partial u'_p}{\partial x'_q}$$

$$= \frac{\partial}{\partial x'_q}(\ell_{pi}u_i) = \ell_{pi}\frac{\partial u_i}{\partial x'_q}$$

$$= \ell_{pi}\frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial x'_q} \quad \text{(by chain rule)}$$

$$= \ell_{pi}\ell_{qj}\frac{\partial u_i}{\partial x_j} = \ell_{pi}\ell_{qj}w_{ij}$$

$$\Rightarrow \qquad w'_{pq} = \ell_{pi}\ell_{qj}w_{ij} \tag{1.7.7}$$

This is a transformation rule for tensors of order two. Hence, w_{ij} is a tensor of order two. Consequently, the gradient of a vector u_i is a tensor of order two.

Property 1.9 Show that the **gradient of a tensor of order n,** $u_{ijk.........n\,terms}$ is a tensor of order (n+1).

Proof: Let $u_{ijk.........n\,terms}$ is a tensor of order n. The gradient of the tensor $u_{ijk.........n\,terms}$ is defined as

$$w_{\tau\, pqr.............} = \frac{\partial u_{ijk..........n\,terms}}{\partial x_\tau} = u_{ijk..........n\,terms,\tau} \tag{1.7.8}$$

Let the tensor $u_{ijk......n\,terms}$ be transformed to the tensor $u'_{prs........n\,terms}$ relative to the new system $ox'_1x'_2x'_3$. Then the transformation law for tensors of order n (1.3.13) yields

$$u'_{pqr...........n\ \ terms} = (\ell_{pi}\ell_{qj}\ell_{rk}............n\,terms)u_{ijk..............n\,terms} \qquad (1.7.9)$$

Suppose 3^n quantities $w_{ijk..........n\,terms}$ relative to new system are transformed to $w'_{pqr..........n\,terms}$. Then

$$w'_{pqr........n\,terms,\tau} = \frac{\partial u'_{pqr..........n\,terms}}{\partial x'_\tau}$$

$$= (\ell_{pi}\ell_{qj}\ell_{rk}.......n\,terms)\frac{\partial u_{ijk.........n\,terms}}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial x'_\tau}$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}.......n\,terms\,\ell_{\tau\alpha}\frac{\partial u_{ijk.............n\,terms}}{\partial x_\alpha}$$

$$= \ell_{pi}\ell_{qj}\ell_{rk}..........n\,terms\,\ell_{\alpha\tau}\times u_{ijk.......nterms,\tau} \qquad (1.7.10)$$

$\Rightarrow$ $$w'_{pqr.........n\,terms,\tau} = \ell_{pi}\ell_{qj}\ell_{rk}..........n\,terms\,\ell_{\alpha\tau}u_{ijk..........,\tau}$$

This is a transformation rule for tensors of order (n+1). Hence, $w_{ijk.........(n+1)terms}$ is a tensor of order (n+1). Consequently, the gradient of a tensor of order n is a tensor of order (n+1).

Books Recommended:

1. **Y.C.Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey, 1965.
2. **Saad, A.S.** Elasticity-Theory and Applications, Pergamon Press, Inc. NY, 1994.
3. **Sokolnikoff, I.S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

CHAPTER-II

ANALYSIS OF TENSOR

Consider an ordered set of ***N*** real variables $x_1, x_2, x_3, ...x_i, ...x_N$; these variables will be called the **coordinates** of a point. (The suffixes *1, 2, 3,……,i,…..N*, which we shall call superscripts, merely serve as labels and do not possess any signification as power indices. Later we shall introduce quantities of the a_i and again the i, which we shall call a subscript, will act only as a label.) Then all the point corresponding to all values of the coordinates are said to form an ***N*-dimensional space,** denoted by V_N. Several or all of the coordinates may be restricted in range to ensure a one-one correspondence between points of the V_N, and sets of coordinates.

A curve in the V_N is defined as the assemblage of points which satisfy the ***N*** equations

$$x_i = x_i(u), \quad (i = 1, 2, 3, ...N)$$

where u is a parameter and $x_i(u)$ are N functions of u, which obey certain continuity conditions. In general, it will be sufficient that derivatives exist up to any order required. **A subspace** V_M of V_N is defined for $M < N$ as the collection of points which satisfy the ***N*** equations

$$x_i = x_i(u_1, u_2, ..., u_M), \quad (i = 1, 2, 3, ...N)$$

where there are M parameters $u_1, u_2, ..., u_M$. The $x_i(u_1, u_2, ..., u_M)$ are N functions of the $u_1, u_2, ..., u_M$ satisfying certain conditions of continuity. In addition the $M \times N$ matrix formed from the partial derivatives $\frac{\partial x_i}{\partial u_j}$ is assumed to be of rank M^*. When $M = N - 1$, the subspace is called a **hyper surface**.

Let us consider a space V_N with the coordinate system $x_1, x_2, x_3, ...x_N$. The N equations

$$\vec{x}_i = \varphi_i(x_1, x_2, ...x_N), \quad (i = 1, 2, 3, ...N) \qquad (2.1)$$

where the φ_i are single-valued continuous differentiable functions of the coordinates, define a new coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3 ...., \bar{x}_N$. Equations (2.1) are said to define **a transformation of coordinates**. It is essential that the N functions φ_i be independent. A necessary and sufficient condition is that the **Jacobian determinant** formed from the partial derivatives $\frac{\partial \bar{x}_i}{\partial x_j}$ does not vanish. Under this condition we can solve equations (2.1) for the x_i as functions of the $\bar{x}_i$ and obtain

$$x_i = \varphi_i(\bar{x}_1, \bar{x}_2, \bar{x}_3, ...\bar{x}_N) \quad (i = 1, 2, 3, ...N)$$

2.1 The Symbol δ_{ij}

We will now introduce the following two conventions:

1) Latin indices, used either as subscripts or superscripts, will take all values from 1 to N unless the contrary is specified. Thus equations (2.1) are briefly written $\bar{x}_i = \varphi_i(x_1, x_2, ...x_N)$, the convention informing us that there are N equations.
2) If a Latin index is repeated in a term, then it is understood that a summation with respect to that index over the range 1, 2, 3, …. N is implied. Thus instead of the expression $\sum_{i=1}^{N} a_i x_i$, we merely write $a_i x_i$. Now differentiation of (2.1) yields

$$d\bar{x}_i = \sum_{r=1}^{N} \frac{\partial \varphi_i}{\partial x_r} dx_r = \sum_{r=1}^{N} \frac{\partial \bar{x}_i}{\partial x_r} dx_r, \qquad (i = 1, 2, 3, ..., N)$$

which simplify, when the above conventions are used, to

$$d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_r} dx_r \,. \tag{2.2}$$

The repeated index r is called a **dummy index,** as it can be repeated by any other Latin index, except 'i' in this particular case. That is, equations (2.2) can equally well be written $d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_m} dx_m$ or for that matter $d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_r} dx_r$. In order to avoid confusion, the same index must not be used more than twice in any single term. For example; $\left(\sum_{i=1}^{N} a_i x_i\right)^2$ will not be written $a_i x_i a_i x_i$, but rather $a_i x_i a_j x_j$. It will always be clear from the context, usually powers will be indicated by the use of brackets; thus $(x_N)^2$ mean the square of x_N. The reason for using superscripts and subscripts will be indicated in due course. Let us introduce the **Kronecker delta.** It is defined as

$$\delta_{ij} = \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & if \ i = j \\ 0 & if \ i \neq j \end{cases} \tag{2.1.1}$$

That is, $\delta_{11} = \delta_{22} = \delta_{33} = 1$; $\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$.The symbol δ_{ij} is known as the Kronecker δ symbol, named after the German Mathematician Leopold Kronecker (1827-1891). The following property is inherent in the definition of δ_{ij}.

1) Kronecker δ is symmetric i.e $\delta_{ij} = \delta_{ji}$ (2.1.2)
2) Summation convention $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ (2.1.3)
3) The unit matrix of order 3 is $I_3 = (\delta_{ij})$ and $\det(\delta_{ij}) = 1$ (2.1.4)
4) The orthonormality of the base unit vectors $\hat{e}_i$ can be written as

$$\hat{e}_i \,.\, \hat{e}_j = \delta_{ij} \tag{2.1.5}$$

2.1.1 Tensor Equation:- An equation of type $\alpha_{ijk} - \beta_{ij} u_k = 0$ is called a tensor equation, for checking the correctness of a tensor equation, we have the following rule

(i) In a correctly tensor equation no suffixes shall appear more than twice in any term, otherwise the operation will not be define. For example $u'_j = \alpha_{ij} u_j v_j$ is not a tensor equation.

(ii) If a suffixes appears only once in a term then it must appear only once in the remaining term also. For example, an equation $u'_j - \ell_{ij} u_i = 0$ is not a tensor equation. Hence j appears once in the first term while it appears twice in the second term.

Property 2.1 Prove the following (**Known as substitution properties of** δ_{ij})

(i) $u_j = \delta_{ij} u_i$ (ii) $\delta_{ij} u_{jk} = u_{ik}$; $\delta_{ij} u_{ik} = u_{jk}$ (iii) $\delta_{ij} u_{ij} = u_{kk} = u_{11} + u_{22} + u_{33}$

Proof. (i) Now $\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} u_3$

$$\Rightarrow u_j + \sum_{\substack{i=j \\ i \neq j}}^{3} \delta_{ij} u_i = u_j \tag{2.1.6}$$

(ii) $\delta_{ij} u_{jk} = \sum_{j=1}^{3} \delta_{ij} u_{jk}$

$= \delta_{ii} u_{ik}$ (for $j \neq i, \delta_{ij} = 0$), here summation over i is not taken

$$= u_{ik} \tag{2.1.7}$$

(iii) $\delta_{ij} u_{ij} = \sum_i \left[\sum_j \delta_{ij} u_{ij} \right]$

$= \sum_i (1.u_{ii})$, in u_{ii} summation is not being taken

$$= \sum_i u_{ii} = u_{11} + u_{22} + u_{33} = u_{kk} \tag{2.1.8}$$

Example 2.1 Given that $a_{ij} = \alpha \delta_{ij} b_{kk} + \beta b_{ij}$,where $\beta \neq 0,\ 3\alpha + \beta \neq 0$, find b_{ij} in terms of a_{ij} .

Solution. Setting $i = j$ in the relation $a_{ij} = \alpha\delta_{ij}b_{kk} + \beta b_{ij}$ and summing accordingly, we obtain

$$a_{ii} = \alpha.3.b_{kk} + \beta.b_{ii}$$

$$= (3\alpha + \beta)b_{kk} \qquad (\because b_{kk} = b_{ii})$$

$$\Rightarrow \qquad b_{kk} = \frac{1}{3\alpha + \beta} a_{kk}$$

Hence, $$b_{ij} = \frac{1}{\beta}\left[a_{ij} - \alpha\delta_{ij}b_{kk}\right] = \frac{1}{\beta}\left[a_{ij} - \frac{\alpha}{3\alpha + \beta}\delta_{ij}a_{kk}\right] \qquad (2.1.9)$$

Property 2.2 Prove that (i) $\ell_{pi}\ell_{qi} = \delta_{pq}$ (ii) $\ell_{pi}\ell_{pj} = \delta_{ij}$ (iii) $|\ell_{ij}| = 1$, $(\ell_{ij})^{-1} = (\ell_{ij})'$

Proof. We know the transformation law of the coordinate system (1.3.10), we have

$$x'_p = \ell_{pi}x_i \text{ and } x_i = \ell_{qi}x'_q \qquad (2.1.10)$$

Now, (i) $x'_p = \ell_{pi}x_i$

$$\Rightarrow x'_p = \ell_{pi}(\ell_{qi}x'_q) \qquad (2.1.11)$$

using the relation (2.1.6) on the L.H.S. of (2.1.11)

$$\Rightarrow \delta_{pq}x'_q = \ell_{pi}\ell_{qi}x'_q$$

$$\Rightarrow (\ell_{pi}\ell_{qi} - \delta_{pq})x'_q = 0$$

$$\Rightarrow \ell_{pi}\ell_{qi} = \delta_{pq} \qquad (2.1.12)$$

(ii) Similarly, $x_i = \ell_{pi}x'_p$

$$\Rightarrow \qquad = \ell_{pi}\ell_{pj}x_j$$

Also $$x_i = \delta_{ij}x_j \qquad (2.1.13)$$

Hence, $$\delta_{ij}x_j = \ell_{pi}\ell_{pj}x_j$$

$$(\delta_{ij} - \ell_{pi}\ell_{pj})x_j = 0$$

$$\Rightarrow \qquad \delta_{ij} = \ell_{pi}\ell_{pj} \qquad (2.1.14)$$

(iii) Using (2.1.12) gives, in the expanded form,

$$\ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 = 1, \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 = 1, \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 1$$

$$\ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} = 0, \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{33} = 0, \ell_{31}\ell_{11} + \ell_{32}\ell_{12} + \ell_{33}\ell_{13} = 0$$

The relations (2.1.12) and (2.1.14) are referred as the orthonormal relations for ℓ_{ij}. In matrix notation, the above said relations may be represented respectively, as follows

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.1.15)$$

or $\quad LL' = L'L = 1$

these expressions show that the matrix L

Property 2.3 Show that δ_{ij} and ℓ_{ij} are tensors, each of order two.

Proof: Let u_i be any tensor of order one,

i> by the substitution property of the Kroneceker delta tensor δ_{ij}, we have

$$u_i = \delta_{ij} u_j \qquad (2.1.16)$$

Now u_i and u_j are each of tensor order one. Therefore, by quotient law, we conclude that δ_{ij} is a tensor of rank two.

ii> The transformation law for the first order tensor is

$$u'_p = \ell_{pi} u_i \qquad (2.1.17)$$

where u_i is a vector and $\ell_{pi} u_i$ is a vector by contraction property. Therefore, by quotient law, the quantities ℓ_{pi} are components of a second order tensor.

Note 1: The tensor δ_{ij} is called a unit tensor or an identity tensor of order two.

2. We may call the tensor ℓ_{ij} as the transformation tensor of rank two.

2.2 The Symbol $\in_{ijk}$

Euclidean geometry investigates the properties of figures which are invariant with respect to translations and rotations in space. It may be subdivided into Algebraic methods the theory applicable to entire configurations such as the class or degree of a curve. The latter discusses by means of the calculus those properties which depend on a restricted portion of the figure. For example, the total curvature of a surface at that point. Succinctly we may say that differential geometry is the study of geometry in small. This chapter is not intended to be a complete course on the subject. However, sufficient theory is developed to indicate the scope and power of the tensor method.

The symbol $\in_{ijk}$ is known as the Levi-civita $\in$-symbol, named after the Italian mathematician Tullio Levi-civita (1873-1941). The $\in$-symbol is also referred to as the **Permutation symbol/alternating symbol** or **alternator**. In terms of mutually orthogonal unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ along the Cartesian axes, it defined as

$$\hat{e}_i.(\hat{e}_j \times \hat{e}_k) = \in_{ijk} \quad \forall i, j, k = 1,2,3 \tag{2.2.1}$$

Thus, the symbol $\in_{ijk}$ gives

$$\in_{ijk} = \begin{cases} 1 & : \textit{if } i, j, k \textit{ take values in the cyclic order} \\ -1 & : \textit{if } i, j, k \textit{ take values in the acyclic order} \\ 0 & : \textit{if any or all of } i, j, k \textit{ take the same value} \end{cases} \tag{2.2.2}$$

These relations are 27 in number. The $\in$-symbol is useful in expressing the vector product of two vectors and scalar triple product.

(i) We have $\hat{e}_i \times \hat{e}_j = \in_{ijk} \hat{e}_k$. (2.2.3)

(ii) For two vectors a_i and b_i ,we write

$$\vec{a} \times \vec{b} = (a_i \hat{e}_i) \times (b_j \hat{e}_j) = a_i b_j (\hat{e}_i \times \hat{e}_j) = \in_{ijk} a_i b_j \hat{e}_k \tag{2.2.4}$$

(iii) $\vec{a} = a_i \hat{e}_i, \vec{b} = b_j \hat{e}_j, \vec{c} = c_k \hat{e}_k$

We have

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = (\vec{a}\times\vec{b}).\vec{c} = (\in_{ijk}\ a_i\ b_j\ \hat{e}_k).(c_k\ \hat{e}_k)$$

$$=\in_{ijk}\ a_i\ b_j\ c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (2.2.5)$$

Property 2.4 Show that $\in_{ijk}$ is a tensor of order 3.

Proof: Let $\vec{a} = a_i$ and $\vec{b} = b_i$ be any two vectors. Let

$$\vec{c} = c_i = \vec{a}\times\vec{b}\,.$$

Then, $c_i =\in_{ijk}\ a_j b_k$ (2.2.6)

Now $a_j b_k$ is a tensor of order 2 and $\in_{ijk}\ a_j b_k$ (by 2.2.6) is a tensor of order one. Therefore, by quotient law, $\in_{ijk}$ is a tensor of order 3.

Example 2.2 Show that $w_{ij} =\in_{ijk}\ u_k$ is a skew-symmetric tensor, where u_k is a vector and $\in_{ijk}$ is an alternating tensor

Solution: Since $\in_{ijk}$ is a tensor of order 3 and u_k is a tensor of order one, so by contraction, the product $\in_{ijk}\ u_k$ is a tensor of order 2. Further

$$w_{ij} =\in_{ijk}\ u_k$$

$$= -\in_{jik}\ u_k$$

$$= -w_{ji} \qquad (2.2.7)$$

This shows that w_{ij} is a tensor which is skew-symmetric.

Example 2.3 Show that u_{ij} is symmetric iff $\in_{ikj}\ u_{ij} = 0$

Solution: We find

$$\in_{ij1}\ u_{ij} =\in_{231}\ u_{23} + \in_{321}\ u_{32} = u_{23} - u_{32}$$

$$\in_{ij2} u_{ij} = \in_{312} u_{31} + \in_{132} u_{13} = u_{31} - u_{13}$$

$$\in_{ij3} u_{ij} = \in_{123} u_{12} + \in_{213} u_{21} = u_{12} - u_{21} \qquad (2.2.8)$$

Thus, u_{ij} is symmetric iff

$$u_{ij} = u_{ji} \text{ or } u_{12} = u_{21}, u_{13} = u_{31}, u_{23} = u_{32} \qquad (2.2.9)$$

2.3. Isotropic Tensors

Definition: A tensor is said to be an isotropic tensor if its components **remain unchanged/invariant** however the axes are rotated.

Note. 1. An isotropic tensor possesses no directional properties. Therefore a non-zero vector (or a non-zero tensor of rank 1) can never be an isotropic tensor. Tensor of higher orders, other than one, can be isotropic tensors.

2. Zero tensors of all orders are isotropic tensors.

3. By definition, a scalar (or a tensor of rank zero) is an isotropic tensor.

4. A scalar multiple of an isotropic tensor is an isotropic tensor.

5. The sum and the differences of two isotropic tensors is an isotropic tensor.

Property 2.5 Prove that substitution tensor δ_{ij} and alternating tensor $\in_{ijk}$ are isotropic tensors

Proof: A>Let the components δ_{ij} relative to x_i-system are transformed to quantities δ'_{pq} relative to x'_i-system. Then, the tensorial transformation rule is

$$\delta'_{pq} = \ell_{pi} \ell_{qj} \delta_{ij} \qquad (2.3.1)$$

Now R.H.S of (2.3.1)

$$= \ell_{pi} \left[\ell_{qj} \delta_{ij} \right] = \ell_{pi} \ell_{qi}$$

$$= \delta_{pq} = \begin{cases} 0 & if \quad p \neq q \\ 1 & if \quad p = q \end{cases} \qquad (2.3.2)$$

Relation (2.3.1) and (2.3.2) show that the components δ_{ij} are transformed into itself under all co-ordinate transformations. Hence, by definition, δ_{ij} is an isotropic tensor.

B> We know that $\in_{ijk}$ is a system of 27 numbers. Let

$$\in_{ijk} = \left[\hat{e}_i \ \hat{e}_j \ \hat{e}_k\right] = \hat{e}_i.(\hat{e}_j \times \hat{e}_k) \tag{2.3.3}$$

Be related to the x_i -axis. Then, the third order tensorial law of transformation (1.3.9) gives

$$\in'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk} \in_{ijk} \tag{2.3.4}$$

where ℓ_{pi} is defined in (1.3.9). We have already check that

$$\ell_{pi}\ell_{qj}\ell_{rk} \in_{ijk} = \begin{vmatrix} \ell_{p1} & \ell_{p2} & \ell_{p3} \\ \ell_{q1} & \ell_{q2} & \ell_{q3} \\ \ell_{r1} & \ell_{r2} & \ell_{r3} \end{vmatrix} \tag{2.3.5}$$

and

$$\left[\hat{e}'_p, \hat{e}'_q, \hat{e}'_r\right] = \begin{vmatrix} \ell_{p1} & \ell_{p2} & \ell_{p3} \\ \ell_{q1} & \ell_{q2} & \ell_{q3} \\ \ell_{r1} & \ell_{r2} & \ell_{r3} \end{vmatrix} \tag{2.3.6}$$

Using (2.3.4, 2.3.5 and 2.3.6), we get

$$\in'_{pqr} = \left[\hat{e}'_p, \hat{e}'_q, \hat{e}'_r\right] = \hat{e}'_p.(\hat{e}'_q \times \hat{e}'_r) = \begin{cases} 1 \ :if \ \ p,q,r \, are \ in \ cyclic \ order \\ -1 \ :if \ \ p,q,r \, are \ in \ anticyclic \ order \\ 0 : if \ any \ two \ or \ all \ suffices \ are \ same \end{cases} \tag{2.3.7}$$

This shows that components $\in_{ijk}$ are transformed into itself under all coordinate transformations. Thus, the third order tensor $\in_{ijk}$ is an isotropic.

Property 2.6 If u_{ij} is an isotropic tensor of second order, then show that $u_{ij} = \alpha\delta_{ij}$ for some scalar α.

Proof: As the given tensor is isotropic, we have

$$u'_{ij} = u_{ij} \tag{2.3.8}$$

for all choices of the x_i' -system. In particular, we choose

$$x_1' = x_2, x_2' = x_3, x_3' = x_1 \tag{2.3.9}$$

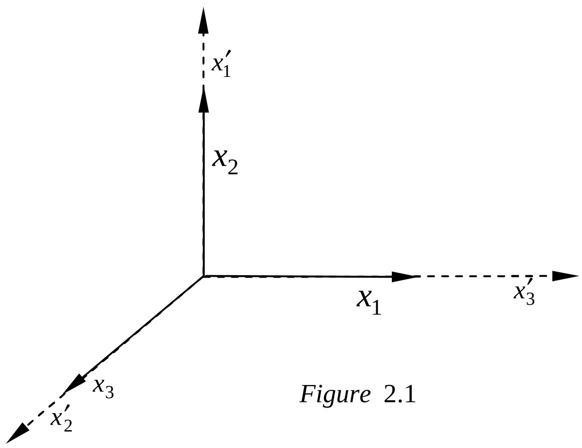


Figure 2.1

Then
$$\ell_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \tag{2.3.10}$$

and law of transformation (1.3.9), as

$$u'_{pq} = \ell_{pi}\ell_{qj}u_{ij} \tag{2.3.11}$$

Now
$$u'_{11} = \ell_{1i}\ell_{1j}u_{ij} = \ell_{1i}(\ell_{11}u_{i1} + \ell_{12}u_{i2} + \ell_{13}u_{i3})$$

$$= \ell_{1i}(0u_{i1} + \ell_{12}u_{i2} + 0u_{i3}) = \ell_{1i}\ell_{12}u_{i2}$$

$$= \ell_{12}(\ell_{11}u_{12} + \ell_{12}u_{22} + \ell_{13}u_{32}) = u_{22}$$

$\Rightarrow$
$$u'_{11} = u_{22} \tag{2.3.12}$$

Similarly,

$$u'_{22} = u_{33}, u'_{12} = u_{23}, u'_{12} = u_{23}, \; u'_{23} = u_{31}, u'_{13} = u_{21}, u'_{21} = u_{32} \tag{2.3.13}$$

Now, we consider the transformation: $x_1' = x_2, x_2' = -x_1, x_3' = x_3$ (2.3.14)

Then $$\ell_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (2.3.15)$$

Using law of transformation defined in (2.3.11), we get

$$u'_{13} = u_{13} = u_{23}, u'_{23} = u_{23} = -u_{13}$$

$$\Rightarrow u'_{13} = -u_{13},\ u_{13} = 0 \text{ and } u_{23} = 0 \qquad (2.3.16)$$

using (2.3.13) and (2.3.16), we obtain

$$\ell_{ij} = \alpha\delta_{ij} \qquad \text{where } \alpha = \ell_{11} = \ell_{22} = \ell_{33} \qquad (2.3.17)$$

Note 1: If ℓ_{ijk} are components of an isotropic tensor of third order, then $\ell_{ijk} = \alpha \in_{ijk}$ for some scalar α.

Note 2: If ℓ_{ijkm} are components of a fourth-order isotropic tensor, then

$$\ell_{ijkm} = \alpha\delta_{ij}\delta_{km} + \beta\delta_{ik}\delta_{jm} + \gamma\delta_{im}\delta_{jk} \text{ for some scalars } \alpha, \beta, \gamma.$$

2.4 Contravariant tensors (vectors)

A set of N functions f_i of the N coordinates x_i are said to be the components of a **contravariant vector** if they transform according to the equation.

$$\bar{f}_i = \frac{\partial \bar{x}_i}{\partial x_j} f_j \qquad (2.4.1)$$

on change of the coordinates x_i to $\bar{x}_i$. This means that any N functions can be chosen as the components of a contravariant vector in the coordinate system x_i, and the equations (2.4.1) define the N components in the new coordinate system $\bar{x}_i$. On multiplying equations (2.4.1) by $\frac{\partial x_k}{\partial \bar{x}_i}$ and summing over the index 'i' from 1 to N, we obtain

$$\frac{\partial x_k}{\partial \bar{x}_i}\bar{f}_i = \frac{\partial x_k}{\partial \bar{x}_i}\frac{\partial \bar{x}_i}{\partial x_j} f_j = \frac{\partial x_k}{\partial x_j} f_j = \delta_{ij} f_j = f_k \tag{2.4.2}$$

Hence the solution of equations (2.4.1) is

$$f_k = \frac{\partial x_k}{\partial \bar{x}_i}\bar{f}_i . \tag{2.4.3}$$

When we examine equations $d\,\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_r} dx_r$ (where repeated index r is called **dummy index**) we see that the differentials dx_i from the components of a contravariant vector, whose components in any other system are the differentials $d\bar{x}_i$ of the system. It follow immediately that dx_i / du is also a contravariant vector, called the **tangent vector** to the curve $x_i = x_i(u)$.

Consider now a further change of coordinates $x_i' = g_i(\bar{x}_1, \bar{x}_2, ...\bar{x}_n)$. Then the new components

$$f_i' = \frac{\partial x_i'}{\partial \bar{x}_j}\bar{f}_j = \frac{\partial x_i'}{\partial \bar{x}_j}\frac{\partial \bar{x}_j}{\partial x_k} f_k = \frac{\partial x_i'}{\partial x_k} f_k \tag{2.4.4}$$

This equation is of the same form as (2.4.1), which shows that the transformations of contravariant vectors form a group.

2.5 Covariant vectors

A set of N functions f_i of the N coordinates x_i are said to be the components of a **covariant vector** if they transform according to the equation.

$$\bar{f}_i = \frac{\partial x_j}{\partial \bar{x}_i} f_j \tag{2.5.1}$$

on change of the coordinates x_i to $\bar{x}_i$. Any N functions can be chosen as the components of a covariant vector in the coordinate system x_i, and the equations

(2.5.1) define the N components in the new coordinate system $\bar{x}_i$. On multiplying equations (2.5.1) by $\frac{\partial \bar{x}_i}{\partial x_k}$ and summing over the index 'i' from 1 to N, we obtain

$$\frac{\partial \bar{x}_i}{\partial x_k} \bar{f}_i = \frac{\partial \bar{x}_i}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} f_j = \frac{\partial x_j}{\partial x_k} f_j = \delta_{jk} f_j = f_k \qquad (2.5.2)$$

Since, $\frac{\partial \Gamma}{\partial \bar{x}_i} = \frac{\partial \Gamma}{\partial x_j} \frac{\partial x_j}{\partial \bar{x}_i}$, it follows immediately from (2.5.1) that the quantities $\frac{\partial \Gamma}{\partial x_i}$ are the components of a covariant vector, whose components in any other system are the corresponding partial derivatives $\frac{\partial \Gamma}{\partial x_i}$. Such a covariant vector is called the gradient of Γ. We now show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to transformations of the type

$$\bar{x}_i = a_{im} x_m + b_i, \qquad (2.5.3)$$

where b_i are N constants which do not necessarily form the components of a contravariant vector and a_{im} are constants (not necessary forming a tensor) such that

$$a_{ir} a_{im} = \delta_{rm} \qquad (2.5.4)$$

We multiply equations (2.5.3) by a_{ir} and sum over the index i from 1 to N and obtain

$$x_r = a_{ir} \bar{x}_i - a_{ir} b_i .$$

Thus,
$$\frac{\partial \bar{x}_i}{\partial x_j} = \frac{\partial x_j}{\partial \bar{x}_i} = a_{ij} \qquad (2.5.5)$$

This shows that the equations (2.4.1) and (2.5.1) define the same type of entity.

Books Recommended:

1. **Y. C. Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey,1965.
2. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
3. **Barry Spain** Tensor Calculus A Concise Course, Dover Publication, INC. Mineola, New York.

CHAPTER-III

APPLICATONS OF TENSOR

3.1 EIGENVALUES AND EIGEN VACTORS

Definition: Let u_{ij} be a **second order** symmetric tensor. A scalar λ is called an **eigenvalue** of the tensor u_{ij} if there exists a non-zero vector v_i such that

$$u_{ij}v_j = \lambda v_i \qquad \forall i, j = 1, 2, 3 \tag{3.1.1}$$

The non-zero vector v_i is then called an eigenvector of tensor u_{ij} corresponding to the eigen value λ. We observe that every (**non-zero**) scalar multiple of an eigenvector is also an eigen vector.

Property 3.1 Show that it is always possible to find three mutually orthogonal eigenvectors of a second order symmetric tensor.

Proof. Let u_{ij} be a second order symmetric tensor and λ be an eigen value of u_{ij}. Let v_i be an eigenvector corresponding to λ. Then

$$u_{ij}v_j = \lambda v_i \tag{3.1.2}$$

or

$$(u_{ij} - \lambda\delta_{ij})v_j = 0 \tag{3.1.3}$$

This is a set of three homogeneous simultaneous linear equations in three unknown v_1, v_2, v_3. These three equations are

$$\left.\begin{aligned} (u_{11}-\lambda)v_1 + u_{12}v_2 + u_{13}v_3 &= 0 \\ u_{21}v_1 + (u_{22}-\lambda)v_2 + u_{23}v_3 &= 0 \\ u_{31}v_1 + u_{32}v_2 + (u_{33}-\lambda)v_3 &= 0 \end{aligned}\right\} \tag{3.1.4}$$

This set of equations possesses a non-zero solution when

$$\begin{vmatrix} u_{11}-\lambda & u_{12} & u_{13} \\ u_{21} & u_{22}-\lambda & u_{23} \\ u_{31} & u_{32} & u_{33}-\lambda \end{vmatrix} = 0 \tag{3.1.5}$$

or $$|u_{ij} - \lambda\delta_{ij}| = 0 \tag{3.1.6}$$

expanding the determinant in (3.1.6), we find

$$\begin{aligned}(u_{11}-\lambda)[(u_{22}-\lambda)(u_{33}-\lambda)-u_{32}u_{23}]\\ -u_{12}[u_{12}(u_{33}-\lambda)-u_{31}u_{23}]\\ +u_{13}[u_{12}u_{32}-u_{31}(u_{22}-\lambda)]=0\end{aligned}$$

or $$\begin{aligned}&-\lambda^3+(u_{11}+u_{22}+u_{33})\lambda^2\\ &-(u_{11}u_{22}+u_{22}u_{33}+u_{33}u_{11}-u_{23}u_{32}-u_{31}u_{13}-u_{12}u_{21})\lambda\\ &+[u_{11}(u_{22}u_{33}-u_{23}u_{32})-u_{12}(u_{21}u_{33}-u_{31}u_{23})+u_{13}(u_{21}u_{32}-u_{31}u_{22})]=0\end{aligned} \tag{3.1.7}$$

we write (3.1.7) as

$$-\lambda^3+\lambda^2 I_1-\lambda I_2+I_3=0 \tag{3.1.8}$$

where $I_1 = u_{11}+u_{22}+u_{33} = u_{ii}$

$$I_2 = u_{11}u_{22}+u_{22}u_{33}+u_{33}u_{11}-u_{12}u_{21}-u_{23}u_{32}-u_{13}u_{31} = \frac{1}{2}[u_{ii}u_{jj}-u_{ij}u_{ji}]$$

$$I_3 = |u_{ij}| = \in_{ijk} u_{i1}u_{j2}u_{k3} \tag{3.1.9}$$

Equation (3.1.8) is a cubic equation in λ .Therefore it has three roots, say $\lambda_1, \lambda_2, \lambda_3$ which may not be distinct (real or imaginary). These roots (which are scalar) are the three eigenvalues of the symmetric tensor u_{ij} .

Further $$\lambda_1+\lambda_2+\lambda_3 = I_1 \tag{3.1.10}$$

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = I_2 \tag{3.1.11}$$

$$\lambda_1\lambda_2\lambda_3 = I_3 \tag{3.1.12}$$

Each root λ_i , when substituted in equation (3.1.4), gives a set of three linear equations (homogeneous) which are not all independent. By discarding one of equations and using the condition

$$v_1^2 + v_2^2 + v_3^2 = 1 \qquad (3.1.13)$$

for unit vectors, the eigenvector $\vec{v}_i$ is determined.

Property 3.2 Eigen values of a real symmetric tensor u_{ij} are real.

Proof. Let λ be eigenvalue with corresponding eigenvector v_j.

Then $$u_{ij} v_j = \lambda v_i \qquad (3.1.14)$$

Taking the complex conjugate on both sides of (3.1.14), we find

$$\bar{u}_{ij} \bar{v}_j = \bar{\lambda} \bar{v}_i$$

$$u_{ij} \bar{v}_j = \bar{\lambda} \bar{v}_i \qquad (3.1.15)$$

since u_{ij} is a real tensor. Now

$$\begin{aligned} u_{ij} \bar{v}_j v_i &= (u_{ij} \bar{v}_j) v_i \\ &= (\bar{\lambda}\, \bar{v}_j) v_i \\ &= (\bar{\lambda}\, \bar{v}_i) v_i \end{aligned} \qquad (3.1.16)$$

Taking complex conjugate of (3.1.16) both side

$$\begin{aligned} \overline{u_{ij} \bar{v}_j v_i} &= \bar{u}_{ij} v_j \bar{v}_i \\ &= u_{ij} v_j \bar{v}_i \quad \text{(by changing the role of } i \text{ and } j) \\ &= u_{ji} v_i \bar{v}_j \\ &= u_{ij} \bar{v}_j v_i \end{aligned} \qquad (3.1.17)$$

This shows that quantity $u_{ij} \bar{v}_j v_i$ is real. Hence $\bar{\lambda} \bar{v}_i v_i$ is real. Since $\bar{v}_i v_i$ is always real, it follows that $\bar{\lambda}$ is real.

Property 3.3 Eigen vector corresponding to two distinct eigen values of the sysmmetric tensor u_{ij} are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of u_{ij}. Let A_i and B_i be the corresponding non-zero eigenvectors. Then

$$u_{ij}A_j = \lambda_1 A_i\,, u_{ij}B_j = \lambda_1 B_i \tag{3.1.18}$$

We obtain

$$u_{ij}A_j B_i = \lambda_1 A_i B_i\,,\; u_{ij}B_j A_i = \lambda_2 A_i B_i \tag{3.1.19}$$

Interchanging the role of i and j

$$u_{ij}A_j B_i = u_{ji}A_i B_j = u_{ij}B_j A_i \tag{3.1.20}$$

From (3.1.19) and (3.1.20), we get

$$\lambda_1 A_i B_i = \lambda_2 A_i B_i$$

$$(\lambda_1 - \lambda_2)A_i B_i = 0$$

$$\Rightarrow \qquad A_i B_i = 0 \qquad (\because\; \lambda_1 \neq \lambda_2) \tag{3.1.21}$$

Hence, eigenvectors A_i and B_i are mutually orthogonal. This completes the proof.

Note: Now we consider various possibilities about eigenvalues λ_1, λ_2, λ_3.

Case 1: if $\lambda_1 \neq \lambda_2 \neq \lambda_3$, i.e., when all eigenvalues are different and real. Then, by property 3.3, three eigenvectors corresponding to λ_i are mutually orthogonal. Hence the results holds.

Case 2: if $\lambda_1 \neq \lambda_2 = \lambda_3$. Let $\vec{v}_{1i}$ be the eigenvector of the tensor u_{ij} corresponding to the eigenvalue λ_1 and $\vec{v}_{2i}$ be the eigenvector corresponding to λ_2. Then

$$\vec{v}_{1i}\cdot\vec{v}_{2i} = 0 \tag{3.1.22}$$

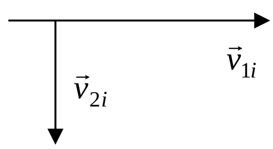


figure 3.1

Let $\vec{p}_i$ be a vector orthogonal to both $\vec{v}_{1i}$ and $\vec{v}_{2i}$. Then

$$\vec{p}_i \cdot \vec{v}_{1i} = \vec{p}_i \cdot \vec{v}_{2i} = 0 \qquad (3.1.23)$$

and
$$u_{ij}\vec{v}_{1j} = \lambda_1 \vec{v}_{1i}, \; u_{ij}\vec{v}_{2j} = \lambda_2 \vec{v}_{2i} \qquad (3.1.24)$$

Let
$$u_{ij} p_j = q_j = \text{a tensor of order 1} \qquad (3.1.25)$$

We shall show that p_i and q_i are parallel.

Now
$$q_i \vec{v}_{1i} = u_{ij} p_j \vec{v}_{1i}$$

$$= u_{ji} p_i \vec{v}_{1j} \quad \text{(By interchanging the role of } i \text{ and } j\text{)}$$

$$= \lambda_1 p_i \vec{v}_{1j} = 0 \qquad (3.1.26)$$

Similarly,
$$q_i \vec{v}_{2i} = 0 \qquad (3.1.27)$$

Thus, q_i is orthogonal to both orthogonal eigenvectors $\vec{v}_{1i}$ and $\vec{v}_{2i}$. Thus q_i must be parallel to p_i. So, we write

$$u_{ij} p_i = q_i = \alpha \, p_i \qquad (3.1.28)$$

for some scalar α.

Relation (3.1.28) shows that α must be an eigenvalue and p_i must be the corresponding eigenvector of u_{ij}.

$$\vec{v}_{3i} = \frac{p_i}{|p_i|} \qquad (3.1.29)$$

Since u_{ij} has only three eigenvalues $\lambda_1, \lambda_2 = \lambda_3$, so α must be equal to $\lambda_2 = \lambda_3$. Thus $\vec{v}_{3i}$ is an eigenvector which is orthogonal to both $\vec{v}_{1i}$ and $\vec{v}_{2i}$, where $\vec{v}_{1i} \perp \vec{v}_{2i}$. Thus, there exists three mutually orthogonal eigenvectors.

Further, let $\vec{w}_i$ be any vector which lies in the plane containing the two eigenvectors $\vec{v}_{2i}$ and $\vec{v}_{3i}$ corresponding to the repeated eigenvalues. Then

$$\vec{w}_i = k_1\vec{v}_{2i} + k_2\vec{v}_{3i}$$ for some scalars k_1 and k_2 and

$$\vec{w}_i \cdot \vec{v}_{1i} = k_1\vec{v}_{2i} \cdot \vec{v}_{1i} + k_2\vec{v}_{3i} \cdot \vec{v}_{1i} = 0 \qquad (3.1.30)$$

and

$$u_{ij}\vec{w}_i = u_{ij}(k_1\vec{v}_{2i} + k_2\vec{v}_{3i})$$

$$= k_1 u_{ij}\vec{v}_{2i} + k_2 u_{ij}\vec{v}_{3i}$$

$$= k_1\lambda_2\vec{v}_{2i} + k_2\lambda_3\vec{v}_{3i} \qquad (\lambda_2 = \lambda_3)$$

$$= \lambda_2(k_1\vec{v}_{2i} + k_2\vec{v}_{3i}) = \lambda_2 w_i \qquad (3.1.31)$$

Thus w_i is orthogonal to $\vec{v}_{1i}$ and w_i is an eigenvector corresponding to λ_2. Hence, any two orthogonal vectors those lie on the plane normal to $\vec{v}_{1i}$ can be chosen as the other two eigenvectors of u_{ij}.

Case 3: if $\lambda_1 = \lambda_2 = \lambda_3$

In this case, the cubic equation in λ becomes

$$(\lambda - \lambda_1)^3 = 0 \qquad (3.1.32)$$

or

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_1 - \lambda & 0 \\ 0 & 0 & \lambda_1 - \lambda \end{vmatrix} = 0 \qquad (3.1.33)$$

Comparing it with equation (3.1.6), we have

$$u_{ij} = 0 \quad \text{for } i \neq j$$

and

$$u_{11} = u_{22} = u_{33} = \lambda_1$$

Thus,

$$u_{ij} = \lambda_1\delta_{ij} \qquad (3.1.34)$$

Let $\vec{v}_i$ be any non-zero vector. Then

$$u_{ij}\vec{v}_j = \lambda_1\delta_{ij}\vec{v}_j$$

$$= \lambda_1\vec{v}_i \qquad (3.1.35)$$

This shows that $\vec{v}_i$ is an eigenvector corresponding to λ_1. Thus, every non-zero vector in space is an eigenvector which corresponds to the same eigenvalue λ_1. Of these vectors, we can certainly choose (at least) there vectors $\vec{v}_{1i}, \vec{v}_{2i}, \vec{v}_{3i}$ that are mutually orthogonal. Thus, in every case, there exists (at least) three mutually orthogonal eigenvectors of u_{ij}.

Example 1. Consider a second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

It is clear, the tensor u_{ij} is not symmetric. We shall find eigenvalues and eigenvectors of u_{ij}.

Solution. The characteristic equation is $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

or $\quad (1-\lambda)[(2-\lambda)(3-\lambda)-2]-1[2-2(2-\lambda)]=0$

or $\quad (1-\lambda)(2-\lambda)(3-\lambda)=0$

Hence, eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, all are different. (3.1.36)

We find that an unit eigenvector corresponding to $\lambda = 1$ is $\hat{v}_{1i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, the unit vector corresponding to $\lambda = 2$ is $\hat{v}_{2i} = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$, the unit vector corresponding to $\lambda = 3$ is $\hat{v}_{3i} = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$. We note that $\hat{v}_{1i} \cdot \hat{v}_{2i} \neq 0, \hat{v}_{2i} \cdot \hat{v}_{3i} \neq 0, \hat{v}_{1i} \cdot \hat{v}_{3i} \neq 0$. This happens due to non-symmetry of the tensor u_{ij}.

Example 2. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij} .

Solution. We note that the tensor is symmetric. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or $$\lambda(1-\lambda)(4-\lambda) = 0$$

Hence, eigenvalues are $\lambda_1 = 0, \lambda_2 = 1,\ \lambda_3 = 4$, all are different. (3.1.37)

Let $\hat{v}_{i1}$ be the unit eigenvector corresponding to eigenvalue $\lambda_1 = 0$. Then, the system of homogeneous equations is

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{v}_{1i} \\ \hat{v}_{2i} \\ \hat{v}_{3i} \end{bmatrix} = 0 \qquad (3.1.38)$$

This gives $\hat{v}_{1i} + \hat{v}_{2i} = 0$, $\hat{v}_{1i} + \hat{v}_{2i} = 0$, $\hat{v}_{3i} = 0$

We find $\hat{v}_{1i} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right)$,

Similarly, $\hat{v}_{2i} = (0,0,1)$ and $\hat{v}_{3i} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$ are eigen vectors corresponding to $\lambda_2 = 1$ and $\lambda_3 = 4$, respectively, Moreover, these vector are mutually orthogonal.

Practice 1. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij} .

Practice 2. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 3 & -2 & 0 \\ 0 & 5 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij} .

Practice 3. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 1 & -5 & 2 \\ 1 & -3 & 1 \\ -1 & 2 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij} .

Practice 4. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 0 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij} .

Practice 4. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & -3 \\ 1 & 2 & -5 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 5. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 3 & 5 & 0 \\ 1 & -1 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Practice 6. Let the matrix of the components of the second order tensor u_{ij} whose matrix representation is

$$\begin{bmatrix} 2 & 5 & 0 \\ 1 & -4 & 1 \\ 1 & 6 & -3 \end{bmatrix}$$

Find eigenvalues and eigenvectors of u_{ij}.

Books Recommended:

1. **Y. C. Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey,1965.
2. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
3. **Barry Spain** Tensor Calculus A Concise Course, Dover Publication, INC. Mineola, New York.
4. **Shanti Narayan** Text Book of Cartesian Tensors, S. Chand & Co.,1950.

CHAPTER-IV

ANALYSIS OF STRAIN

4.1 INTRODUCTION

Rigid Body: A rigid body is an ideal body such that the ***distance between every pair of its points remains unchanged under the action of external forces***. The possible displacementsin a ***rigid body***are translation and rotation. These displacements are called rigid displacements. In translation, each point of the rigid body moves in a fixed direction. In rotation about a line, every point of the body (rigid) moves in a circular path about the line in a plane perpendicular to the line.

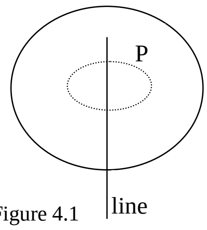


Figure 4.1

In a rigid body motion, there is a uniform motion throughout the body.

Elastic Body: A body is called elastic if it possesses the property of recovering its original shape and size when the forces causing deformation are removed.

Continuous Body: In a continuous body, the atomistic structure of matter can be disregarded and the body is replaced a continuous mathematical region of the space whose geometrical points are identified with material points of the body.

The mechanics of such continuous elastic bodies is called mechanics of continuous. This branch covers a vast range of problem of elasticity, hydromechanics, aerodynamics, plasticity and electrodynamics, seismology, etc.

Deformation of Elastic Bodies: The change in the relative position of points in a continuous is called deformation, and the body itself is then called a strained body. The study of deformation of an elastic body is known a s the analysis of strain. The deformation of the body is due to relative movements or distortions within the body.

4.2 TRANSFORMATION OF AN ELASTIC BODY

We consider the undeformed and deformed both positions of an elastic body. Let $ox_1x_2x_3$ be mutually orthogonal Cartesian coordinates fixed in space. Let a continuous body B, referred to system $ox_1x_2x_3$, occupies the region R in the undeformed state. In the deformed state, the points of the body B will occupy some region say R'.

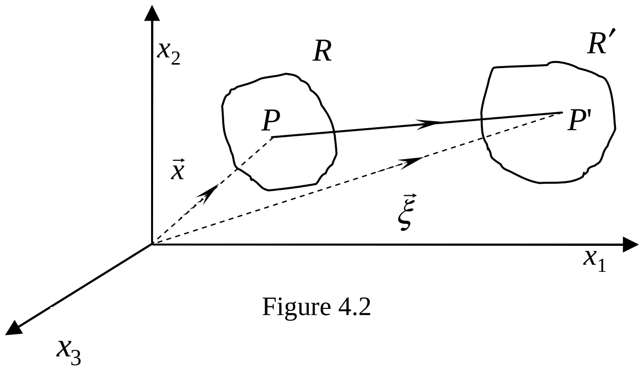


Figure 4.2

Let $P(x_1, x_2, x_3)$ be the coordinate of a material point P of the elastic body in the initial or unstained state. In the transformation or deformed state, let this material point occupies the geometric point $P'(\xi_1, \xi_2, \xi_3)$. We shall be concerned only with continuous deformation of the body from region R into the region R' and we assume that the deformation is given by the equation

$$\begin{aligned}\xi_1 &= \xi_1(x_1, x_2, x_3)\\ \xi_2 &= \xi_2(x_1, x_2, x_3)\\ \xi_3 &= \xi_3(x_1, x_2, x_3)\end{aligned} \tag{4.2.1}$$

The vector $\vec{PP'}$ is called the displacement vector of the point P and is denoted by u_i.

Thus,

$$u_i = \xi_i - x_i : i = 1, 2, 3 \tag{4.2.2}$$

or

$$\xi_i = u_i + x_i : i = 1, 2, 3 \tag{4.2.3}$$

Equation (4.2.1) expresses the coordinates of the points of the body in the transformed state in terms of their coordinates in the initial undeformed state. This type of description of deformation is known as the Lagrangian method of describing the transformation of a coordinate medium.

Another method, known as Euler's method expresses the coordinates in the undeformed state in terms of the coordinates in the deformed state.

The transformation (4.2.1) is invertible when

$$J \neq 0$$

Then, we may write

$$x_i = x_i(\xi_1, \xi_2, \xi_3) : i = 1, 2, 3 \tag{4.2.4}$$

In this case, the transformation from the region R into region R' is one to one. Each of the above description of deformation of the body has its own advantages. It is however; more convenient in the study of the mechanics of solids to use Lagrangian approach because the undeformed state of the body often possesses certain symmetries which make it convenient to use a simple system of coordinates.

A part of the transformation defined by equation (4.2.1) may represent rigid body motion. (i.e.translations and rotations) of the body as a whole. This part of the deformation leaves unchanged the length of every vector joining a pair of points within the body and is of no interest in the analysis of strain. The remaining part of the transformation (4.2.1) will be called **pure deformation**. Now, we shall learn how to distinguish between pure deformation and rigid body motions when the latter are present in the transformation equation (4.2.1)

4.3. LINEAR TRANSFORMATION OR AFFINE TRANSFORAMTION

Definition: The transformation

$$\xi_i = \xi_i(x_1, x_2, x_3)$$

is called a linear transformation or affine transformation when the function ξ_i are **linear functions**of the coordinates x_1, x_2, x_3. In order to distinguish between rigid motion and pure deformation, we consider the simple case in which the transformation (4.2.1) is linear.

We assume that the general form of the linear transformation (4.2.1) is of the type

$$\left.\begin{aligned} \xi_1 &= \alpha_{10} + (\alpha_{11}+1)x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ \xi_2 &= \alpha_{20} + \alpha_{21}x_1 + (1+\alpha_{22})x_2 + \alpha_{23}x_3, \\ \xi_3 &= \alpha_{30} + \alpha_{31}x_1 + \alpha_{32}x_2 + (1+\alpha_{33})x_3, \end{aligned}\right\} \tag{4.3.1}$$

or

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \;\; ; \; i, j = 1, 2, 3 \tag{4.3.2}$$

where the coefficients α_{ij} are constants and are well known.

Equation (4.3.2) can written in the matrix form as

$$\begin{bmatrix} \xi_1 - \alpha_{10} \\ \xi_2 - \alpha_{20} \\ \xi_3 - \alpha_{30} \end{bmatrix} = \begin{bmatrix} 1+\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.3.3}$$

or

$$\begin{bmatrix} u_1 - \alpha_{10} \\ u_2 - \alpha_{20} \\ u_3 - \alpha_{30} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.3.4}$$

We can look upon the matrix $(\alpha_{ij} + \delta_{ij})$ as an operator acting on the vector $\vec{x} = x_i$ to give the vector α_{i0}.

If the matrix $(\alpha_{ij} + \delta_{ij})$ is non-singular, then we obtain

$$\left(\alpha_{ij} + \delta_{ij}\right)^{-1} \begin{bmatrix} \xi_1 - \alpha_{10} \\ \xi_2 - \alpha_{20} \\ \xi_3 - \alpha_{30} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.3.5}$$

which is also linear as inverse of a linear transformation is linear. In fact, matrix algebra was developed basically to express linear transformations in a concise and lucid manner.

Example1. Sum of two linear transformations is a linear transformation.

Solution. Let

and $$\left.\begin{aligned} \xi_i &= \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \\ \varsigma_i &= \beta_{i0} + (\beta_{ij} + \delta_{ij})x_j \end{aligned}\right\} ; \quad i, j = 1, 2, 3 \tag{4.3.6}$$

are two linear transformation and suppose $\zeta_i = \xi_i + \varsigma_i$.

Now,

$$\begin{aligned} \zeta_i &= \xi_i + \varsigma_i \\ &= (\alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j) + (\beta_{i0} + (\beta_{ij} + \delta_{ij})x_j) \\ &= (\alpha_{i0} + \beta_{i0}) + 2\{(\alpha_{ij} + \beta_{ij})/2 + \delta_{ij}\}x_j \end{aligned}$$

$\because (\delta_{ij} x_j = x_i)$

$$\zeta_i = \vartheta_{i0} + (\vartheta_{ij} + \delta_{ij})x_j \tag{4.3.7}$$

where $\vartheta_{ij} = \alpha_{ij} + \beta_{ij}$; $i, j = 1, 2, 3$ relation (4.3.7) is a linear transformation by definition of linear transformation as defined in relation (4.3.2). Hence sum or difference of linear transformation is linear transformation.

Practice1. Show that product of two linear transformation is a linear transformation which is not commutative

Example2. Under a linear transformation, a plane is transformed into a plane.

Solution. Let

$$lx + my + mz + c = 0 \quad (4.3.8)$$

be an equation of plane which is not passes through (0,0,0) in the undeformed state and (l, m, n) are direction ratios of the plane. Let

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (4.3.9)$$

Be the linear transformation of points. Let its inverse be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \quad (4.3.10)$$

Then the equation of the plane is transformed to

$$l(L_1\xi_1 + M_1\xi_2 + N_1\xi_3) + m(L_2\xi_1 + M_2\xi_2 + N_2\xi_3) + n(L_3\xi_1 + M_3\xi_2 + N_3\xi_3) + c = 0 \quad (4.3.11)$$

or $(lL_1 + mL_2 + nL_3)\xi_1 + (lM_1 + mM_2 + nM_3)\xi_2 + (lN_1 + mN_2 + nN_3)\xi_3 + c = 0$

$$\alpha\xi_1 + \beta\xi_2 + \gamma\xi_3 + c = 0 \quad (4.3.12)$$

Relation (4.3.12) is again an equation of a plane in terms of new coordinates (ξ_1, ξ_2, ξ_3) . Hence the result.

Practice2.A linear transformation carries line segments into line segments. Thus, it is the linear transformation that allows us to assume that a line segment is transformed to a line segment and not to a curve.

4.4. SMALL/ INFINITESIMAL LINEAR DEFORMATIONS

Definition: A linear transformation of the type $\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j$; $i, j = 1,2,3$ is said to be a small linear transformation of the coefficients α_{ij} are so small that their products can be neglected in comparison with the linear terms.

Note 1: The product of two small linear transformations is small linear transformation which is **commutative** and the product transformation is obtained by superposition of the original transformations and the result is independent of the order in which the transformations are performed.

Note 2: In the study of fine deformation (as compared to the infinitesimal affine deformation), the principle of superposition of effects and the independent of the order of transformations are no longer valid.

If a body is subjected to large linear transformation, a straight line element seldom remains straight. A curved element is more likely to result. The linear transformation then expresses the transformation of elements P_1P_2 to the tangent $P_1'T_1'$ to the curve at P_1' for the curve itself.

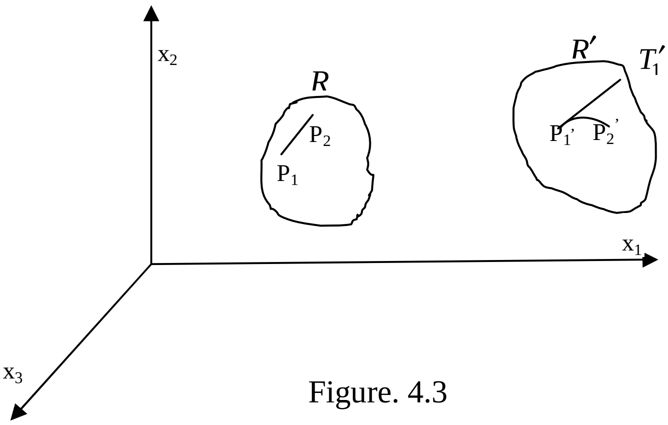


Figure. 4.3

For this reason, a linear transformation is sometimes called linear tangent transformation. It is obvious that the smaller the element P_1P_2, the better approximation of $P_1'P_2'$ by its tangent $P_1'T_1'$.

4.5 HOMOGENEOUS DEFORMATION

Suppose that a body B, occupying the region R in the undeformed state, is transformed to the region R' under the linear transformation.

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})x_j \qquad (4.5.1)$$

referred to orthogonal Cartesian system $ox_1x_2x_3$. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the unit base vectors directed along the coordinate axes x_1, x_2, x_3.

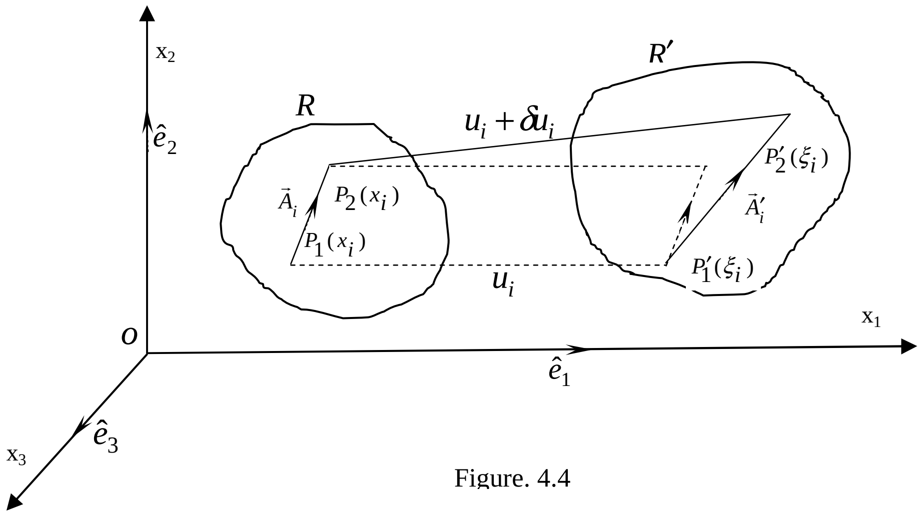


Figure. 4.4

Let $P_1(x_{11}, x_{12}, x_{13})$ and $P_2(x_{21}, x_{22}, x_{23})$ be two points of the elastic body in the initial state. Let the positions of these points in the deformed state, due to linear transformation (4.3.2), be $P_1'(\xi_{11}, \xi_{12}, \xi_{13})$ and $P_2'(\xi_{21}, \xi_{22}, \xi_{23})$. Since transformation (4.3.2) is linear, so the line segment $\overline{P_1P_2}$ is transformed into a line segment $\overline{P_1'P_2'}$.

Let the vector $\overrightarrow{P_1P_2}$ has component A_i and vector $\overrightarrow{P_1'P_2'}$ has components A_i'. Then

$$\overrightarrow{P_1P_2} = A_i\hat{e}_i, \qquad A_i = x_{2i} - x_{1i} \; (4.5.2)$$

and

$$\overrightarrow{P_1'P_2'} = A_i'\hat{e}_i, \qquad A_i' = \xi_{2i} - \xi_{1i} \qquad (4.5.3)$$

Let $$\delta A_i = A_i' - A_i \qquad (4.5.4)$$

be change in vector A_i. The vectors A_i and A_i', in general, differ in direction and magnitude. From equations (4.5.1), (4.5.2) and (4.5.3), we write

$$
\begin{aligned}
A_i' &= \xi_{2i} - \xi_{1i} \\
&= [\alpha_{i0} + (\alpha_{ij} + \delta_{ij}) x_{2j}] - [\alpha_{i0} + (\alpha_{ij} + \delta_{ij}) x_{1j}] \\
&= (x_{2i} - x_{1i}) + \alpha_{ij}(x_{2j} - x_{2i}) \\
&= A_i + \alpha_{ij} A_j \\
A_i' - A_i &= \alpha_{ij} A_j \\
\delta A_i &= \alpha_{ij} A_j
\end{aligned}
\tag{4.5.5}
$$

Thus, the linear transformation (4.3.2) changes the vector A_i into vector A_i' where

$$
\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} 1+\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+\alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+\alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{4.5.6}
$$

or

$$
\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{4.5.7}
$$

Thus, the linear transformation (4.3.2) or (4.5.6) or (4.5.7) are all equivalent. From equation (4.5.6), it is clear that two vectors A_i and B_i whose components are equal transform into two vectors A_i' and B_i' whose components are again equal. Also two parallel vectors transform into parallel vectors transformation into parallel vectors.

Hence, two equal and similarly oriented rectilinear polygons located in different part of the region R will be transformed into equal and similarly oriented polygons in the transformed region R' under the linear transformation (4.5.1).

Thus, the different parts of the body B, when the latter is subjected to the linear transformation (4.5.1), experience the same deformation independent of the position of the part of the body.

For this reason, the linear deformation (4.5.1) is called a homogeneous deformation.

Theorem: Prove that the necessary and sufficient condition for an infinitesimal affine transformation

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j$$

to represent a rigid body motion is that the matrix α_{ij} is skew-symmetric

Proof: With reference to an orthogonal system $ox_1x_2x_3$ fixed in space, let the line segment $\overline{P_1P_2}$ of the body in the undeformed state be transferred to the line segment $\overline{P_1'P_2'}$ in the deformed state due to infinitesimal affine transformation

$$\xi_i = \alpha_{i0} + (\alpha_{ij} + \delta_{ij})\, x_j \tag{4.5.8}$$

In which α_{ij} are known as constants. Let A_i be vector $\overline{P_1P_2}$ and A_i' be the vector $\overline{P_1'P_2'}$

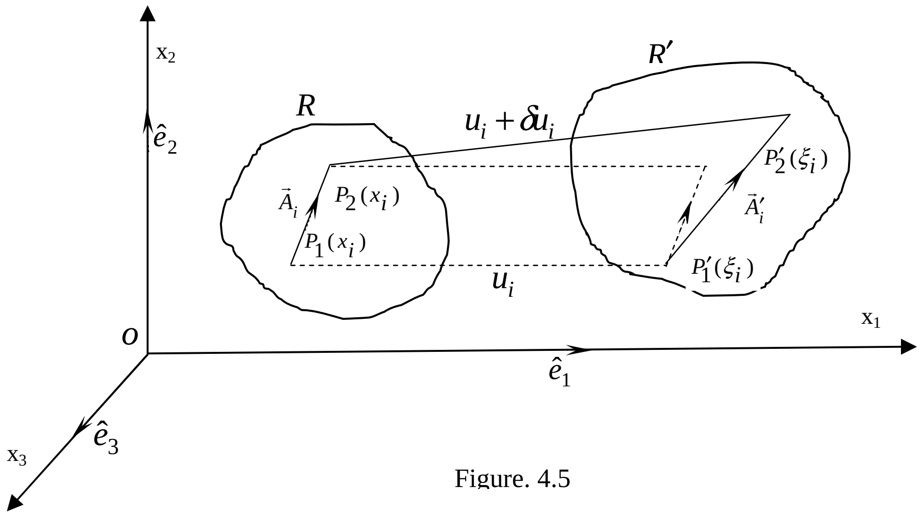


Figure. 4.5

Then

$$A_i = x_i - x_{io},\, A_i' = \xi_i - \xi_{i0} \tag{4.5.9}$$

Let

$$\delta A_i = A_i' - A_i \tag{4.5.10}$$

From (4.5.9) and (4.5.10), we find

$$A'_i = \xi_i - \xi_{i0}$$

$$= (\alpha_{i0} + \alpha_{ij}x_j + x_i) - (\alpha_{i0} + \alpha_{ij}x_{jo} + x_{i0})$$

$$= (x_i - x_{i0}) + \alpha_{ij}(x_j - x_{j0})$$

$$= A_i + \alpha_{ij}A_j$$

This gives

$$\delta A_i = A'_{i'} - A_i = \alpha_{ij}A_j \,. \qquad (4.5.11)$$

Let A denotes the length of the vector. Then

$$A = |A_i| = \sqrt{A_iA_i} = \sqrt{A_1^{\,2} + A_2^{\,2} + A_3^{\,2}} \qquad (4.5.12)$$

Let δA denotes the change in length A due to deformation. Then

$$\delta A = |A'_i| - |A_i| \qquad (4.5.13)$$

It is obvious that $\delta A \neq |\delta A_i|$,but

$$\delta A = \sqrt{(A_i + \delta A_i)(A_i + \delta A_i)} - \sqrt{A_iA_i}$$

This imply

$$(\mathrm{A}+\delta \mathrm{A})^2 = (A_i + \delta A_i)(A_i + \delta A_i)$$

Or

$$(\delta A)^2 + 2A\delta A = (\delta A_i)(\delta A_i) + 2A_i(\delta A_i) \qquad (4.5.14)$$

Since the linear transformation (4.5.8) or (4.5.11) is small,the term $(\delta A)^2$ and $(\delta A_i)(\delta A_i)$ are to be neglected in (4.5.14). Therefore,after neglecting these terms in(4.5.14), we write

$$2A\delta A = 2A_i\delta A_i \,,$$

or

$$A\delta A = A_i \delta A_i = A_1\delta A_1 + A_2\delta A_2 + A_3\delta A_3 \quad (4.5.15)$$

Using (4.5.11), equation (4.5.15) becomes

$$A\delta A = A_i(\alpha_{ij}A_j)$$

$$= \alpha_{ij}A_i A_j$$

$$= \alpha_{11}A_1^{\ 2} + \alpha_{22}A_2^{\ 2} + \alpha_{33}A_3^{\ 3} + (\alpha_{12}+\alpha_{21})A_1A_2 + (\alpha_{13}+\alpha_{31})A_3A_1 + (\alpha_{23}+\alpha_{32})A_2A_3 \quad (4.5.16)$$

Case 1: suppose that the infinitesimal linear transformation (4.5.9) represent a rigid body motion.Then, the length of the vector A_i before deformation and after deformation remains unchanged.

That is

$$\delta A = 0 \qquad (4.5.17)$$

For all vectors A_i

Using (4.5.16), we then get

$$\alpha_{11}A^2{}_1 + \alpha_{22}A_2^{\ 2} + \alpha_{33}A_3^{\ 2} + (\alpha_{12}+\alpha_{21})A_1A_2 + (\alpha_{23}+\alpha_{32})A_2A_3 + (\alpha_{13}+\alpha_{31})A_3A_1 \quad (4.5.18)$$

For all vectors A_i .This is possible only when

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0,$$

$$\alpha_{12}+\alpha_{21} = \alpha_{13}+\alpha_{31} = \alpha_{23}+\alpha_{32} = 0 \quad ,$$

i.e., $\alpha_{ij} = -\alpha_{ji}$, for all i& j (4.5.19)

i.e. , the matrix α_{ij} is skew- symmetric.

Case 2:suppose α_{ij} is skew-symmetric. Then, equation (4.5.16) shows that

$$A\delta A = 0 \qquad (4.5.20)$$

For all vectors A_i . This implies

$$\delta A = 0 \qquad (4.5.21)$$

For all vectors A_i

This shows that the transformation (4.5.8) represents a rigid body linear small transformation.

This completes the proof of the theorem.

Remarks :when the quantities α_{ij} are skew –symmetric , then the linear infinitesimal transformation.

$$\delta A_i = \alpha_{ij} A_j$$

Equation (4.5.19) takes the form

$$\delta A_1 = -\alpha_{21} A_2 + \alpha_{13} A_3$$

$$\delta A_2 = \alpha_{21} A_1 - \alpha_{32} A_3$$

$$\delta A_2 = -\alpha_{13} A_1 + \alpha_{32} A_2 \qquad (4.5.22)$$

Let

$$w_1 = \alpha_{32} = -\alpha_{23}$$

$$w_2 = \alpha_{13} = -\alpha_{31}$$

$$w_3 = \alpha_{21} = -\alpha_{12} \qquad (4.5.23)$$

Then, the transformation (4.5.22) can be written as the vectors product

$$\overline{\delta A} = \overline{w} \times \overline{A} \,,$$

(4.5.24) Where $\overline{w} = w_i$ is the infinitesimal rotation vector. Further

$$\delta A_i = A_i' - A_i$$

$$= (\xi_i - \xi_i^0) - (x_i - x_i^0)$$

$$== \delta x_i - \delta x_i^0 \qquad (4.5.25)$$

This yield

$$\delta x_i = \delta x_i^0 + \delta A_i ,$$

$\delta x_i = \delta x_i^0 + \delta A_i ,$

or

$$\delta x_i = \delta x_i^0 + (\overline{w} + \overline{A}) \qquad (4.5.26)$$

Here, the quantities

$$\delta x_i^0 = \xi_i^0 - x_i^0$$

arethe components of the displacement vector representing the translation of the point P^0 and the remaining terms of (4.5.26) represent rotation of the body about the point P^0.

4.6 PURE DEFORMATION AND COMPONENTS OF STRAIN TENSOR

We consider the infinitesimal linear transformation

$$\delta A_i = \alpha_{ij} A_j \, (4.6.1)$$

Let $$w_{ij} = 1/2 (\alpha_{ij} - \alpha_{ji}) \qquad (4.6.2)$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) \qquad (4.6.3)$$

Then the matrix w_{ij} is anti-symmetric while e_{ij} is symmetric.

Moreover,

$$\alpha_{ij} = e_{ij} + w_{ij} \, (4.6.4)$$

and this decomposition of α_{ij} as a sum of s symmetric and skew-symmetric matrices is unique.

From (4.6.1) and (4.6.4), we write

$$\delta A_i = e_{ij}A_j + w_{ij}A_j \quad (4.6.5)$$

This shows that the transformation of the components of a vector A_i given by

$$\delta A_i = w_{ij}A_j \quad (4.6.6)$$

represent rigid body motion with the component of rotation vector w_i given by

$$w_1 = w_{32}, w_2 = w_{13}, w_3 = w_{21} \quad (4.6.7)$$

and the transformation

$$\delta A_i = e_{ij}A_j\,, \quad (4.6.8)$$

with $$e_{ij} = e_{ji}\,, \quad (4.6.9)$$

represents a pure deformation.

STRAIN COMPONENTS: The symmetric coefficients, e_{ij} , in the pure deformation

$$\delta A_i = e_{ij}A_j$$

are called the strain components.

Note (1): These components of straincharacterizepure deformation of the elastic body. Since A_j and δA_i are vectors (each is a tensor of order 1),therefore,by quotient law, the strains components e_{ij} form a tensor of order 2.

Note 2: For most materials / structures, the strains are of the order 10^{-3}, such strains certainly deserve to be called small.

Note 3: The strain components e_{11}, e_{22}, e_{33} are called normal strain components while $e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}$ are called shear strain components,

Example: For the deformation defined by the linear transformation

$$\xi_1 = x_1 + x_2, \xi_2 = x_1 - 2x_2, \xi_3 = x_1 + x_2 - x_3\,,$$

Find the inverse transformation of rotation and strain tensor, and axis of rotation.

Solution: The given transformation is express as

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (4.6.10)$$

and its inverse transformation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \qquad (4.6.11)$$

giving

$$x_1 = \frac{1}{3}(2\xi_1 + \xi_2),$$

$$x_2 = \frac{1}{3}(\xi_1 - \xi_2)$$

$$x_3 = \xi_1 - \xi_3 \qquad (4.6.12)$$

comparing (4.6.10) with

$$\xi_i = (\alpha_{ij} + \delta_{ij}) x_j \qquad (4.6.13)$$

We find

$$\alpha_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \qquad (4.6.14)$$

Then

$$w_{ij} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad (4.6.15)$$

and

$$e_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$$

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & -3 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix} \qquad (4.6.16)$$

and

$$\alpha_{ij} = w_{ij} + e_{ij} \qquad (4.6.17)$$

The axis of rotation is

$$\overline{w} = w_i \hat{e}_i$$

where

$$w_1 = w_{32} = \frac{1}{2},$$

$$w_2 = w_{13} = -\frac{1}{2},$$

$$w_3 = w_{21} = 0 \qquad (4.6.18)$$

4.7 GEOMETRICAL INTERPRETATION OF THE COMPONENTS OF STRAIN

Normal strain component e_{11}:

Let e_{ij} be the components of strains the pure infinitesimal linear deformation of a vector A_i is given by

$$\delta A_i = e_{ij} A_j \tag{4.7.1}$$

with $e_{ij} = e_{ji}$.

Let edenotes the extension (or change) in length per unit length of the vector A_i with magnitude A.Then, by definition,

$$e = \frac{\delta A}{A} \tag{4.7.2}$$

We note that e is positive or negative upon whether the material line element A_i experiences an extension or a contraction. Also, $e = 0$, if and only if the vector $\overline{A}$ retains its length during a deformation.This number e is referred to as the normal strain of the vector A_i.Since the deformation is linear and infinitesimal, we have (proved earlier)

$$A\delta A = A_i \delta A_i \tag{4.7.3}$$

Or

$$\frac{\delta A}{A} = \frac{A_i \delta A_i}{A^2}$$

Now from (4.7.1) and (4.7.3), we write

$$e = \frac{\delta A}{A} = \frac{A_i \delta A_i}{A^2}.$$

This implies

$$e = \frac{1}{A^2}\left[e_{11}A_2^{\ 2} + e_{22}A_2^{\ 2} + e_{33}A_3^{\ 2} + 2e_{12}A_1A_2 + 2e_{13}A_1A_3 + 2e_{23}A_2A_3\right] \qquad (4.7.4)$$

Since $e_{ij} = e_{ji}$

In particular,we consider the case in which the vector A_i in the underformed state is parallel to the x_1-axis. Then

$$A_1 = A, A_2 = A_3 = 0 \qquad (4.7.5)$$

Using (4.7.5), equation (4.7.4) gives

$$e = e_{11}\,. \qquad (4.7.6)$$

Thus, the component e_{11} of the strain tensor, to a good approximation to the extension or change in length of a material line segment (or fiber of the material) originally placed parallel to the x_1-axis in the undeformed state.

Similarly, normal strains e_{22} and e_{33} are to be interpreted.

Illustration: let $e_{ij} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then all unit vectors parallel to the x_1-axis will be extended by an amount e_{11}. In this case, one has a homogeneous deformation of material in the direction of the x_1-axis. A cube of material whose material whose edges before deformation are L unit along will become (after deformation due to e_{ij}) a rectangular parallelepiped whose dimension in the direction of the x_2- and x_3- axes are unchanged.

Remark: The vector

$$\overline{A} = A_i = (A,0,0)$$

is changed to (due to deformation)

$$\overline{A'} = (A + \delta A_1)\hat{e}_1 + \delta A\hat{e}_2 + \delta A_3\hat{e}_3$$

in which

$$\delta A_i = e_{ij} A_j = e_{i1} A_1$$

gives

Thus $$\overline{A'} = (A + e_{11}A,\ e_{12}A,\ \ e_{13}A)$$

this indicates that vector $A_i = (A,0,0)$ upon deformation,in general, changes its orientation also.This length of the vector due to deformation becomes $(1+e_{11})A$.

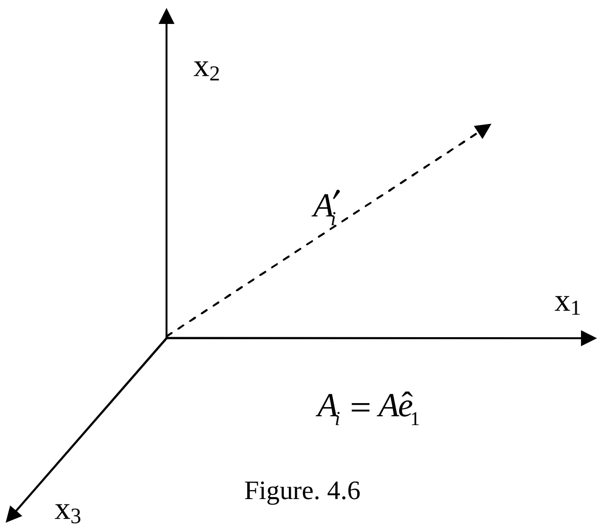


Figure. 4.6

Question: From the relation $\delta A_i = e_{ij} A_j$, find δA and δA_i for a vector lying initially along x-axis (i.e., $\overline{A} = A\hat{e}_1$) and justify the fact that $\dfrac{\delta A}{A} = e_{11}$. Does δA_i lie along the x-axis?

Answer: It is given that $A_i = (A,0,0)$. The given relation

$$\delta A_i = e_{ij} A_j \tag{4.7.7}$$

Gives

$$\delta A_1 = e_{11}A, \delta A_2 = e_{12}A, \delta A_3 = e_{13}A \qquad (4.7.8)$$

Thus, in general, the vector δA_i does not lie along the x-axis.

Further

$$(A+\delta A) = \sqrt{\left[A(1+e_{11})^2 + (e_{12}A)^2 + (e_{13}A)^2\right]}$$

$$= A\sqrt{1+2e_{11}+e_{11}{}^2+e_{12}{}^2+e_{13}{}^2}\,. \qquad (4.7.9)$$

Neglecting square terms as deformation is small, equation (4.7.9) gives

$$(A+\delta A)^2 = A^2(1+2e_{11}),$$

$$A^2 + 2A\delta A = A^2 + 2A^2 e_{11,}$$

$$2A\delta A = 2A^2 e_{11}$$

$$\frac{\delta A}{A} = e_{11}\,. \qquad (4.7.10)$$

This shows that e_{11} gives the extension of a vector (A, 0, 0) per unit length due to deformation.

Remarks: the strain components e_{ij} refer to the chosen set of coordinate axes. If the axes changed, the strain component e_{ij} will, in general, changes as per tensor transformation laws.

Geometrical interpretation of shearing Stress e_{23}:

The shearing strain component e_{23} may be interpreted by considering intersecting vectors initially parallel to two coordinate axes - x_2 -and x_3 -axis

Now, we consider in the undeformed state two vectors.

$$\overline{A} = A_2\hat{e}_2\,,$$

$$\overline{B} = B_3 \hat{e}_3 \tag{4.7.11}$$

directed along x_2 -and x_3 -axis, respectively.

The relations of small linear deformation are

$$\delta A_i = e_{ij} A_j \,,$$

$$\delta B_i = e_{ij} B_3 \,, \tag{4.7.12}$$

Further, the vectors A_i and B_i due to deformation become (figure 4.7)

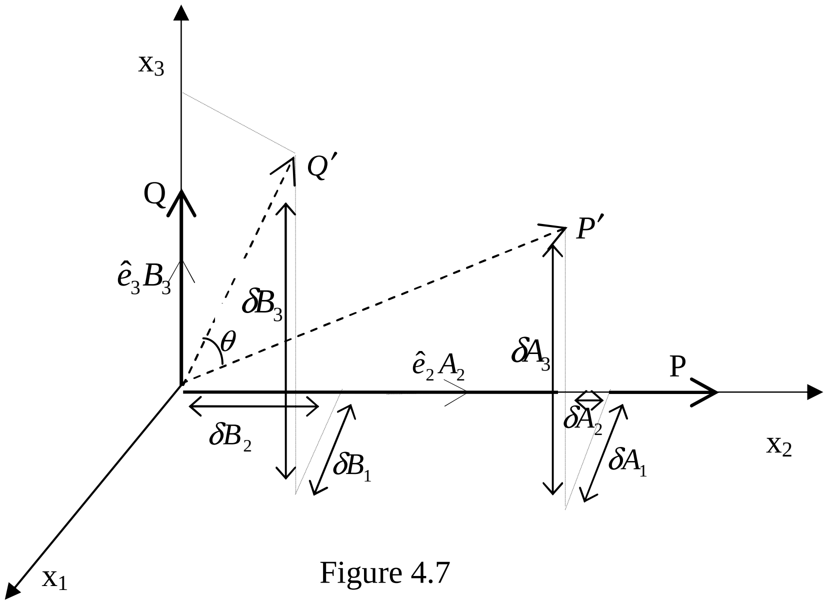


Figure 4.7

$$\begin{aligned} \overline{A'} &= \delta A_1 \,\hat{e}_1 + \left(A_2 + \delta A_2\right)\hat{e}_2 + \delta A_3 \,\hat{e}_3 \\ \overline{B'} &= \delta B_1 \,\hat{e}_1 + \delta B_2 \,\hat{e}_2 + \left(B_3 + \delta B_3\right)\hat{e}_3 \end{aligned} \tag{4.7.13}$$

Let θ be the angle between $\overline{A'}$ and $\overline{B'}$. Then

$$\cos\theta = \frac{\overline{A'}.\overline{B'}}{A'B'} = \frac{\delta A_1 \delta B_1 + \left(A_2 + \delta A_2\right)\delta B_2 + \delta A_3\left(B_3 + \delta B_3\right)}{\sqrt{(\delta A_1)^2 + \left(A_2 + \delta A_2\right)^2 + \left(\delta A_3\right)^2}\sqrt{\left(\delta B_1\right)^{12} + \left(\delta B_1\right)^2 + \left(B_3 \delta B_3\right)^2}} \tag{4.7.14}$$

Since , the deformation is small , we may neglect the product of the changes in the components of the vector A_i and B_i.Neglecting these product , equation (4.7.11) gives

$$\cos\theta = (A_2\delta B_2 + B_3\delta A_3)(A_2 + \delta A_2)^{-1}(B_3 + \delta B_3)^{-1}$$

$$= \frac{A_2\delta B_2 + B_3\delta A_3}{A_2 B_3}(1 + \frac{\delta A_2}{A_2})^{-1}\left(1 + \frac{1+\delta B_3}{B_3}\right)^{-1}$$

$$= \left(\frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2}\right)\left(1 - \frac{\delta A_2}{A_2}\right)\left(1 - \frac{\delta B_3}{B_3}\right),$$

Neglecting other terms,this gives

$$\cos\theta = \frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2} \qquad (4.7.15)$$

Neglecting the product terms involving changes in the components of the vectors A_i and B_i.

Since in formula (4.7.15), all increments in the components of initial vectors on assuming (without loss of generality)

$$\delta A_1 = \delta A_2 \equiv 0\ ,$$

And $\delta B_1 = \delta B_3 \equiv 0$,

can be represented as shown in the figure below (it shows that vector A_i' and B_i' lie in the x_2x_3-plane). We call that equation (4.7.13) now may be taken as

$$\overline{A'} = A_2\hat{e}_2 + \delta A_3\hat{e}_3,$$

$$\overline{B'} = \delta B_2\hat{e}_2 + B_3\hat{e}_3 \qquad (4.7.16)$$

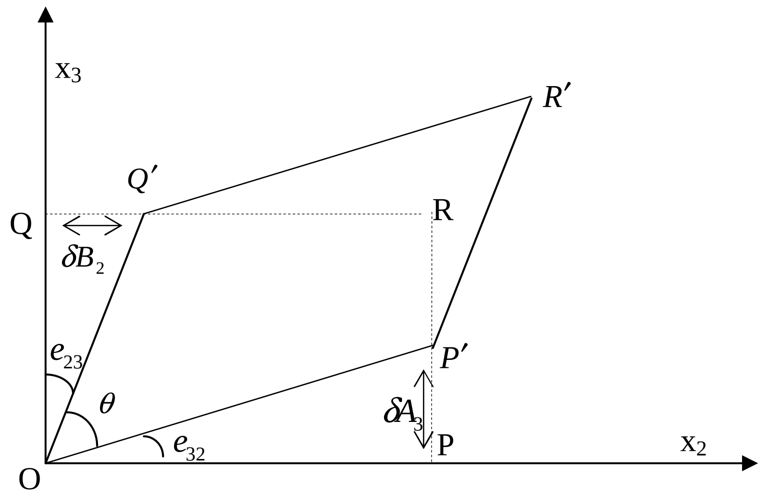


Figure 4.8

Form equation (4.7.11) and 4.7.12), we obtain

$$\delta A_3 = e_{32} A_2 ,$$

$$\delta B_2 = e_{23} B_3 \qquad (4.7.17)$$

This gives

$$e_{32} = \frac{\delta A_3}{A_2} = \tan \angle P'OP \qquad (4.7.18)$$

$$e_{23} = \frac{\delta B_2}{B_3} = TAN \angle Q'OQ \qquad (4.7.19)$$

since strain $e_{23} = e_{32}$ are small, so

$$\angle P'OP = \angle Q'OQ \cong e_{23} ,$$

And here

$$2e_{23} \cong 90^o - \theta = \frac{\pi}{2} - \theta \qquad (4.7.20)$$

Thus, a positive value of $2e_{23}$ represents the decrease in the right angle between the vectors A_i and B_i due to small linear deformation which were initially directed along the positive x_2 and x_3-axes. The quantity / strain component e_{23} is called the shearing strain.

A similar interpretation can be made for the shear strain components of material arcs.

Remarks 1: By rotating the parallelogram $R'OP'Q'$ throw an angle e_{23} about the origin (in the $x_2 x_3$ -plane), we obtain the following configurations (figure 4.9)

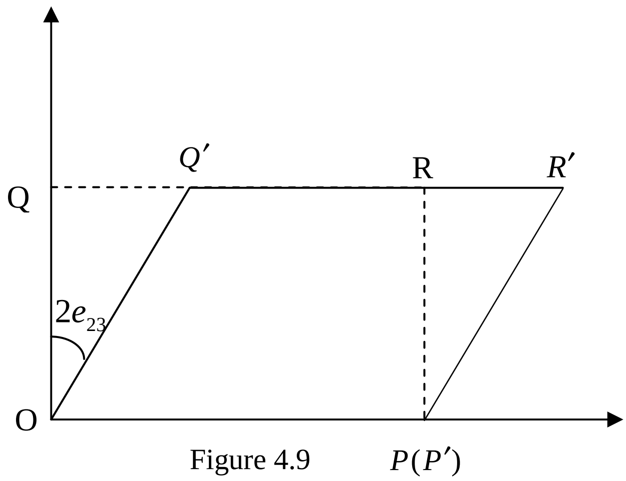


Figure 4.9

Thisfigure shows a slide or a shear of planar elements parallel to the $x_1 x_2$ – plane.

Remarks 2: Figure shows that areas of rectangle OQRP and the parallelogram $OQ'R'P'$ are equal as they have the same height and same base in the $x_2 x_3$-plane.

Remarks 3: For the strain tensor $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & e_{32} & 0 \end{bmatrix}$,

A cubical element is deformed into a parallelepiped and the volumes of the cube and parallelepiped remain the same.Such a small linear deformation is called a pure shear.

4.8 NORMAL AND TANGENTIAL DISPLACEMENTS

Consider a point P (x_1, x_2, x_3) of the material. Let it be moved to Q under a small linear transformation. Let the components of the displacement vector $\overline{PQ}$ be u_1, u_2, u_3. In the plane OPQ, let $\overline{PN} = \bar{n}$ be the projection of $\overline{PQ}$ on the line OPN and let $\overline{PT} = \bar{t}$ be the tangential of $\overline{PQ}$ in the plane of OPQ or PQN.

Definition: vectors $\bar{n}$ and $\bar{t}$ are, respectively, called the normal and the tangentialcomponents of the displacement of P.

Note: The magnitude n of normal displacement $\bar{n}$ is given by the dot product of vectors $\overline{OP} = (x_1, x_2, x_3)$ and $\overline{PQ} = (u_1, u_2, u_3)$.

the magnitude t of tangential vector $\bar{t}$ is given the vector product of vectors $\overline{OP}$ and $\overline{PQ}$ (this does not give the direction of $\bar{t}$).

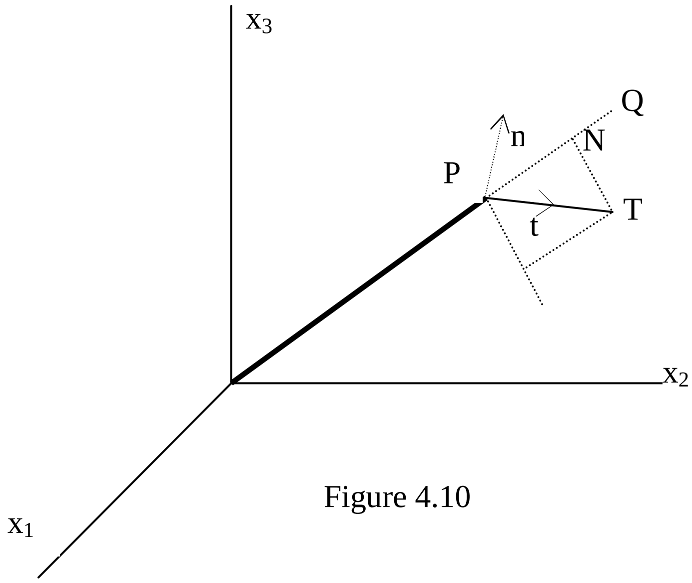


Figure 4.10

Thus

$$n = \cos \angle NPQ = \frac{\overline{OP}.\overline{PQ}}{\left|\overline{OP}\right|},$$

$$t = PQ \sin \angle NPQ = \frac{(OP)(PQ)\sin(NPQ)}{OP} = \frac{\left|\overline{OP} \times \overline{PQ}\right|}{\left|\overline{OP}\right|},$$

And

$$n^2 + t^2 = u_1^2 + u_2^2 + u_3^2.$$

Books Recommended:

1. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

2. **Shanti Narayan** Text Book of Cartesian Tensors, S. Chand & Co.,1950.

CHAPTER-V

STRAIN QUADRIC OF CAUCHY

5.1 Strain Quadric of Cauchy

Let $P^0\left(x_1^0, x_2^0, x_3^0\right)$ be any fixed point of a continuous medium with reference axis $0\,x_1x_2x_3$ fixed in space. We introduce a local system of axis with origin at point P^0 and with axes parallel to the fixed axes (figure 5.1)

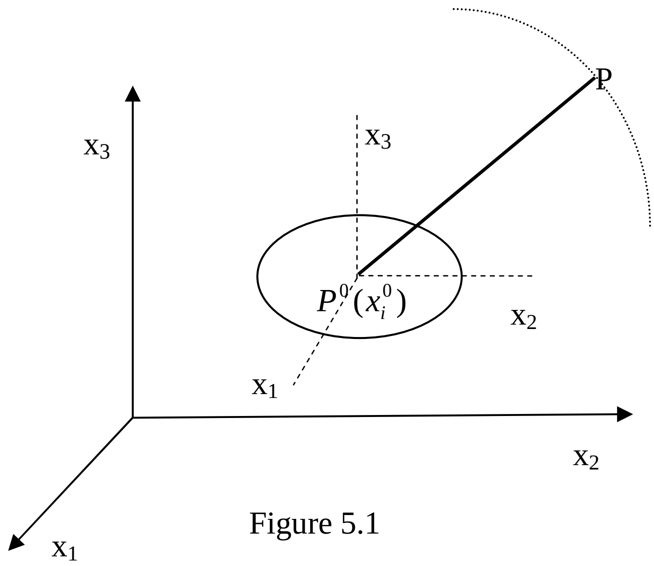


Figure 5.1

with reference to these axes, consider the equation

$$e_{ij}x_ix_j = \pm k^2 \tag{5.1.1}$$

where k is a real constant and is the strain tensor at P^0. This equation represents a quadric of Cauchy. The sign + or – in equation (5.1.1) be chosen so that the quadric surface (5.1.1) becomes a real one. The nature of this quadratic surface depends on the value of the strain e_{ij}.

If $|e_{ij}| \neq 0$, the quadratic is either an ellipsoid or a hyperboloid.

If $|e_{ij}| = 0$, the quadratic surface degenerates into a cylinder of the elliptic or hyperbolic type or else into two parallel planes symmetrically situated with respect to the quadric surface.

This strain quadric is completely determined once the strain components e_{ij} at point $P^{(0)}$ are known. Let $\overline{P^0P}$ be the radius vector A_i of magnitude A to any point $P(x_1, x_2, x_3)$, referred to local axis, on the strain quadric surface (5.1.1). Let e be the extension of the vector A_i due to some linear deformation characterized by

$$\delta A_i = e_{ij} A_j , \tag{5.1.2}$$

Then, by definition,

$$e = \frac{\delta A}{A} = \frac{A\,\delta A}{A^2} = \frac{A_i \delta A_i}{A^2}$$

This gives

$$e = \frac{e_{ij} A_i A_j}{A^2} \tag{5.1.3}$$

using (5.1.2)

Since $\overline{P^0P} = A_i$ and the coordinate of point P, on the surface (5.1.1), relative to P^0 are (x_1, x_2, x_3), it follows that

$$A_i = x_i \tag{5.1.4}$$

From equation (5.1.1), (5.1.2) and (5.1.4); we obtain

$$eA^2 = e_{ij} A_i A_j = e_{ij} x_i x_j = \pm k^2$$

Or

$$e = \pm \frac{k^2}{A^2} \tag{5.1.5}$$

Result (1): Relation (5.1.5) shows that the extension or elongation of any radius vector A_i of the strain quadric of Cauchy, given by equation (5.1.1), is inversely proportional to the length 'A' of any radius vector this deformation the elongation of any radius vector of the strain quadric at the point $P^0(x_i^0)$.

Result (2): we know that the length 'A' of the radius vector A_i of strain quadric (5.1.1) at the point $P^0(x_i^0)$ has maximum and minimum values along the axes of the quadric. In general, axes of the strain quadric (5.1.1) differs from the coordinates axes through $P^0\left(x_i^0\right)$. Therefore, the maximum and minimum extensions or elongation of the radius vectors of strain quadric (5.1.1) will be along its axes.

Result (3): Another interesting property of the strain quadric (5.1.1) is that normal v_i to this surface at the end point P of the vector $\overline{P^0P} = A_i$ is parallel to the displacement vector δA_i.

To prove this property, let us write equation (5.1.1) in the form

$$G = e_{ij}x_jx_i \pm k^2 = 0 \tag{5.1.6}$$

Then the direction of the normal $\hat{v}$ to the strain quadric (5.1.6) is given by the gradient of the scalar function G. The components of the gradient are

$$\frac{\partial G}{\partial x_k} = e_{ij}\delta_{ik}x_j + e_{ij}x_i\delta_{kj}$$

$$= e_{kj}x_j + e_{ik}x_i$$

$$= 2e_{kj}x_j$$

Or

$$\frac{\partial G}{\partial x_k} = 2\delta A_k \tag{5.1.7}$$

This shows that vector $\frac{\partial G}{\partial x_k}$ and vector δA_k are parallel. Hence, the vector $\overline{\delta A}$ is directed along the normal at P to the strain quadric of Cauchy.

5.2 STRAIN COMPONENTS AT A POINT IN A ROTATION OF COORDINATE AXES

Let new axes $0x_1' x_2' x_3'$ be obtained from the old reference system $0x_1x_2x_3$ by a rotation. Let the directions of the new axes x_i' be the specified relative to the old system x_i by the following table of direction cosines in which ℓ_{pi} is the cosine of the angle between the $x_{p'}$ -and x_i axis.

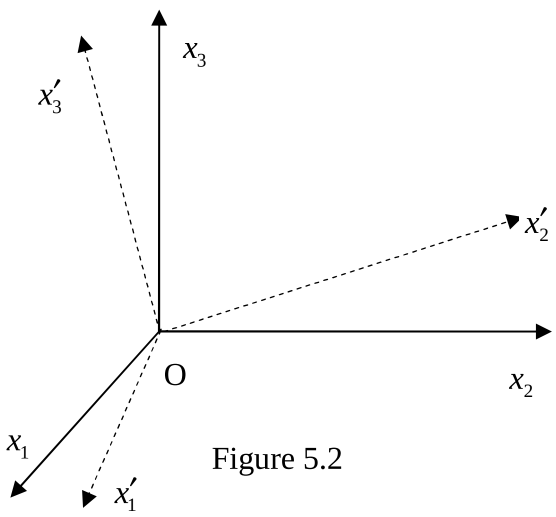


Figure 5.2

That is $$\ell_{pi} = \cos(x_p', x_i).$$

Thus

	x_1	x_2	x_3
x_1'	ℓ_{11}	ℓ_{12}	ℓ_{13}
x_2'	ℓ_{21}	ℓ_{22}	ℓ_{23}
x_3'	ℓ_{31}	ℓ_{32}	ℓ_{33}

Then the transformation law for coordinates is

$$x_i = \ell_{pi} x'_p \tag{5.2.1}$$

Or $$x'_p = \ell_{pi} x_i \tag{5.2.2}$$

The well –known orthogonality relations are

$$\ell_{pi}\ell_{qi} = \delta_{pq} \tag{5.2.3}$$

$$\ell_{pi}\ell_{pj} = \delta_{ij} \tag{5.2.4}$$

with reference to new x'_p -system, a new set of strain components e'_{pq} is determined at the point o while e_{ij} are the components of strain at o relative to old axes $ox_1x_2x_3$.

Let

$$e_{ij}x_ix_j = \pm k^2 \tag{5.2.5}$$

be the equation of the strain quadric surface relative to old axis. The equation of quadric surface with reference to new prime system becomes

$$e'_{pq}x'_px'_q = \pm k^2 \tag{5.2.6}$$

As we know that quadric form is invariant w. r. t. an orthogonal transformation of coordinates. Further, equation (5.2.2) to (5.2.6) together yield

$$\begin{aligned} e'_{pq}x'_px'_q &= e_{ij}x_ix_j \\ &= e_{ij}\left(\ell_{pi}x'_p\right)\left(\ell_{qj}x'_q\right) \\ &= \left(e_{ij}\ell_{pi}\ell_{qj}\right)x'_px'_q \end{aligned}$$

Or

$$(e'_{pq} - \ell_{pi}\ell_{qj}e_{ij})x'_px'_q = 0 \tag{5.2.7}$$

Since equation (5.2.7) is satisfied for arbitrary vector x'_p, we must have

$$e'_{pq} = \ell_{pi}\ell_{qj}e_{ij} \tag{5.2.8}$$

Equation (5.2.8) is the law of transformation for second order tensors. We, therefore, conclude that the components of strain form a second order tensor.

Similarly, it can be verified that

$$e_{ij} = \ell_{pi}\ell_{qj}e'_{pq} \tag{5.2.9}$$

Question: Assuming that e_{ij} is a tensor of order 2, show that quadratic form $e_{ij}x_i x_j$ is an invariant.

Solution: We have

$$e_{ij} = \ell_{pi}\ell_{qj}e'_{pq}$$

So,

$$e_{ij}x_i x_j = \ell_{pi}\ell_{qj}e'_{pq}x_i x_j$$

$$= e'_{pq}\left(\ell_{pi}x_i\right)\left(\ell_{qj}x_j\right)$$

$$= e'_{pq}x'_p x'_q\,. \tag{5.2.10}$$

Hence the result

5.3 PRINCIPAL STRAINS AND INVERIANTS

From a material point $P^0\left(x_i^0\right)$, there emerge infinitely many material arcs/ filaments, and each of these arcs generally changes in length and orientation under a deformation. We seek now the lines through $P^0\left(x_i^0\right)$ whose orientation is left unchanged by the small linear deformation given by

$$\delta A_i = e_{ij}A_j \tag{5.3.1}$$

where the strain components e_{ij} are small and constant. In this situation, vectors A_i and δA_i are parallel and, therefore,

$$\delta A_i = eA_i \tag{5.3.2}$$

for some constant e.

Equation (5.3.2) shows that the constant e represents the extension.

$$\left(e = \frac{|\delta A_i|}{|A_i|} = \frac{\delta A}{A} \right)$$

of vector A_i. From equation (4.11.1) and (4.11.2), we write

$$e_{ij} A_j = e A_i$$

$$= e \delta_{ij} A_j \qquad (5.3.3)$$

This implies

$$(e_{ij} - e\delta_{ij}) A_j = 0 \qquad (5.3.4)$$

We know that e_{ij} is a real symmetric tensor of order 2. The equation (5.3.3) shows that the scalar e is an eigen value of the real symmetric tensor e_{ij} with corresponding eigenvector A_i. Therefore, we conclude that there are precisely three mutually orthogonal direction are not changed on account of deformation and these direction coincide with the three eigenvectors of the strain tensor e_{ij} .These directions are known as principle direction of strain. Equation (5.3.4) gives us a system of three homogeneous equations in the unknown A_1, A_2, A_3. This system possesses a non-trivial solution if and only if the determination of the coefficients of the A_1, A_2, A_3 is equal to zero, i.e.,

$$\begin{vmatrix} e_{11} - e & e_{12} & e_{13} \\ e_{21} & e_{22} - e & e_{23} \\ e_{31} & e_{32} & e_{33} - e \end{vmatrix} = 0 \qquad (5.3.5)$$

which is cubic equation in e.

Let e_1, e_2, e_3 be the three roots of equation (5.3.5), these are known as principal strains. Evidently, the principal strains are the eigenvalues of the second order real symmetric strain tensor e_{ij}. Consequently, these principal strains are real (not necessarily distinct). Physically, the principal strains e_1, e_2, e_3 (all different) are the extensions of the vectors, say $\overline{A_i}$, in the principal / invariant of strain. So, vectors A_i, δA_i, $A + \delta A_i$ are collinear. At the point P^0 consider the strain quadric

$$e_{ij} x_i x_j = \pm k^2 \quad (5.3.6)$$

For every principal direction of strain A_i, we know that δA_i is normal to the quadric surface (5.3.6). Therefore, the principal directions of strain are also normal to the strain quadric of Cauchy. Here, principal direction of strain must be the three principal axes of the strain quadric of Cauchy. If some of the principal strains e_i are equal, then the associated directions become indeterminate but one can always select three directions that all mutually orthogonal If the $e_1 \neq e_2 = e_3$, then the quadric surface of Cauchy is a surface revolution and our principal direction, say $\tilde{A}_1$, will be directed along the axis of revolution.

In this case, any two mutually perpendicular vectors lying in the plane normal to $\tilde{A}_1$ may be taken as the other two principal directions of strain.

If $e_1 = e_2 = e_3$, then strain quadric of Cauchy becomes a sphere and any three orthogonal directions may be chosen as the principal directions of strain.

Result: If the principal directions of strain are taken as the coordinate axes, then

$$e_{11} = e_1, e_{22} = e_2, e_{33} = e_3$$

And

$$e_{12} = e_{13} = e_{23} = 0,$$

As a vector initially along an axis remains in the same direction after deformation (so change in right angles are zero). In this case, the strain quadric Cauchy has the equation.

$$e_1 x_1^2 + e_2 x_2^2 + e_3^2 x_3^2 = \pm k^2 \quad (5.3.7)$$

Result 2: Expanding the cubic equation (5.3.5), we write

$$-e^3 + v_1 e^2 - v_2 e + v_3 = 0$$

where $v_1 = e_{11} + e_{22} + e_{33}$

$$= e_{ii} = tr(E), \tag{5.3.8}$$

$$v_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{23}^2 - e_{13}^2 - e_{12}^2$$

$$= tr(E^2) = \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ji}), \tag{5.3.9}$$

$$v_3 = \varepsilon_{ijk} e_{1i} e_{2j} e_{3k}$$

$$= |e_{ij}| = tr(E^3) \tag{5.3.10}$$

Also e_1, e_2, e_3 are roots of a cubic equation (5.3.8), so

$$\left.\begin{aligned} v_1 &= e_1 + e_2 + e_3 \\ v_2 &= e_1e_2 + e_2e_3 + e_3e_1 \\ v_3 &= e_1e_2e_3 \end{aligned}\right\} \tag{5.3.11}$$

We know that eigenvalues of a second order real symmetric tensor are independent of the choice of the coordinate system.

It follows that v_1, v_2, v_3 are given by (5.3.10) three invariants of the strain tensor e_{ij} with respect to an orthogonal transformation of coordinates.

Geometric meaning of the first strain invariant $\vartheta = e_{ii}$

The quantity $\vartheta = e_{ii}$ has a simple geometric meaning. Consider a volume element in the form of rectangle parallelepiped whose edges of length l_1, l_2, l_3 are parallel to the direction of strain. Due to small linear transformation /deformation, this volume element becomes again rectangle parallelepiped with edges of length $l_1(1+e_1), l_2(1+e_2), l_3(1+e_3)$, where e_1, e_2, e_3 are principal strains. Hence, the change δV in the volume V of the element is

$$\delta V = l_1 l_2 l_3 (1+e_1)(1+e_2)(1+e_3) - l_1 l_2 l_3$$

$$= l_1 l_2 l_3 (1+e_1+e_2+e_3) - l_1 l_2 l_3, \qquad \text{ignoring small strains } e_i.$$

$$= l_1 l_2 l_3 (e_1+e_2+e_3)$$

This implies

$$\frac{\delta V}{V} = e_1 + e_2 + e_3 = \vartheta$$

Thus the first strain invariant ϑ represents the change in volume per unit initial volume due to strain produced in the medium. The quantity ϑ is called the cubical dilatation or simply the dilatation.

Note: If $e_1 > e_2 > e_3$ then e_3 is called the minor principal strain, e_2 is called the intermediate principal strain, and e_1 is called the major principal strain.

Question: For small linear deformation, the strains e_{ij} are given by

$$(e_{ij}) = \alpha \begin{bmatrix} x_2 & \frac{(x_1+x_2)}{2} & x_3 \\ \frac{(x_1+x_2)}{2} & x_1 & x_3 \\ x_3 & x_3 & 2(x_1+x_2) \end{bmatrix}, \qquad \alpha = \text{constant}$$

Find the strain invariants, principal strain and principal direction of strain at the point P(1,1,0).

Solution: The strain matrix at the point P(1,1,0) becomes

$$(e_{ij}) = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 4\alpha \end{bmatrix},$$

whose characteristics equation becomes

$$e(e-2\alpha)(e-4\alpha) = 0.$$

Hence, the principal strains are

$$e_1 = 0, e_2 = 2\alpha, e_3 = 4\alpha \,.$$

The three scalar invariants are

$$v_1 = e_1 + e_2 + e_3 = 6\alpha, v_2 = 8\alpha^2, v_3 = 0$$

The three principal unit directions are found to be

$$A_i^1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \qquad A_i^2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad A_i^3 = (0,0,1)$$

Exercise: The strain field at a point P(x, y, z) in an elastic body is given by

$$e_{ij} = \begin{bmatrix} 20 & 3 & 2 \\ 3 & -10 & 5 \\ 2 & 5 & -8 \end{bmatrix} \times 10^{-6} \,.$$

Determine the strain invariant and the principal strains.

Question: Find the principal directions of strain by finding the extremal value of the extension ϑ. OR, Find the direction in which the extension ϑ is stationary.

Solution: Let ϑ be the extension of a vector A_i due to small linear deformation

$$\delta A_i = e_{ij} A_j \tag{5.3.12}$$

Then

$$\vartheta = \frac{\delta A}{A} \tag{5.3.13}$$

We know that for an infinitesimal linear deformation (5.3.12), we have

$$A \delta A = A_i \delta A_i \tag{5.3.14}$$

Thus

$$\vartheta = \frac{A\delta A}{A^2} = \frac{A_i \delta A_i}{A^2} = \frac{e_{ij} A_i A_j}{A^2} \tag{5.3.15}$$

Let

$$\frac{A_i}{A} = a_i \tag{5.3.16}$$

Then $a_i a_i = 1$ (5.3.17)

And equation (5.3.15) then gives

$$e(a_1, a_2, a_3) = e_{ij} a_i a_j \quad (5.3.18)$$

Thus the extension e_i is a function of a_1, a_2, a_3 which are not independent because of relation (5.3.17). The extreme/stationary (or max/min) values of the extension e are to be found by making use of Lagrange's method of multipliers. For this purpose, we consider the auxiliary function

$$F(a_1, a_2, a_3) = e_{ij} a_i a_j - \lambda(a_i a_i - 1) \quad (5.3.19)$$

where λ is a constant.

In order to find the values of a_i, a_2, a_3 for which the function (5.3.18) may have a maximum or minimum, we solve the equations.

$$\frac{\partial F}{\partial a_k} = 0, \quad \text{k=1, 2, 3.} \quad (5.3.20)$$

Thus, the stationary values of e are given by

$$e_{ij}(\delta_{ik} a_j + a_i \delta_{jk}) - \lambda 2 a_i \delta_{ik} = 0$$

Or $e_{kj} a_j + e_{ik} a_i - 2\lambda a_k = 0$

Or $2 e_{ki} a_i - 2\lambda a_k = 0$

Or $e_{ki} a_i = \lambda a_k$. (5.3.21)

This shows that λ is an eigenvalue of the strain tensor e_{ij} and a_i is the corresponding eigenvector. Therefore, equation in (5.3.21) determines the principal strains and the stationary/extreme values are precisely the principal strains.

Thus, the extension e assumes the stationary values along the principal direction of strain and the stationary/extreme values are precisely the principal strains.

Remarks: Let M be the square matrix with eigenvectors of the strain tensor e_{ij} as columns. That is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Then $$e_{ij}A_{1j} = e_1 A_{1i}$$

$$e_{ij}A_{2j} = e_2 A_{2i}$$

$$e_{ij}A_{3j} = e_3 A_{3i}$$

The matrix M is called the modal matrix of strain tensor e_{ij}.

Let

$$E = (e_{ij}), D = dia(e_1, e_2, e_3).$$

Then, we find

EM=MD

Or $$M^{-1}EM = D.$$

This shows that the matrices E and D are similar.

We know that two similar matrices have the same eigenvalues. Therefore, the characteristic equation associated with $M^{-1}EM$ is the same as the one associated with E. Consequently, eigenvalues of E and D are identical.

Question: Show that, in general, at any point of the elastic body there exists (at least) three mutually perpendicular principal directions of the strain due to an infinitesimal linear deformation.

Solution: Let e_1, e_2, e_3 be the three principal strains of the strain tensor e_{ij}. Then, they are the roots of the cubic equation

$$(e - e_1)(e - e_2)(e - e_3) = 0$$

And $$e_1 + e_2 + e_3 = e_{11} + e_{22} + e_{33} = e_{ii},$$

$$e_1e_2 + e_2e_3 + e_3e_1 = \frac{1}{2}\left(e_{ii}e_{jj} - e_{ij}e_{ji}\right),$$

$$e_1e_2e_3 = \left|e_{ij}\right| = \in_{ijk} e_{1i}e_{2j}e_{3k}.$$

We further assume that coordinate axes coincide with the principal directions of strain. Then, the strain components are given by

$$e_{11} = e_1, e_{22} = e_2, e_{33} = e_3,$$

$$e_{12} = e_{13} = e_{23} = 0,$$

and the strain quadric of Cauchy becomes

$$e_1x_1^2 + e_2x_2^2 + e_3x_3^2 = \pm k^2. \quad (5.3.22)$$

Now, we consider the following three possible cases for principal strains.

Case: 1 When $e_1 \neq e_2 \neq e_3$. In this case, it is obvious that there exists three mutually orthogonal eigenvectors of the second order real symmetric strain tensor e_{ij}. These eigenvectors are precisely the three principal directions that are mutually orthogonal.

Case: 2 When $e_1 \neq e_2 = e_3$.

Let A_{1i} and A_{2i} be the corresponding principal orthogonal directions corresponding to strains (distinct) e_1 and e_2, respectively. Then

$$e_{ij}A_{1j} = e_1A_{1i}$$

$$e_{ij}A_{2j} = e_2A_{2i} \quad (5.3.23)$$

Let p_i be a vector orthogonal to both A_{1i} and A_{2i}. Then

$$p_iA_{1i} = p_iA_{2i} = 0 \quad (5.3.24)$$

Let $$e_{ij}p_i = q_j \quad (5.3.25)$$

Then $$q_j A_{1j} = (e_{ij} p_i) A_{1j} = (e_{ij} A_{1j}) p_i = e_1 A_{1i} p_i = 0 \qquad (5.3.26a)$$

similarly $$q_j A_{2j} = 0 \qquad (5.3.26b)$$

This shows that the vector q_j is orthogonal to both A_{1j} and A_{2j}. Hence, the vectors q_i and p_i must be parallel. Let

$$q_i = \alpha\, p_i \qquad (5.3.27)$$

for some scalar α. From equation (5.3.25) and (5.3.27), we write

$$e_{ij} p_j = q_i = \alpha p_i \qquad (5.3.28)$$

which shows that the scalar α is an eigenvalue /principal strain tensor e_{ij} with corresponding principal direction p_i. Since e_{ij} has only three principal strains e_1, e_2, α and two of these are equal, so α must be equal to $e_2 = e_3$. We denote the normalized form of p_i by A_{3i}. This shows the existence of three mutually orthogonal principal directions in this case. Further, let v_i be any vector normal to A_{1i}. Then v_i lies in the plane containing principal directions A_{2i} and A_{3i}. Let

$$v_i = k_1 A_{2i} + k_2 A_{3i} \quad \text{for some constant } k_1 \text{ and } k_2 \qquad (5.3.29)$$

Now
$$\begin{aligned} e_{ij} v_j &= e_{ij}(k_1 A_{2j} + k_2 A_{3j}) \\ &= k_1 (e_{ij} A_{2j}) + k_2 (e_{ij} A_{3j}) \\ &= k_1 (e_2 A_{2i}) + k_2 (e_3 A_{3i}) \\ &= e_2 (k_1 A_{2i} + k_2 A_{3i}) \ (\because e_2 = e_3) \\ &= e_2 v_i \end{aligned}$$

This shows that the direction v_i is also a principal directions strain e_2. Thus, in this case, any two orthogonal (mutually) vectors lying on the plane normal to A_{1i} can be chosen as the other two principal directions. In this case, the strain quadric surface is a surface of revolution.

Case3: when $e_1 = e_2 = e_3$, then the strain quadric of Cauchy is a sphere with equation

$$e_1\left(x_1^2 + x_2^2 + x_3^2\right) = \pm k^2$$

Or
$$x_1^2 + x_2^2 + x_3^2 = \pm \frac{k^2}{e_1}$$

and any three mutually orthogonal directions can be taken as the coordinate axes which are coincident with principal directions of strain. Hence, the result.

5.4 GENERAL INFINITESIMAL DEFORMATION

Now we consider the general functional transformation and relation to the linear deformation. Consider an arbitrary material point $P^0\left(x_i^0\right)$ in a continuous medium. let the same material point assume after deformation the point $Q^0\left(\xi_i^0\right)$. Then

$$\xi_i^0 = x_i^0 + u_1\left(x_1^0, x_2^0, x_3^0\right) \qquad (5.4.1)$$

where u_i are the components of the displacement vector $\overline{P^0Q^0}$. We assume that as well as their partial derivatives is a continuous function. The nature of the deformation in the neighborhood of the point P^0 can be determined by considering the change in the vector $\overline{P^0}P = A_i$; in undeformed state.

Let $Q(\xi_1, \xi_2, \xi_3)$ be the deformed position of P. then the displacement u_i at the point P is

$$u_i(x_1, x_2, x_3) = \xi_i - x_i \qquad (5.4.2)$$

The vector
$$A_i = x_i - x_i^0 \qquad (5.4.3)$$

Has now deformed to the vector

$$\xi_i - \xi_i^0 = A_i' \qquad \text{(say)}$$

(5.4.4)

Therefore,
$$\delta A_i = A_i' - A_i$$
$$= \left(\xi_i - \xi_i^0\right) - \left(x_i - x_i^0\right)$$
$$= \left(\xi_i - x_i\right) - \left(\xi_i^0 - x_i^0\right)$$
$$= u_i\left(x_1, x_2, x_3\right) - u_i\left(x_1^0, x_2^0, x_3^0\right)$$
$$= u_i\left(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3\right) - u_i\left(x_1^0, x_2^0, x_3^0\right)$$
$$= \left(\frac{\partial u_i}{\partial x_j}\right) A_j \qquad (5.4.5)$$

plus the higher order terms of Taylor's series. The subscript o indicates that the derivatives are to be evaluated at the point P^0. If the region in the neighborhood of P^0 is chosen sufficiently small, i.e. if the vector A_i is sufficiently small, then the product terms like A_i, A_j may be ignored. Ignoring the product terms and dropping the subscript 0 in (5.4.5), we write

$$\delta A_i = u_{i,j} A_j \qquad (5.4.6)$$

where the symbol $u_{i,j}$ has been used for $\frac{\partial u_i}{\partial x_j}$. Result (5.4.6) holds for small vectors A_i.

If we further assume that the displacements u_i as well as their partial derivatives are so small that their products can be neglected, then the transformation (which is linear) given by (5.4.4) becomes infinitesimal in the neighborhood of the point P^0 under consideration and

$$\delta A_i = \alpha_{ij} A_j \qquad (5.4.7)$$

with
$$\alpha_{ij} = u_{i,j} \qquad (5.4.8)$$

Hence, all results discussed earlier are immediately applicable. The transformation (5.4.6) can be spited into deformation and rigid body motion as

$$\delta A_i = u_{i,j} A_j = \left(\frac{u_{i,j} + u_{j,i}}{2} + \frac{u_{i,j} - u_{j,i}}{2} \right) A_j$$

$$= e_{ij} A_j + w_{ij} A_j \qquad (5.4.9)$$

Where $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (5.4.10)$$

$$w_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \qquad (5.4.11)$$

The transformation

$$\delta A_i = e_{ij} A_j \qquad (5.4.12)$$

represents pure deformation and

$$\delta A_i = w_{ij} A_j \qquad (5.4.13)$$

represents rotation. In general, the transformation (5.4.9) is no longer homogeneous as both strain components e_{ij} and components of rotation w_{ij} are function of the coordinates. We find

$$v = e_{ij} = \frac{\partial u_i}{\partial x_i} = u_{i,i} = div \bar{u} \qquad (5.4.14)$$

That is, the cubic dilatation is the divergence of the displacement vector $\bar{u}$ and it differs, in general, from point of the body. The rotation vector w_i is given by

$$w_1 = w_{32}, w_2 = w_{13}, w_3 = w_{21}. \qquad (5.4.15)$$

Question: For the small linear deformation given by

$\bar{u} = \alpha x_1 x_2 \left(\hat{e}_1 + \hat{e}_2\right) + 2\alpha\left(x_1 + x_2\right) x_3 \hat{e}_3$, α =constant.

Find the strain tensor, the rotation and the rotation vector.

Solution: We have

$$u_1 = \alpha x_1 x_2, u_2 = \alpha x_1 x_2, u_3 = 2\alpha(x_1 + x_2)x_3 .$$

Then strains are given by

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \alpha x_2, e_{22} = \frac{\partial u_2}{\partial x_2} = \alpha x_1, e_{33} = \frac{\partial u_3}{\partial x_3} = 2\alpha(x_1 + x_2)$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\alpha}{2}(x_1 + x_2)$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \alpha x_3, e_{23} = \alpha x_3$$

We know that

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{5.4.16}$$

We find

$$w_{11} = w_{22} = w_{33} = 0$$

$$w_{12} = \frac{\alpha}{2}[x_1 - x_2] = -w_{21},\ w_{13} = -\alpha x_3 = -w_{31},\quad w_{23} = -\alpha x_3 = -w_{32}$$

Therefore

$$(w_{ij}) = \alpha\begin{bmatrix} 0 & \frac{(x_1 - x_2)}{2} & -x_3 \\ -\frac{(x_1 - x_2)}{2} & 0 & -x_3 \\ x_3 & x_3 & 0 \end{bmatrix} \tag{5.4.17}$$

The rotation vector $\bar{w} = w_i$ is given by $w_i = \in_{ijk} u_{kj}$. We find

$$w_1 = w_{32} = \alpha x_3, w_2 = w_{13} = -\alpha x_3, w_3 = w_{21} = \frac{\alpha}{2}(x_2 - x_1)$$

So $$\overline{w} = \alpha x_3(\hat{e}_1 - \hat{e}_2) + \frac{\alpha}{2}(x_2 - x_1)\hat{e}_3 \qquad (5.4.18)$$

Exercise 1: For small deformation defined by the following displacement, find the strain tensor, rotation tensor and rotation vector.

(i) $u_1 = -\alpha x_2 x_3, u_2 = \alpha x_1 x_2, u_3 = 0$

(ii) $u_1 = \alpha^2(x_1 - x_3)^2, u_2 = \alpha^2(x_2 + x_3)^2, u_3 = -\alpha x_1 x_2, \alpha = \text{constant}$ (5.4.19)

Exercise 2: the displacement components are given by

$u = -yz, v = xz, w = \phi(x, y)$ calculate the strain components. (5.4.20)

Exercise 3: Given the displacements

$$u = 3x^2 y, v = y^2 + 6xz, w = 6z^2 + 2yz$$

Calculate the strain components at the point (1, 0, 2). What is the extension of a line element (parallel to the x- axis) at this point? (5.4.21)

Exercise 4: Find the strain components and rotation components for the small displacement components given below

(a) Uniform dilation- u=ex, v=ey, w=ez

(b) Simple extension- u=ex, v=w=0

(c) Shearing strain- u=2sy, v=w=0

(d) Plane strain- u=u(x, y), v=v(x, y), w=0 (5.4.22)

5.5 SAINT-VENANT'S EQUATIONS OF COMPATIBILITY

By definition, the strain components e_{ij} in terms of displacement components u_i are given by

$$e_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}] \qquad (5.5.1)$$

Equation (5.5.1) is used to find the components of strain if the components of displacement are given. However, if the components of strain, e_{ij},are given then equation (5.4.1) is a set of six partial differential equations in the three unknown u_1, u_2, u_3.Therefore, the system (5.5.1) will not have single valued solution for u_i unless given strains e_{ij} satisfy certain conditions which are known as the conditions of compatibility or equations of compatibility.

Equations of compatibility

we have $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (5.5.2)$$

so, $$e_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right) \qquad (5.5.3)$$

Interchanging i with k and j with l in equation (5.4.3), we write

$$e_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right) \qquad (5.5.4)$$

adding (5.5.3) and (5.5.4), we get

$$e_{ij,kl} + e_{kl,ij} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}\right) \qquad (5.5.5)$$

Interchanging i and l in (5.5.5), we get

$$e_{lj,ki} + e_{ki,lj} = \frac{1}{2}\left(u_{l,jki} + u_{j,lki} + u_{k,lij} + u_{i,ljk}\right) \qquad (5.5.6)$$

From (5.5.5) and (5.5.6), we obtain

$$e_{ij,kl} + e_{kl,ij} = e_{lj,ki} + e_{ki,lj}$$

Or $$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad (5.5.7)$$

These equations are known as equations of compatibility.

These equations are necessary conditions for the existence of a single valued continuous displacement field. These are 81 equations in number. Because of symmetry in indices i, j and k, l; some of these equations are identically satisfied and some are repetitions. Only 6 out of 81 equations are essential. These equations were first obtained by **Saint-Venant's in 1860.**

A strain tensor e_{ij} that satisfies these conditions is referred to as a possible strain tensor.

Show that the conditions of compatibility are sufficient for the existence of a single valued continuous displacement field.

Let $P^0(x_i^0)$ be some point of a simply connected region at which the displacements u_i^0 and rotations w_{ij}^0 are known. The displacements u_i of an arbitrary point $P'(x_i')$ can be obtained in terms of the known functions e_{ij} by mean of a line integral along a continuous curve C joining the point P^0 and P'.

$$u_j(x_1', x_2', x_3') = u_j^0(x_1^0, x_2^0, x_3^0) + \int_{P^0}^{P'} du_j \tag{5.5.8}$$

If the process of deformation does not create cracks or holes, i.e., if the body remains continuous, the displacements u_j' should be independent of the path of integration. That is, u_j' should have the same value regardless of whether the integration is along curve C or any other curve. We write

$$du_j = \frac{\partial u_j}{\partial x_k} dx_k = u_{j,k} dx_k = (e_{jk} + w_{jk}) dx_k \tag{5.5.9}$$

Therefore

$$u_j' = u_j^0 + \int_{P^0}^{P'} e_{jk} dx_k + \int_{p^0}^{P'} w_{jk} dx_k \,,\ P(x_k) \text{ being point the joining curve.} \tag{5.5.10}$$

Integrating by parts the second integral, we write

$$\int_{P^0}^{P'} w_{jk} dx_k = \int_{P^0}^{P'} w_{jk} d(x_k - x'_k)$$ the point $P'(x'_k)$ being fixed so $dx'_k = 0$

$$= \{(x_k - x_k')w_{jk}^0\}_{P^o}^{P'} - \int_{P^0}^{P'} (x_k - x_k')w_{jk,l} dx_l \qquad (5.5.11)$$

From equations (5.5.10) and (5.5.11), we write

$$u_j(x_1', x_2', x_3') = u_j^0 + (x_k' - x_k^0)w_{jk}^0 + \int_{P^o}^{P'} e_{jk} dx_k + \int_{P^o}^{P'} (x_k' - x_k)w_{jk,l} dx_l$$

$$= u_j^0 + (x_k' - x_k^0)w_{jk}^0 + \int_{P^0}^{P'} [e_{jl} + (x_k' - x_k)w_{jk,l}] dx_l \qquad (5.5.12)$$

where the dummy index k of e_{jk} has been changed to l.

but $$w_{jk,l} = \frac{1}{2}\frac{\partial}{\partial x_l}[u_{j,k} - u_{k,j}]$$

$$= \frac{1}{2}[u_{j,kl} - u_{k,jl}]$$

$$= \frac{1}{2}[u_{j,kl} + u_{l,jk}] - \frac{1}{2}[u_{l,jk} - u_{k,jl}]$$

$$= e_{jl,k} - e_{lk,j} \qquad (5.5.13)$$

using (5.5.13), equation (5.5.12) becomes

$$u_j(x_1', x_2', x_3') = u_j^0 + (x_k' - x_k^0)w_{jk}^0 + \int_{P^0}^{P'} [e_{jl} + \{x_k' - x_k\}\{e_{jl,k} - e_{kl,j}\}] dx_l$$

$$= u_j^0 + (x_k' - x_k^0)w_{jk}^0 + \int_{P^0}^{P'} U_{jl} dx_l \qquad (5.5.14)$$

where for convenience we have set

$$U_{jl} = e_{jl} + (x_k' - x_k)(e_{jl,k} - e_{kl,j}) \tag{5.5.15}$$

which is known function as e_{ij} are known. The first two terms in the side of equation (5.5.14) are independent of the path of integration. From the theory of line integrals, the third term becomes independent of the path of integration when the integrands $U_{jl} dx_1$ must be exact differentials. Therefore, if the displacements $u_i(x_1', x_2', x_3')$ are to be independent of the path of integration, we must have

$$\frac{\partial U_{jl}}{\partial x_i} = \frac{\partial U_{ji}}{\partial x_l} \qquad \text{for } i, j, l = 1,2,3 \tag{5.5.16}$$

Now

$$U_{jl,i} = e_{jl,i} + (x_k' - x_k)(e_{jl,ki} - e_{kl,ji}) - \delta_{ki}(e_{jl,k} - e_{kl,j})$$
$$= e_{jl,i} - e_{kl,i} + e_{li,j} + (x_k' - x_k)(e_{jl,ki} - e_{kl,ji}) \tag{5.5.17}$$

and

$$U_{jl,i} = e_{ji,l} + (x_k' - x_k)(e_{ji,kl} - e_{ki,jl}) - \delta_{kl}(e_{ji,k} - e_{ki,j})$$
$$= e_{ji,l} - e_{ji,l} + e_{li,j} + (x_k' - x_k)(e_{ji,kl} - e_{ki,jl}) \tag{5.5.18}$$

Therefore, equations (5.5.16) and (5.5.17), (5.5.18) yields

$$(x_k' - x_k)[e_{jl,ki} - e_{kl,ji} - e_{ji,kl} + e_{ki,kl}] = 0$$

Since this is true for an arbitrary choice of $x_k' - x_k$ (as P' is arbitrary), it follows that

$$e_{ji,kl} + e_{kl,ji} - e_{ik,jl} - e_{jl,ki} = 0 \tag{5.5.19}$$

This is true as these are the compatibility relations. Hence, the displacement (5.5.8) independent of the path of integration. Thus, the compatibility conditions (5.5.7) are sufficient also.

Remarks1: The compatibility conditions (5.4.7) are necessary and sufficient for the existence of a single valued continuous displacement field when the strain components are prescribed.

In details form, these 6 conditions are

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{-\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} \right)$$

$$\frac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{-\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} \right)$$

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{-\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} \right)$$

$$\frac{2\partial^2 e_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 e_{11}}{\partial {x_2}^2} + \frac{\partial^2 e_{22}}{\partial {x_1}^2}$$

$$\frac{2\partial^2 e_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 e_{22}}{\partial {x_3}^2} + \frac{\partial^2 e_{33}}{\partial {x_3}^2}$$

$$\frac{2\partial^2 e_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 e_{33}}{\partial {x_1}^2} + \frac{\partial^2 e_{11}}{\partial {x_3}^2} . \qquad (5.5.20)$$

These are the necessary and sufficient conditions for the components e_{ij} to give single valued displacements u_i for a simply connected region.

Definition: A region space is said to be simply connected if an arbitrary closed curve lying in the region can be shrunk to a point, by continuous deformation, without passing outside of the boundaries.

Remarks2: The specification of the strains e_{ij} only does not determine the displacements u_i uniquely because the strains e_{ij} characterize only the pure deformation of an elastic neighborhood of the point x_i .

The displacements u_i may involve rigid body motions which do not affect e_{ij} .

Example1: (i) Find the compatibility condition for the strain tensor e_{ij} if e_{11}, e_{22}, e_{33} are independent of x_3 and $e_{31} = e_{32} = e_{33} = 0$.

(ii) Find the condition under which the following are possible strain components.

$$e_{11} = k\left(x_1^{\,2} - x_2^{\,2}\right), e_{12} = k' x_1 x_2, e_{22} = k x_1 x_2,$$

$$e_{31} = e_{32} = e_{33} = 0, k \;\&\; k' \text{ are constants}$$

(iii) When e_{ij} given above are possible strain components, find the corresponding displacements, given that $u_3 = 0$

Solution: (i) We verify that all the compatibility conditions except one are obviously satisfied. The only compatibility to be satisfied by e_{ij} is

$$e_{11,22} + e_{22,11} = 2e_{12,21}. \qquad (5.5.21)$$

(ii) Five conditions are trivially satisfied. The remaining condition (5.5.20) is satisfied iff

$$k' = k \quad \text{as} \qquad e_{11,22} = -2k, e_{12,12} = k', e_{22,11} = 0$$

(iii) We find

$$e_{11} = u_{1,1} = k\left(x_1^2 - x_2^2\right), u_{2,2} = kx_1x_2, u_{1,2} + u_{2,1} = -2kx_1x_2, \; (\therefore k' = -k)$$

$$u_{2,3} = u_{1,3} = 0$$

This shows that the displacement components u_1 and u_2 are independent of x_3.

We find (exercise)

$$u_1 = \frac{1}{6}(2x_1^{\,3} - 6x_1x_2^{\,2} + x_2^{\,3}) - cx_2 + c_1$$

$$u_2 = \frac{1}{2}kx_1x_2^{\,2} + cx_1 + c_2 \text{ where } c_1, c_2 \text{ and c constants.}$$

Example: Show that the following are not possible strain components

$$e_{11} = k\left(x_1^{\,2} + x_2^{\,2}\right), e_{22} = k\left(x_2^{\,2} + x_3^{\,2}\right), e_{33} = 0$$

$$e_{12} = k' x_1 x_2 x_3, e_{13} = e_{21} = 0, k \; \& \; k' \text{ being constants.}$$

Solution: The given components e_{ij} are possible strain components if each of the six compatibility conditions are satisfied. On substitution, we find

$$2k = 2k' x_3$$

This can't be satisfied for $x_3 \neq 0$. For $x_3 = 0$, this gives k=0 and then all e_{ij} vanish. Hence, the given e_{ij} are not possible strain components.

Exercise1: Consider a linear strain field associated with a simply connected region R such that $e_{11} = Ax_2^{\,2}, e_{22} = Ax_1^{\,2}, e_{12} Bx_1 x_2, e_{13} = e_{23} = e_{33} 0$.find the relationship between constant A and B such that it is possible to obtain a single- valued continuous displacement field which corresponds to the given strain field.

Exercise2: Show by differentiation of the strain displacement relation that the compatibility conditions are necessary condition for the existence of continuous single-valued displacements.

Exercise3: Is the following state of strain possible? (c=constant)

$$e_{11} = c\left(x_1^{\,2} + x_2^{\,2}\right)x_3, e_{22} = cx_2^{\,2} x_3, e_{12} = 2cx_1 x_2 x_3, e_{31} = e_{32} = e_{33} = 0$$

Exercise4: Show that the equations of compatibility represent a set of necessary and sufficient conditions for the existence single-valued displacements. Drive the equations of compatibility for plane strain.

Exercise 5: If $e_{11} = e_{22} = e_{33} = 0, e_{13} = \phi,_2$ and $e_{23} = \phi_{,1}$; where ϕ is a function of x_1 and x_2, show that ϕ must satisfy the equation

$$\nabla^2 \phi = \text{constant}$$

Exercise 6: If e_{13} and e_{23} are the only non-zero strain components and e_{13}, e_{23} are independent of x_3, show that the compatibility condition may be reduced to the following condition

$$e_{13,2} - e_{23,1} = \text{constant.}$$

Exercise 7: Find which of the following values of e_{ij} are possible linear strains

(i) $e_{11} = \alpha\left(x_1^{\,2} + x_2^{\,2}\right), e_{22} = \alpha x_2^{\,2}, e_{12} = 2\alpha x_1 x_2, e_{31} = e_{32} = e_{33} = 0, \alpha = \text{constant.}$

(ii)
$$e_{ij} = \begin{bmatrix} x_1 + x_2 & x_1 & x_2 \\ x_1 & x_2 + x_3 & x_3 \\ x_2 & x_3 & x_1 + x_3 \end{bmatrix}$$

Compute the displacements in the case (i).

5.6 FINITE DEFORMATIONS

All the results reported in the preceding sections of this chapter were that of the classical theory of infinitesimal strains. Infinitesimal transformations permit the application of the derivatives of superposition of effects. Finite deformations are those deformations in which the displacements u_i together with their derivatives are no longer small. Consider an aggregate of particles in a continuous medium. We shell use the same reference frame for the location of particles in the deformed and undeformed states.

Let the coordinates of a particle lying on a curve C_0, before deformation, be denoted by (a_1, a_2, a_3) and let the coordinates of the same particle after deformation (now lying same curve C) be (x_1, x_2, x_3). Then the elements of arc of the curve C_0 and C are given, respectively, by

$$ds_o^{\,2} = da_i da_i \tag{5.6.1}$$

and
$$ds^2 = dx_i dx_i \tag{5.6.2}$$

we consider first the Eulerian description of the strain and write

$$a_i = a_i(x_1, x_2, x_3) \tag{5.6.3}$$

then
$$da_i = a_{i,j} dx_j = a_{i,k} dx_k \tag{5.6.4}$$

substituting from (5.6.3) into (5.6.1), we write

$$ds_0^{\ 2} = a_{i,j} a_{i,k} dx_j dx_k \tag{5.6.5}$$

using the substitution tensor, equation (5.6.2) can be rewritten as

$$ds^2 = \delta_{jk} dx_j dx_k \tag{5.6.6}$$

We know that the measure of the strain is the difference $ds^2 - ds_0^{\ 2}$

from equations (5.6.5) and (5.6.6), we get

$$ds^2 - ds_0^{\ 2} = \left(\delta_{jk} - a_{i,j} a_{i,k}\right) dx_j dx_k$$

$$= 2\eta_{jk} dx_j dx_k \tag{5.6.7}$$

where

$$2\eta_{jk} = \delta_{jk} - a_{i,j} a_{i,k} \tag{5.6.8}$$

We now write the strain components η_{jk} in term of displacement components u_i, where

$$u_i = x_i - a_i \tag{5.6.9}$$

this gives

$$a_i = x_i - u_i$$

Hence

$$a_{i,j} = \delta_{ij} - u_{i,j} \tag{5.6.10}$$

$$a_{i,k} = \delta_{ik} - u_{i,k} \tag{5.6.11}$$

Equations (5.6.8), (5.6.10) and (5.6.11) yield

$$2\eta_{jk} = \delta_{jk} - \left(\delta_{ij} - u_{i,j}\right)\left(\delta_{ik} - u_{i,k}\right)$$

$$= \delta_{jk} - [\delta_{jk} - u_{k,j} - u_{j,k} + u_{i,j} u_{i,k}]$$

$$= \left(u_{j,k} + u_{k,j}\right) - u_{i,j}u_{i,k} \tag{5.6.12}$$

The quantities η_{jk} are called the Eulerian strain components.

If, on the other hand, Lagrangian coordinates are used, and equations of transformation are of the form

$$x_i = x_i\left(a_1, a_2, a_3\right) \tag{5.6.13}$$

then

$$dx_i = x_{i,j}da_i = x_{i,k}da_k \tag{5.6.14}$$

and

$$ds^2 = x_{i,j}x_{i,k}da_i da_k \tag{5.6.15}$$

while

$$ds_0{}^2 = \delta_{ij}da_j da_k \tag{5.6.16}$$

The Lagrangian components of strain $\in_{jk}$ are defined by

$$ds^2 - ds_0{}^2 = 2\in_{jk} da_j da_k \tag{5.6.17}$$

Since

$$x_i = a_i + u_i \tag{5.6.18}$$

Therefore,

$$x_{i,j} = \delta_{ij} + u_{i,j}$$

$$x_{i,k} = \delta_{ik} + u_{i,k}$$

Now

$$ds^2 - ds_0{}^2 = \left(x_{i,j}x_{i,k} - \delta_{jk}\right)da_j da_k$$

$$= \left[\left(\delta_{ij} + u_{i,j}\right)\left(\delta_{ik} + u_{i,k}\right) - \delta_{jk}\right]da_j da_k$$

$$= \left(u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}\right)da_j da_k \tag{5.6.19}$$

Equation (5.6.17) and (5.6.19) give

$$2\in_{jk} = u_{j,k} + u_{k,j} + u_{i,j}u_{i,k} \tag{5.6.20}$$

It is mentioned here that the differentiation in (5.6.12) is carried out with respect to the variable x_i, while in (5.6.19) the 'a_i' are regarded as the independent as the independent variables. To make the difference explicitly clear, we write out the typical expressions η_{jk} and $\in_{jk}$ in unabridged notation,

$$\eta_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{5.6.21}$$

$$2\eta_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{5.6.22}$$

$$\in_{xx} = \frac{\partial u}{\partial a} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial b}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right] \tag{5.6.23}$$

$$2\in_{xy} = \left(\frac{\partial u}{\partial a} + \frac{\partial v}{\partial a}\right) + \left(\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right) \tag{5.6.24}$$

When the strain components are large, it is no longer possible to give simple geometrical interpretations of the strain $\in_{jk}$ and η_{jk}.

Now we consider some particular cases.

Case1: Consider a line element with

$$ds_0 = da_1, da_2 = 0, da_3 = 0 \tag{5.6.25}$$

Define the extension E_1 of this element by

$$E_1 = \frac{ds - ds_0}{ds_0}$$

then

$$ds = (1 + E_1)ds_o \tag{5.6.26}$$

and consequently

$$ds^2 - ds_0{}^2 = 2\in_{jk} da_j da_k$$

$$= 2\in_{11} da_j{}^2 \qquad (5.6.27)$$

Equation (5.6.25) to (5.6.27) yield

$$(1+E_1)^2 - 1 = 2\in_{11}$$

Or $$E_1 = \sqrt{1+2\in_{11}} - 1 \qquad (5.6.28)$$

When the strain $\in_{11}$ is small,(5.6.28) reduced to

$$E_1 \cong \in_{11}$$

As was shown in discussion of strain infinitesimal strains.

Case II: Consider next two line elements

$$ds_0 = da_2, da_1 = 0, da_3 = 0 \qquad (5.6.29)$$

and

$$d\overline{s_0} = d\overline{a_3}, d\overline{a_1} = d\overline{a_2} = 0 \qquad (5.6.30)$$

These two elements lie initially along the a_2-and a_3-axes.

Let θ denote the angle between the corresponding deformed dx_i and $d\overline{x_i}$, of length ds and $d\bar{s}$ respectively. Then

$$ds d\bar{s}\cos\theta = dx_i d\overline{x_i} = x_{i,\alpha}\overline{x_{i,\beta}} da_\alpha d\overline{a_\beta} = x_{i,2}\overline{x_{i,3}} da_2 d\overline{a_3}$$

$$= 2\in da_2 d\overline{a_3} \qquad (5.6.31)$$

Let $$\alpha_{23} = \frac{\pi}{2} - \theta \qquad (5.6.32)$$

Denotes the change in the right angle between the line elements in the initial state. Then, we have

$$\sin\alpha_{23} = 2\in_{23}\left(\frac{da_2}{ds}\right)\left(\frac{d\overline{a_3}}{d\overline{s}}\right) \tag{5.6.33}$$

$$= \frac{2\in_{23}}{\sqrt{1+2\in_{22}}\sqrt{1+2\in_{33}}} \tag{5.6.34}$$

using relations (5.6.26) and (5.6.28).

Again, if the strains $\in_{ij}$ are so small that their products can be neglected, then

$$\alpha_{23} \cong 2\in_{23} \tag{5.6.35}$$

As proved earlier for infinitesimal strains.

Remarks: If the displacements and their derivatives are small, then it is immaterial whether the derivatives are calculated at the position of a point before or after deformation. In this case, we may neglect the nonlinear terms in the partial derivatives in (5.6.12) and (5.6.20) and reduce both sets of formulas to

$$2\eta_{jk} = u_{j,k} + u_{k,j} = 2\in_{jk}$$

Which were obtained for an infinitesimal transformation, It should be emphasized of finite homogeneous strain are not in general commutative and that the simple superposition of effects is no longer applicable to finite deformation.

Books Recommended:

1. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

2. **S. Timoshenko and N. Goodier,** Theory of Elasticity, McGraw Hill, New York, 1970.

CHAPTER-VI

ANALYSIS OF STRESS

6.1 INTRODUCTION

Deformation and motion of an elastic body are generally caused by external forces such as surface loads or internal forces such as earthquakes, nuclear explosions etc. When an elastic body is subjected to such force, its behaviour depends on magnitude of forces, upon their direction and upon the inherent strength of the material of which the body is made. Such forces give rise to interaction between neighbouring portions in the interior parts of the elastic solid. The concept of stress vector on a surface and state of stress at a point of the medium shall be discussed.

An approach to the solutions of problems in elastic solid mechanics is to examine deformation initially and then consider stresses and applied loads. Another approach is to establish relationship between applied loads and internal stresses first and then to consider deformations. Regardless of the approach selected, it is necessary to derive the components relations individually.

6.2 BODY FORCES AND SURFACE FORCES

Consider a continuous medium. We refer the points of this medium to a rectangular Cartesian coordinate system. Let τ represents the region occupied by the body in deformed state. A deformable body may be acted upon by two different types of external forces.

(i) **Body forces:** These forces are those forces which act on every volume element of the body and hence on the entire volume of the body. Forexample, gravitational force and magnetic forces are body forces. Let ρ denotes the density of a volume element$\Delta\tau$ of the bodyτ. Let g be the gravitational force/acceleration. Then the force acting on mass $\rho\Delta\tau$ contained in volume $\Delta\tau$ is g $\rho\Delta\tau$.

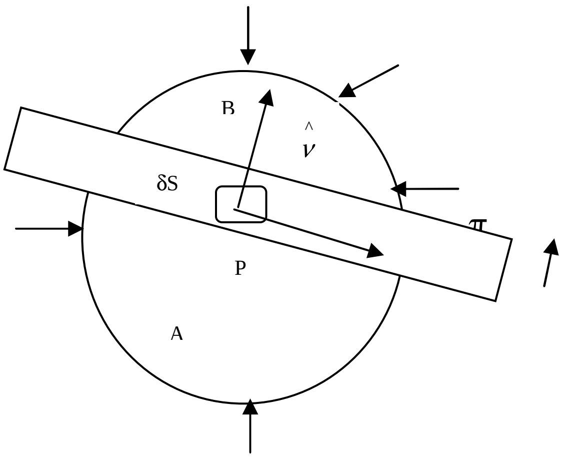
(ii) **Surface forces:** These forces act on every surface element of the body .Such forces are also called contact forces. Loads applied over the external surface or bounding surface are examples of surface forces. Hydrostatic pressure acting on the surface of a body submerged in a liquid /water is a surface force.

(iii) **Internal forces:** Internal forces such as earthquakes, nuclear explosions arise from the mutual interaction between various parts of the elastic body.

Now we consider an elastic body in its unreformed state with no forces acting on it. Let a system of forces applied on it. Due to these forces, the body is deformed and a system of internal forces is set up to oppose this deformation. These internal forces give rise to stress within the body. It is therefore necessary to consider how external forces are transmitted through the medium.

6.3 STRESS VECTOR ON A PLANE AT A POINT

Let us consider an elastic body in equilibrium under the action of a system of external forces.



Let us pass a fictitious plane π through a point P(x_1, x_2, x_3,) in the interior of this body. The body can be considered as consisting of two parts, say, A and B and these parts are in welded contacts at the interface.Part A of the body is in equilibrium under forces(external) and the effect of part B on the plane π. We assume that this effect is continuously distributed over the surface of intersection around the point P, let us consider a small surface δS(on the place π) and let $\hat{\nu}$ be an outward unit normal unit vector (for the part A of the body).The effect of part B on this small surface element can be reduced to a force and a vector couple $\overline{C}$. Now let usshrink in size towards zero in amanner such that the point P always remains aside and remains the normal vector.

$$\underset{\delta S\to 0}{Lim}\frac{\overline{Q}}{\delta S}=\overline{T}(x_1, x_2, x_3),$$

$$\underset{\delta S\to 0}{Lim}\frac{\overline{C}}{\delta S}=0,$$

Now $\overline{T}$ is a surface force per unit area. The force $\overline{T}$is called the stress vector or traction on the plane πat P.

Note 1: Forces acting over the surface of a body are never idealized point forces; they are, in reality, forces per unit area applied over some finite area. These external forces per unit area are also called tractions.

Note 2:Cauchy's stress postulate

If we consider another oriented plane containing same point P(x_i), then the stress vector is likely to have a different direction. For this purpose, Cauchy made the following postulated known as **Cauchy's stress postulate**

"**The stress vector** $\underset{\sim}{T}$ **depends on the orientation of the plane upon which it acts**".

Let $\hat{\nu}$ be the unit normal to the plane π through the point P.This normal characterizes the orientation of the plane upon which the stress vector acts. For this reason, we write the stress vector as $\overset{\hat{\nu}}{\underset{\sim}{T}}$, indicating the dependence on the orientation $\hat{\nu}$.

Cauchy's Reciprocal Relation

When the plane π is in the interior of the elastic body, the normal $\hat{\nu}$ has two possible directions that are opposite to each other and we choose one of these directions.

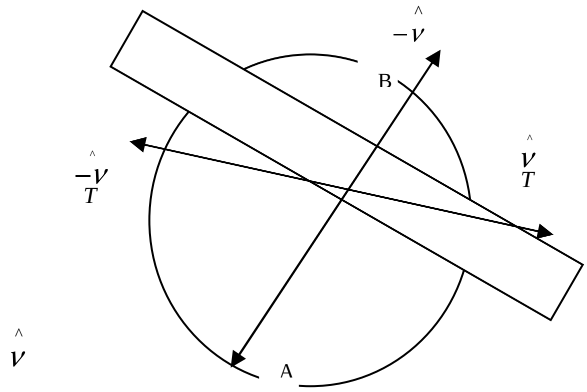


Figure 6.2

For a chosen $\hat{\nu}$,the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is interpreted as the internal surface force per unit area acting on plane π due to the action of part B of the material/body which $\hat{\nu}$ is directed upon the part A across the plane π.

Consequently, $\overset{-\hat{\nu}}{\underset{\sim}{T}}$ is the internal surface force per unit area acting on π due to the action of part A for which $\hat{\nu}$ is the outward drawn unit normal. By Newton's third law of motion, vectors $\overset{-\hat{\nu}}{\underset{\sim}{T}}$ and $-\overset{\hat{\nu}}{\underset{\sim}{T}}$ balance each other as the body is in equilibrium.

$$\therefore \qquad \overset{-\hat{\nu}}{\underset{\sim}{T}} = -\overset{\hat{\nu}}{\underset{\sim}{T}}$$

which is known as Cauchy's Reciprocal Relation.

Homogenous State of Stress

If π and π' are any two parallel planes through any two points P and P' of a continuous elastic body, and if the stress vector on π at P is equal to the stress on π' at P', then the state of stress in the body is said to be a homogeneous state of stress.

6.4 NORMAL AND TANGENTIAL STRESSES

In general, the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is inclined to the plane on which it acts and need not be in the direction of unit normal. The projection of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ on the normal $\hat{\nu}$ is called **the normal stress**. It is denoted by σ or σ_n. The projection of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ on the plane π, in the plane of $\overset{\hat{\nu}}{\underset{\sim}{T}}$ and $\hat{\nu}$, is called **the tangential or shearing stress**. It is denoted by τ or σ_t.

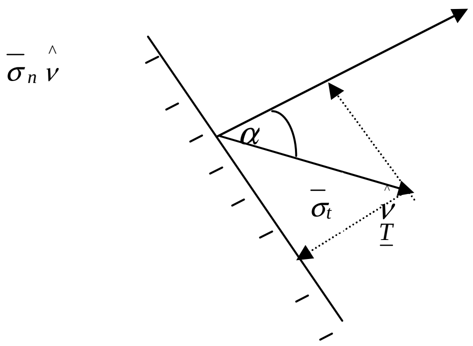


Figure 6.3

Thus,

$$\sigma = \sigma_n = \overset{\hat{\nu}}{\underset{\sim}{T}}.\hat{\nu}$$

$$\tau = \sigma_t = \overset{\hat{\nu}}{\underset{\sim}{T}}.\hat{t} \qquad (6.4.1)$$

$$\left|\overset{\hat{\nu}}{\underset{\sim}{T}}\right|^2 = \sigma_n^2 + \sigma_t^2 \qquad (6.4.2)$$

where $\overset{\hat{\nu}}{\underset{\sim}{T}}$ unit vector normal to $\hat{\nu}$ and lies in the place π.

A stress in the direction of the outward normal is considered positive (i.e.σ> 0) and is called a **tensile stress**. A stress in the opposite direction is considered negative (σ< 0) and is called a **compressible stress.**

If σ = 0, $\underset{\sim}{\overset{\hat{\nu}}{T}}$ is perpendicularto $\hat{\nu}$.The stress vector $\underset{\sim}{\overset{\hat{\nu}}{T}}$ is called a **pure shear stress** or a **pure tangential stress.**

If τ = 0, then $\underset{\sim}{\overset{\hat{\nu}}{T}}$ is parallel to $\hat{\nu}$.The stress vector $\underset{\sim}{\overset{\hat{\nu}}{T}}$ is then called **pure normal stress**.When $\underset{\sim}{\overset{\hat{\nu}}{T}}$ acts opposite to the normal $\hat{\nu}$,then the pure normal stress is called **pressure** (σ <0, τ= 0).

From (6.4.1), we can write $\underset{\sim}{\overset{\hat{\nu}}{T}} = \sigma \hat{\nu} + \tau \hat{t}$ (6.4.3)

$$\tau = \sqrt{\left|\underset{\sim}{\overset{\hat{\nu}}{T}}\right|^2 - \sigma^2}$$

(6.4.4)

Note: $\tau = \sigma_t = \left|\overset{\nu}{T}\right| Sin\,\alpha$ (6.4.5)

$$|\sigma| = \left| \overset{\hat{\nu}}{T} \times \hat{\nu} \right| \quad \text{as} \quad |\hat{\nu}| = 1$$

Thisτ in magnitude is given by the magnitude of vector product of $\underset{\sim}{\overset{\hat{\nu}}{T}}$ and $\hat{\nu}$.

6.5 STRESS COMPONENTS

Let $P(x_i)$ be any point of the elastic medium whose coordinates are (x_1 , x_2 , x_3) relative to rectangular Cartesian system $ox_1x_2\,x_3$.

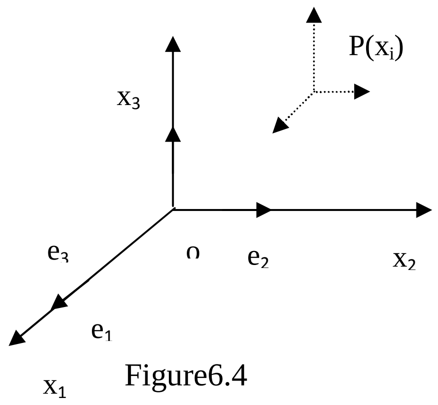


Figure6.4

Let $\overset{\hat{1}}{T}$ denote the stress vector on theplane, with normal along x_1 −axis, at the point P. Let the stress vector $\overset{\hat{1}}{T}$ has components $\tau_{11}, \tau_{12}, \tau_{13}$, i.e.

$$\overset{\hat{1}}{\underset{\sim}{T}} = \tau_{11}\hat{e}_1 + \tau_{12}\hat{e}_2 + \tau_{13}\hat{e}_3 = \tau_{1j}\hat{e}_j \qquad 6.5.1)$$

Let $\overset{\hat{2}}{T}$ denote the stress vector on the plane, with normal along x_2 −axis, at the point P.

$$\overset{\hat{2}}{\underset{\sim}{T}} = \tau_{21}\hat{e}_1 + \tau_{22}\hat{e}_2 + \tau_{23}\hat{e}_3 = \tau_{2j}\hat{e}_j \qquad (6.5.2)$$

Similarly $$\overset{\hat{3}}{\underset{\sim}{T}} = \tau_{31}\hat{e}_1 + \tau_{32}\hat{e}_2 + \tau_{33}\hat{e}_3 = \tau_{3j}\hat{e}_j \qquad (6.5.3)$$

Equations (6.5.1) to (6.5.3) can be condensed in the following form

$$\overset{\hat{i}}{\underset{\sim}{T}} = \tau_{ij}\hat{e}_j \qquad (6.5.4)$$

$$\overset{\hat{i}}{\underset{\sim}{T}}.\hat{e}_k = (\tau_{ij}\hat{e}_j).\hat{e}_k = \tau_{ij}\delta_{jk} = \tau_{ik} \qquad (6.5.5)$$

Thus, for given i & j, the quantity τ_{ij} represent the jth components of the stress vector $\overset{\hat{i}}{\underset{\sim}{T}}$ acting on a plane having $\hat{e}_i$ as the unit normal. Here, the first suffix i indicates the direction of the normal to the plane through P and the second suffix j indicates the direction of the stress component. In all, we have 9 components τ_{ij} at the point $P(x_i)$ in the $ox_1x_2\,x_3$ system. These quantities are called stress — components. The matrix

$$(\tau_{ij}) = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \qquad (6.5.6)$$

whose rows are the components of the three stress vectors, is called the matrix of the state of stress at P. The dimensions of stress components are force/(length)^2=$ML^{-1}T^{-2}$. The stress components $\tau_{11}, \tau_{22}, \tau_{33}$ are called **normal stresses** and other components $\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}$ are called as **shearing stresses(** $\overset{1}{\underset{\sim}{T}}\hat{e}_1 = e_{11}, \overset{1}{\underset{\sim}{T}}.\hat{e}_2 = e_{12}$ $etc.$). In CGS system, the stress is measured in **dyne per square centimetre.**In English system, it measured in **pounds per square inch or tons per square inch.**

DYADIC REPRESENTATION OF STRESS

It may be helpful to consider the stress tensor as a vector - like quantity having a magnitude and associated direction (s), specified by unit vector. The dyadic is such a representation. We write the **stress tensor or stress dyadic** as

$$\begin{aligned}\tau = \tau_{ij}\,\hat{e}_i\,\hat{e}_j &= \tau_{11}\,\hat{e}_1\,\hat{e}_1 + \tau_{12}\,\hat{e}_1\,\hat{e}_2 + \tau_{13}\,\hat{e}_1\,\hat{e}_3 + \tau_{21}\,\hat{e}_2\,\hat{e}_1 + \tau_{22}\,\hat{e}_2\,\hat{e}_2 \\ &\quad + \tau_{23}\,\hat{e}_2\,\hat{e}_3 + +\tau_{31}\,\hat{e}_3\,\hat{e}_1 + \tau_{32}\,\hat{e}_3\,\hat{e}_2 + \tau_{33}\,\hat{e}_3\,\hat{e}_3\end{aligned} \qquad (6.5.7)$$

where the juxtaposed double vectors are called **dyads.**

The stress vector $\overset{i}{\underset{\sim}{T}}$ acting on a plane having normal along $\hat{e}_i$ is evaluated as follows:

$$\overset{i}{\underset{\sim}{T}} = \overline{\sigma}\,\hat{e}_i = (\tau_{jk}\,\hat{e}_j\,\hat{e}_k).\hat{e}_i = \tau_{jk}\,\hat{e}_j\,\delta_{ki} = \tau_{ji}\,\hat{e}_j = \tau_{ij}\,\hat{e}_j \qquad (6.5.8)$$

6.6 STATE OF STRESS AT A POINT-THE STRESS TENSOR

We shall show that the state of stress at any point of an elastic medium on an oblique plane is completely characterized by the stress components at P.

ANALYSIS OF STRESS

Let $\overset{\hat{\nu}}{\underset{\sim}{T}}$ be the stress vector acting on an oblique plane at the material point P, the unit normal to this plane being $\hat{\nu} = \nu_i$.

Through the point P, we draw three planar elements parallel to the coordinate planes. A fourth plane ABC at a distance h from the point P and parallel to the given oblique plane at P is also drawn. Now, the tetrahedron PABC contains the elastic material.

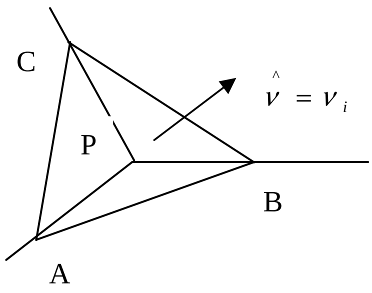


Figure6.5

Let τ_{ij} be the components of stress at the point P regarding the signs (negative or positive) of scalar quantities τ_{ij}, we adopt the following convention.

If one draws an exterior normal (outside the medium) to a given face of the tetrahedron PABC ,then the positive values of components τ_{ij} are associated with forces acting in the positive directions of the coordinate axes. On the other hand, if the exterior normal to a given face is pointing in a direction opposite to that of the coordinate axes, then the positive values of τ_{ij} are associated with forces directed oppositely to the positive directions of the coordinate axes.

Let σ be the area of the face ABC of the tetrahedron in figure. Let σ_1, σ_2, $\sigma_3$be the areas of the plane faces PBC, PCA and PAB (having normal's along $x_1 -$, $x_2 -$& $x_3 -$axes) respectively.

Then $$\sigma_i = \sigma \cos(x_i, \hat{\nu}) = \sigma \nu_i \tag{6.6.1}$$

The volume of the tetrahedron is

$$v = \frac{1}{3} h\sigma \tag{6.6.2}$$

Assuming the continuity of the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}} = \overset{\nu}{T}_i$, the x_i component of the stress force acting on the face ABC of the tetrahedron PABC (made of elastic material) is

$(\overset{\nu}{T}_i + \varepsilon_i)\sigma$

provided $$\lim_{h\to 0} \varepsilon_i = 0 \tag{6.6.3}$$

Here ε_i are inserted because the stress force acts at points of the oblique plane ABC and not on the given oblique plane through P. Under the assumption of continuing of stress field, quantities ε_1' are infinitesimals. We note that the plane element PBC is a part of the boundary surface of the material contained in the tetrahedron. As such, the unit outward normal to PBC is $-\hat{e}_i$. Therefore, the xi component of force due to stress acting on the face PBC of area σ_i is

$$(\tau_{1i} + \varepsilon_{1i})\sigma_1 \tag{6.6.4a}$$

where $\lim_{h\to 0} \varepsilon_{1i} = 0$

Similarly forces on the face PCA and PAB are

$$(\tau_{2i} + \varepsilon_{2i})\sigma_2,\ (\tau_{3i} + \varepsilon_{3i})\sigma_3$$

with $$\lim_{h\to 0}\varepsilon_{2i} = \lim_{h\to 0}\varepsilon_{3i} = 0$$

(6.6.4b)

On combining (6.6.4a) and (6.6.4b) , we write

$$(-\tau_{ji} + \varepsilon_{ji})\sigma_j \tag{6.6.5}$$

as the x_i -- component of stress force acting on the face of area provided $\lim_{h\to 0}\varepsilon_{ji} = 0$

In equation (6.6.5), the stress components are taken with the negative sign as the exterior normal to a face of area σj is in the negative direction of the xj axis.Let F_i be the body force per unit volume at the point P. Then the $x_i$component of the body force acting on the volume of tetrahedron PABC is

$$\frac{1}{3}h\sigma(F_i + \varepsilon_i') \tag{6.6.6}$$

where ε_i' 's are infinitesimal and

$$\lim_{h\to 0}\varepsilon_i' = 0$$

Since the tetrahedral element PABC of the elastic body is in equilibrium, therefore, the resultant force acting on the material contained in PABC must be zero. Thus

$$(\overset{\nu}{T}_i + \varepsilon_i)\sigma + (-\tau_{ji} + \varepsilon_{ji})\sigma_j + \frac{1}{3}h\sigma(F_i + \varepsilon')h = 0$$

Using (6.6.1), above equation (after cancellation of σ) becomes

$$(\overset{\nu}{T}_i + \varepsilon_i) + (-\tau_{ji} + \varepsilon_{ji})\nu_j + \frac{1}{3}h\sigma(F_i + \varepsilon')h = 0 \tag{6.6.7}$$

As we take the $\lim h \to 0$ in (6.6.7), the oblique face ABC tends to the given oblique plane at P. Therefore, this limit gives

$$\overset{\nu}{T}_i - \tau_{ji}\nu_j = 0$$

or $$\overset{\nu}{T}_i = \tau_{ji}\nu_j \tag{6.6.8}$$

This relation connecting the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ and the stress components τ_{ij} is known as **Cauchy's law or formula.**

It is convenient to express the equation (6.6.8) in the matrix notation. This has the form

$$\begin{bmatrix} \overset{\nu}{T_1} \\ \overset{\nu}{T_2} \\ \overset{\nu}{T_3} \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \qquad (6.6.8a)$$

As $\overset{\hat{\nu}}{\underset{\sim}{T}}$ and ν_i are vectors. Equation (6.6.8) shows, by quotient law for tensors, that **new components form a second order tensor.**

This stress tensor is called the **CAUCHY'S STRESS TENSOR.**

We note that, through a given point, there exist infinitely many surface plane elements. On every one of these elements we can define a stress vector. The totality of all these stress vectors is called the state of stress at the point. The relation (6.6.8) enables us to find the stress vector on any surface element at a point by knowing the stress tensor at that point. As such, the state of stress at a point is completely determined by the stress tensor at the point.

Note: In the above, we have assumed that stress can be defined everywhere in a body and secondly that the stress field is continuous. These are the basic assumptions of continuum mechanics. Without these assumptions, we can do very little. However, in the further development of the theory, certain mathematical discontinuities will be permitted / allowed.

6.7 BASIC BALANCE LAWS

(A) Balance of Linear Momentum:

So far, we have discussed the state of stress at a point. If it is **desired to move from one point to another,** the stress components will change. Therefore, it is necessary to investigate the equations / conditions which control the way in which they change.

While the strain tensor e_{ij} has to satisfy six compatibility conditions, the components of stress tensor must satisfy three linear **partial differential equations of the first order.** The principle of balance of linear momentum gives us these differential equations. This law, consistent with the Newton's second law of motion, states that **the time rate of change of linear momentum is equal to the resultant force on the elastic body.**

Consider a continuous medium in equilibrium with volume τ and bounded by a closed surface σ. Let F_i be the components of the **body force per unit volume** and $\overset{\nu}{T_i}$ be the component of the surface force in the x_i direction. For equilibrium of the medium, the resultant force acting on the matter within τ must vanish i.e.

$$\int_\tau F_i d\tau + \int_\sigma \overset{\nu}{T_i}\, d\sigma = 0 \qquad \text{for i = 1,2,3} \qquad (6.7.1)$$

We know the following Cauchy's formula

$$\overset{\nu}{T_i} = \tau_{ji}\nu_j \qquad \text{for i = 1,2,3} \qquad (6.7.2)$$

whereτ_{ij} is the stress tensor and ν_j is the unit normal to the surface. Using (6.7.2) into equation (6.7.l), we obtain

$$\int_\tau F_i d\tau + \int_\sigma \tau_{ji}\nu_j d\sigma = 0 \qquad \text{for i = 1,2,3} \qquad (6.7.3)$$

We assume that stresses τ_{ij} and their first order partial derivatives are also continuous and single valued in the region τ . Under these assumptions, Gauss-divergence theorem can be applied to the surface integral in (3) and we find

$$\int_{\tau} \tau_{ji,j} d\tau = \int_{\sigma} \tau_{ji} \nu_j d\sigma \qquad (6.7.4)$$

From equations (6.7.3) and (6.7.4), we write

$$\int_{\tau} (\tau_{ji} + F_i) d\tau = 0 \qquad (6.7.5)$$

for each i = l, 2 , 3. Since the region τ of integration is arbitrary (every part ofthe medium is in equilibrium) and the integrand is continuous, so, we must have

$$\tau_{ji.j} + F_i = 0 \qquad (6.7.6)$$

for each i = 1,2,3 .and at every interior point of the continuous elastic body. These equations are

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + F_1 = 0,$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + F_2 = 0, \qquad (6.7.7)$$

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + F_3 = 0,$$

These equations are referred to as **Cauchy's equations of equilibrium**. These equations are also called **stress equilibrium equations**. These equations are associated with undeformed Cartesian coordinates. These equations were obtained by Cauchy in 1827.

Note 1: In the case of motion of an elastic body, these equations (due to balance of linear momentum) take the form

$$\tau_{ji.j} + F_i = \rho \ddot{u}_i \qquad (6.7.8)$$

where $\ddot{u}_i$ is the acceleration vector and p is the density (mass per unit volume) of the body.

Note 2: When body force F_i is absent (or negligible), equations of equilibrium reduce to

$$\tau_{ji.j} = 0 \qquad (6.7.9)$$

Example: Show that for zero body force, the state of stress for an elastic body given by

$$\tau_{11} = x^2 + y + 3z^2, \tau_{22} = 2x + y^2 + 2z, \ \tau_{33} = -2x + y + z^2$$

$$\tau_{12} = \tau_{21} = -xy + z^3, \ \tau_{13} = \tau_{31} = y^2 - xz, \ \ \tau_{23} = \tau_{32} = x^2 - yz \text{ is possible.}$$

Example: Determine the body forces for which the following stress field describes a state of equilibrium

$$\tau_{11} = -2x^2 - 3y^2 - 5z \ , \tau_{22} = -2y^2 + 7, \ \tau_{33} = 4x + y + 3z - 5$$

$$\tau_{12} = \tau_{21} = z + 4xy - 6, \ \tau_{13} = \tau_{31} = -3x + 2y + 1, \ \ \tau_{23} = \tau_{32} = 0$$

Example: Determine whether the following stress field is admissible in an elastic body when body forces are negligible.

$$[\tau_{ij}] = \begin{bmatrix} yz + 4 & z^2 + 2x & 5y + z \\ . & xz + 3y & 8x^3 \\ . & . & 2xyz \end{bmatrix}$$

(B) Balance of Angular momentum

The principle of balance of angular momentum for an elastic solid is **"The time rate of change of angular momentum about the origin is equal to the resultant moment about of origin of body and surface forces."** This law assures the symmetry of the stress tensor τ_{ij}.

Let a continuous elastic body in equilibrium occupies the region τ bounded by surface σ. Let F_i be the body force acting at a point $P(x_i)$ of the body, Let the position vector of the point P relative to the origin be $\bar{r} = x_i \hat{e}_i$ Then , the moment of force F is $\bar{r} \times \bar{F} = \varepsilon_{ijk} x_j F_k$, where ε_{ijk} is the alternating tensor.

As the elastic body is in equilibrium, the resultant moment due to body and surface forces must be zero. So

$$\int_\tau \varepsilon_{ijk} x_j F_k \, d\tau + \int_\sigma \varepsilon_{ijk} x_j \overset{\nu}{T}_k \, d\sigma = 0 \text{ for each i = 1,2,3} \qquad (6.7.9)$$

Since, the body is in equilibrium, so the Cauchy's equilibrium equationsgive

$$F_k = -\tau_{lk,l} \qquad (6.7.10)$$

The stress vector $\overset{\nu}{T}_k$ in terms of stress components is given by $\overset{\nu}{T}_k = \tau_{lk} \nu_l$ (6.7.11)

The Gauss divergence theorem gives us

$$\int_\sigma \varepsilon_{ijk} x_j \tau_{lk} \nu_l \, d\sigma = \int_\tau \left[\varepsilon_{ijk} x_j \tau_{lk}\right]_{,l} d\tau$$

$$= \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \delta_{jl} \tau_{lk}] d\tau$$

$$= \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] d\tau \qquad (6.7.12)$$

From equations (6.7.9), (6.7.10) and (6.7.12); we write

$$\int_\tau \varepsilon_{ijk} x_j (-\tau_{lk,l}) d\tau + \int_\tau \varepsilon_{ijk} [x_j \tau_{lk,l} + \tau_{jk}] d\tau = 0 \qquad (6.7.13)$$

This gives

$$\int_\tau \varepsilon_{ijk} x_j \tau_{jk} \, d\tau = 0 \qquad (6.7.14)$$

fori = 1, 2 , 3. Since the integrand is continuous and the volume is arbitrary, so

$$\varepsilon_{ijk} \tau_{jk} = 0 \qquad (6.7.15)$$

fori = 1, 2 , 3 and at each point of the elastic body. Expanding (6.7.5) , we write

$$\varepsilon_{123}\tau_{23} + \varepsilon_{132}\tau_{32} = 0$$

$$\Rightarrow \tau_{23} - \tau_{32} = 0$$

$$\varepsilon_{213}\tau_{13} + \varepsilon_{231}\tau_{31} = 0$$

$$\Rightarrow \tau_{23} - \tau_{32} = 0$$

$$\varepsilon_{312}\tau_{12} + \varepsilon_{321}\tau_{21} = 0$$

$$\Rightarrow \tau_{12} - \tau_{21} = 0$$

i.e. $\Rightarrow \tau_{ij} = \tau_{ji}$ $\quad$ *for* $i \neq j$ at every point of the medium. (6.7.16)

This proves the symmetry of stress tensor. This law is also referred to **as Cauchy's second law**. It is due to **Cauchy** in 1827.

Note 1: On account of this symmetry, the state of stress at every point is specified by six instead of nine functions of position.

Note 2: In summary, the six components of the state of the stress must satisfy three partial differential equations $\tau_{ij,j} + F_i = 0$ within the body and the three relations ($\overset{\nu}{T_i} = \tau_{ji,j}\nu_j$) on the bounding surface. The equations $\overset{\nu}{T_i} = \tau_{ji,j}\nu_j$ are called the boundary conditions.

Note 3: Because of symmetry of the stress tensor, the equilibrium equations may be written as $\tau_{ij,j} + F_i = 0$

Note 4: Since $\overset{i}{T_j} = \tau_{ji}$, equations of equilibrium (using symmetry of τ_{ij}) may also be expressed as $\overset{i}{T}_{j,j} = -F_i$ or $div\overset{i}{\underset{\sim}{T}} = -F_i$

Note 5: Because of the symmetry of τ_{ij} , the boundary conditions can be expressed as

$\overset{\nu}{T}_i = \tau_{ij}\nu_j$

Remark: It is obvious that the three equations of equilibrium do not suffice for the determination of the six functions that specify the stress field. This may be expressed by the statement that the stress field is statistically indeterminate. To determine the stress field, the equations of equilibrium must be supplemented by other relations that can't be obtained from static considerations.

6.8 TRANSFORMATION OF COORDINATES

We have defined earlier the components of stress with respect to Cartesian system $ox_1x_2x_3$. Let $ox'_1x'_2x'_3$ be any other Cartesian system with thesame origin but oriented differently. Let these coordinates be connected by the linear relations

$$x'_p = \ell_{pi} x_i \tag{6.8.1}$$

where ℓ_{pi} are the direction cosines of the x'_p - axis with respect to the x_1 - axis.

i.e $$\ell_{pi} = \cos(x'_p, x_i) \tag{6.8.2}$$

Let τ'_{pq} be the components of stress in the new reference system (Figure 6.6)

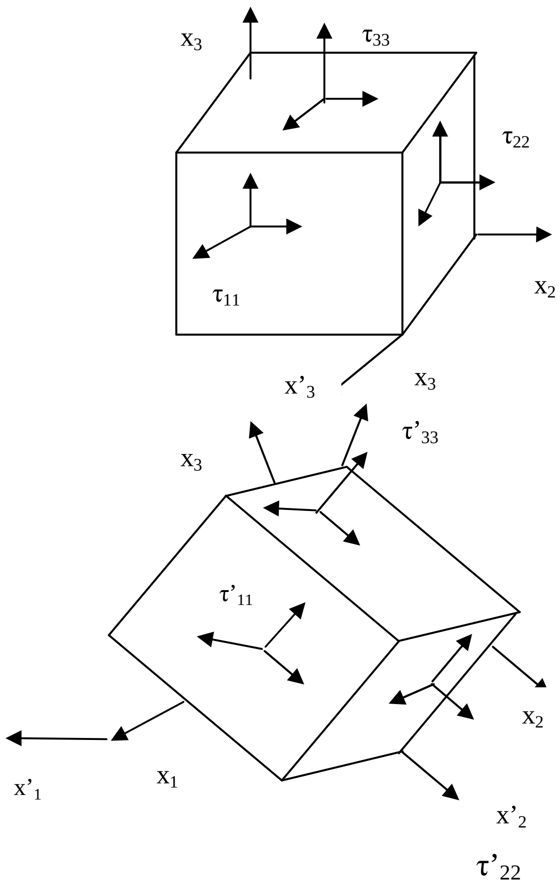


Figure 6.6& 6.7

Figure 6.7, Transformation of stress components under rotation of co-ordinates system.

Theorem: let the surface element $\Delta\sigma$ and $\Delta\sigma'$, with unit normal $\hat{\nu}$ and $\hat{\nu}'$, pass through the point P. Show that the component of the stress vector $\underset{\sim}{\overset{\hat{\nu}}{T}}$ acting on $\Delta\sigma$ in the direction of $\hat{\nu}'$ is equal to the component of the stress vector $\underset{\sim}{\overset{\hat{\nu}'}{T}}$ acting on $\Delta\sigma'$ in the direction of $\hat{\nu}$

Proof: In this theorem, it is required to show that

Thus, $\overset{\hat{\nu}'}{\underset{\sim}{T}}.\hat{\nu} ' = \overset{\hat{\nu}}{\underset{\sim}{T}}.\hat{\nu}$ (6.8.3)

The Cauchy's formulagives us

$$\overset{\hat{\nu}}{\underset{\sim}{T}} = \tau_{ji}\nu_j \qquad (6.8.4)$$

and

$$\overset{\hat{\nu}'}{\underset{\sim}{T}} = \tau_{ji}\nu'_j \qquad (6.8.5)$$

due to symmetry of stress tensors as with

$$\hat{\nu} = \nu_j, \; \hat{\nu}' = \nu'_j$$

Now $\overset{\hat{\nu}'}{\underset{\sim}{T}}.\hat{\nu} = \overset{\hat{\nu}'}{T_i}.\hat{\nu}_i$

$$= (\tau_{ij}\nu'_j)\nu_i$$

$$= (\tau_{ji}\nu'_j)\nu_i$$

$$= \overset{\hat{\nu}}{T_i}\,\nu'_i \qquad (6.8.6)$$

This completes the proof of the theorem.

Article: Use the formula (6.8.3) to derive the formulas of transformation of the components of the stress tensor τ_{ij}.

Solution: Since the stress components τ'_{pq} is the projection on the x'_p — axis of the stress vector acting on a surface element normal to the x'_p — axis (by definition), we can write

$$\tau'_{pq} = \overset{P}{T_q} = \overset{\hat{\nu}'}{T}.\hat{\nu} \qquad (6.8.7)$$

where

$$\hat{\nu}' \text{ is parallel to the } x'_p\text{-axis} \quad (6.8.8)$$

$\hat{\nu}$ is parallel to the x'$_q$ - axis

Equations (6.8.6) and (6.8.7) imply

$$\tau'_{pq} = \tau_{ij}\nu'_i\nu_j \quad (6.8.9)$$

Since

$$\nu'_i = \cos(x'_p, x_i) = \ell_{pi} \quad (6.8.10)$$

$$\nu_i = \cos(x'_q, x_i) = \ell_{qi}$$

Equation (6.8.9) becomes

$$\tau'_{pq} = \tau_{ij}\nu'_i\nu_j \quad (6.8.11)$$

Equation (6.8.11) and definition of a tensor of order 2, show that the stress components τ_{ij} transform like a Cartesian tensor of order 2. Thus, the physical concept of stress which is described by τ_{ij}agrees with the mathematical definition of a tensor of order 2 in a Euclidean space.

6.9 Theorem: Show that the quantity

$\Theta = \tau_{11} + \tau_{22} + \tau_{33}$ is invariant relative to an orthogonal transformation of Cartesian coordinates.

Proof: Let τ_{ij} be the tensor relative to the Cartesian system $ox_1x_2x_3$. Let these axes be transformed to $ox'_1x'_2x'_3$ under the orthogonal transformation

$$x'_p = \ell_{pi}x_i \quad (6.9.1)$$

where

$$\ell_{pi} = \cos(x'_p, x_i) \quad (6.9.2)$$

Let τ_{pi} be the stress components relative to new axes, then these components are given by the rule for second order tensors.

$$\tau'_{pp} = \ell_{pi} \ell_{pj} \tau_{ij} \tag{6.9.3}$$

Putting q= p and taking summation over the common suffix, we write

This implies $\tau'_{pp} = a_{pi} a_{pj} \tau_{ij}$

$$= \delta_{ij} \tau_{ij} = \tau_{ij}$$

$$\tau'_{11} + \tau'_{22} + \tau'_{33} = \tau_{11} + \tau_{22} + \tau_{33} = \Theta \tag{6.9.4}$$

This proves the theorem.

Remark: This theorem shows that whatever be the orientation of three mutually orthogonal planes passing through a given point, the sum of the normal stresses is independent of the orientation of these planes.

Exercise 1: Prove that the tangential traction parallel to a line l , across a plane at right angles to a line l' , the two lines being at right angles to each other , is equal to the tangential traction, parallel to the line l' , across a plane at right angles to l.

Exercise 2: Show that the following two statements are equivalent.

(a) The components of the stress are symmetric.

(b) Let the surface elements Δσ and Δσ' with respective normal $\hat{\nu}$ and $\hat{\nu}'$ passes through a point P. Then $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu}' = \underset{\sim}{\overset{\hat{\nu}'}{T}} . \hat{\nu}$

Hint: (b) ⇒ (a)

Let $\hat{\nu} = \hat{i}$ and $\hat{\nu}' = \hat{j}$

Then $\underset{\sim}{\overset{\hat{\nu}}{T}} . \hat{\nu}' = \underset{\sim}{\overset{i}{T}} . \hat{j} = \overset{i}{T}_j = \tau_{ij}$

and $\overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu} = \overset{i}{\underset{\sim}{T}} . \hat{i} = \overset{j}{T}_i = \tau_{ji}$

by assumption $\overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu}' = \overset{\hat{\nu}'}{\underset{\sim}{T}} . \hat{\nu}$,

therefore $\tau_{ij} = \tau_{ji}$

This shows that τ_{ij} is symmetric.

Example l: The stress matrix at a point P in a material is given as

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix}.$$

Find

(i) The stress vector on a plane element through P and parallel to the plane $2x_1 + x_2 - x_3 = 1$,

(ii) The magnitude of the stress vector, normal stress and the shear stress.

(iii) The angle that the stress vector makes with normal to the plane.

Solution: (i) The plane element on which the stress vector is required is parallel to the plane $2x_1 + x_2 - x_3 = 1$. Therefore, direction ratios of the normal to the required plane at P are< 2, 1,-1>. So, the d.c.'s of the unit normal $\hat{\nu} = \nu_i$ to the required plane at P are

$$\nu_1 = \frac{2}{\sqrt{6}}, \nu_2 = \frac{1}{\sqrt{6}}, \nu_3 = -\frac{1}{\sqrt{6}}$$

let $\overset{\nu}{\underset{\sim}{T}} . = \overset{\nu}{\underset{\sim}{T_i}}$ be the required stress vector. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T_1} \\ \overset{\nu}{T_2} \\ \overset{\nu}{T_3} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}$$

or $\overset{\nu}{T_1} = \sqrt{3/2}\,, \overset{\nu}{T_2} = 3\sqrt{3/2}\,, \overset{\nu}{T_3} = \sqrt{3/2}$

So, the required stress vector at P is

$$\overset{\nu}{T_1} = \sqrt{3/2}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3) \text{ and } |\overset{\nu}{\underset{\sim}{T}}| = \sqrt{33/2}$$

(ii) The normal stress is given by

$$\sigma = \overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}} . \hat{\nu} = \sqrt{\frac{3}{2}} .. \frac{1}{\sqrt{6}} (2+3-1) = \frac{1}{2} \times 4 = 2$$ the shear stress is given by

$$\tau = \sqrt{|\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}|^2 - \sigma^2} = \sqrt{33/2 - 4} = \frac{5}{\sqrt{2}}$$

(As $\tau \neq 0$,so the stress vector $\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}$. need not be along the normal to the plane element)

iii)letθbe the angle between the stress vector $\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}$. and normal $\hat{\nu}$.

Then

$$\cos\theta = \frac{\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}} . \hat{\nu}}{|\overset{\hat{\nu}}{\underset{\sim}{\mathrm{T}}}| . |\hat{\nu}|} = 2 \Big/ \sqrt{33/2} = \sqrt{8/33}$$

This determines the required inclination.

Example 2: The stress matrix at a point P(x_i) in a material is given by

$$[\tau_{ij}] = \begin{bmatrix} x_3 x_1 & x_3^2 & 0 \\ x_3^2 & 0 & -x_2 \\ 0 & -x_2 & 0 \end{bmatrix}$$

Find the stress vector at the point Q (l, 0, -l) on the surface $x_2^2 + x_3^2 = x_1$

Solution: The stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ is required on the surface element

f(x_1 , x_2 , x_3) $= x_1 - x_2^2 - x_3^2 = 0$, at the point Q(l , 0 , -l). We find $\nabla f = \hat{e}_1 + 2\hat{e}_3$ and $|\nabla f| = \sqrt{5}$ at the point Q.

Hence, the unit outward normal $\hat{\nu} = \nu_i$ to the surface f = 0 at the pointQ(1,0,-1) is

$$\hat{\nu} = \frac{\nabla f}{|\nabla f|} = \frac{1}{5}(\hat{e}_1 + 2\hat{e}_3)$$

giving $$\nu_1 = \frac{1}{\sqrt{5}}, \nu_2 = 0, \nu_3 = \frac{2}{\sqrt{5}}$$

The stress matrix at the point Q(1, 0, -l) is

$$[\tau_{ij}] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

let $\underset{\sim}{\overset{\nu}{T}} = \underset{\sim}{\overset{\nu}{T_i}}$ be the required stress vector at the point Q. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T_1} \\ \overset{\nu}{T_2} \\ \overset{\nu}{T_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

or $$\overset{\nu}{T}_1 = -\sqrt{1/5}\,, \overset{\nu}{T}_2 = \sqrt{1/5}\,, \overset{\nu}{T}_3 = 0$$

So, the required stress vector at P is

$$\overset{\nu}{T}_1 = \frac{1}{\sqrt{5}}(-\hat{e}_1 + \hat{e}_2)$$

Example 3: The stress matrix at a certain point in a given material is given by

$$[\tau_{ij}] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Find the normal stress and the shear stress on the octahedral plane element through the point.

Solution: An octahedral plane is a plant whose normal makes equal angles withpositive directions of the coordinate axes.Hence, the components of the unit normal $\hat{\nu} = \nu_i$ are

$$\nu_1 = \nu_2 = \nu_3 = \frac{1}{\sqrt{3}}$$

let $\underset{\sim}{\overset{\nu}{T}} = \underset{\sim}{\overset{\nu}{T_i}}$ be the required stress vector. Then, Cauchy's formula gives

$$\begin{bmatrix} \overset{\nu}{T}_1 \\ \overset{\nu}{T}_2 \\ \overset{\nu}{T}_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

or $$\overset{\nu}{T}_1 = \sqrt{5/3}\,, \overset{\nu}{T}_2 = \sqrt{3}\,, \overset{\nu}{T}_3 = \sqrt{3}$$

The magnitude of this stress vector is

$$|\underset{\sim}{\overset{\nu}{T}}| = \sqrt{43/3}$$

let σ be the normal stress and τ be the shear stress.Then

$$\sigma = \overset{\hat{\nu}}{\underset{\sim}{T}} . \hat{\nu} = \frac{1}{3}(5+3+3) = \frac{11}{3} \text{ and } \tau = \sqrt{\frac{43}{3} - \frac{121}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Since σ> 0, the normal stress on the octahedral plane is tensile.

Example 4: The state of stress at a point P in cartesian coordinates is given by

τ_{11}=500, τ_{12}= τ_{21}=500, τ_{13}= τ_{31}=800, τ_{22}=1000,τ_{33}= -300, τ_{23}= τ_{32}= -750

Compute the stress vector $\overline{T}$ and the normal and tangential components of stress on the plane passing through P whose outward normal unit vector is $\hat{\nu} = \frac{1}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$

Solution: The stress vectoris given by $\overset{\nu}{T_i} = \tau_{ji}\nu_j$,

We find $\overset{\nu}{T_1} = \tau_{11}\nu_1 + \tau_{21}\nu_2 + \tau_{31}\nu_3 = 250 + 250 + 400\sqrt{2}$ =1064(approx.)

$$\overset{\nu}{T_2} = \tau_{12}\nu_1 + \tau_{22}\nu_2 + \tau_{32}\nu_3 = 250 + 500 + \frac{750}{\sqrt{2}} = 221(approx.)$$

$$\overset{\nu}{T_3} = \tau_{13}\nu_1 + \tau_{23}\nu_2 + \tau_{33}\nu_3 = 400 - 375 + 150\sqrt{2} = 237(approx.)$$

Books Recommended:

1. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

2. **S. Timoshenko and N. Goodier,** Theory of Elasticity, McGraw Hill, New York, 1970.

CHAPTER-VII

STRESS QUADRIC OF CAUCHY

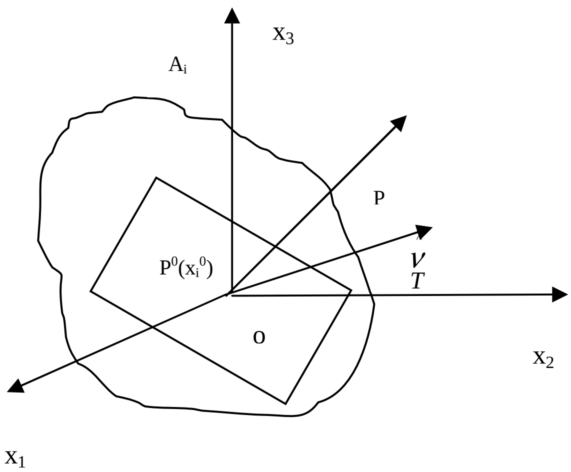
7.1 Stress Quadric of Cauchy

In a rectangular Cartesian coordinate systems $ox_1x_2x_3$, consider the equation

$$\tau_{ij}x_i xj = \pm k^2 \qquad (7.1.1)$$

where (x_1, x_2, x_3) are the coordinates a point P relative to the point P^0 whose Coordinates relative to origin O are (x_1^0, x_2^0 ,x_3^0), τ_{ij} is the stress tensor at the point $P^0(x_i^0)$ and k is a real constant . **The sign + or - is so chosen that the quadric surface (7.1.1) is real.**

The quadric surface (7.1.1) is known as the stress quadric of Cauchy with its centre at the point $P^0(x_i^0)$.



Let A_i be the radius vector, of magnitude A, on this stress quadric surface which is normal on the plane π through the point P^0 having stress tensor τ_{ij}. Let $\hat{\nu}$ be the unit vector along the vector A_i. Then

$$\nu_i = A_i/A = x_i/A \qquad (7.1.2)$$

Let $\overset{\nu}{\underset{\sim}{T}}$ denote the stress vector on the plane at the point P^0. Then, the normal stress N on the plane is given by

$$N = \overset{\nu}{\underset{\sim}{T}} . \overset{\wedge}{\nu} = \overset{\nu}{T}_i \, \nu_i = \tau_{ij}\nu_j\nu_i = \tau_{ij}\nu_i\nu_j \, . \tag{7.1.3}$$

From equations (7.1.l) and (7.1.2), we obtain

$$\tau_{ij}(A\nu_i)(A\nu_j) = \pm k^2$$

$$\tau_{ij}\nu_i\nu_j = \pm k^2 / A^2$$

$$N = \pm k^2 / A^2 \tag{7.1.4}$$

This gives the normal stress acting on the plane π with orientation $\overset{\wedge}{\nu} = \nu_i$ **in terms of the length of the radius vector** of the stress quadric from the point (centre) 0 along the vector. The relation (7.1.4) shows that the normal stress N on the plane π through P^0 along with orientation along A_i is inversely proportional to the square of that radius vector $A_i = \overline{P}^0\overline{P}$ of the stress quadric.

The positive sign in (7.1.1) or (7.1.4) is chosen whenever the normal stress N represents tension (i.e., N > 0) and negative sign when N represents compression (i.e. N < 0).

The Cauchy's stress quadric (7.1.l) possesses another interesting property. This property is "**The normal to the quadric surface at the end of the radius vector Ai is parallel to the stress vector $\overset{\nu}{\underset{\sim}{T}}$ acting on the plane π at P_0.**"

To prove this property, let us write equation (7.1.l) in the form

$$G(x_1, x_2, x_3) = \tau_{ij}\, x_i\, x_j \mp k^2 = 0 \tag{7.1.5}$$

Then the direction of the normal to the stress quadric surface is given by the gradient of the scalar point function $G(x_1, x_2, x_3)$. The components of gradient are

$$\frac{\partial G}{\partial x_n} = \tau_{ij} x_j \frac{\partial x_i}{\partial x_n} + \tau_{ij} x_i \frac{\partial x_j}{\partial x_n} = \tau_{ij} x_j \frac{\partial x_i}{\partial x_n} + \tau_{ij} x_i \frac{\partial x_j}{\partial x_n}$$

$$= \tau_{ij} (\delta_{in}) x_j + \tau_{ij} x_i \delta_{jn} = \tau_{nj} x_j + \tau_{nj} x_j$$

$$= 2\tau_{nj} x_j \quad = 2A\tau_{nj} \nu_j$$

$$= 2A \overset{\nu}{T}_n \tag{7.1.6}$$

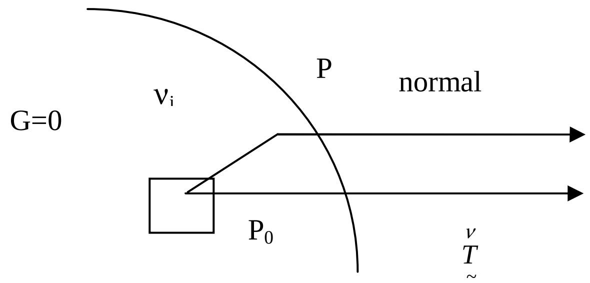


Figure 7.2

Equation (7.1.6) shows that vectors $\overset{\nu}{T}_n$ *and* $\frac{\partial G}{\partial x_n}$ are parallel. Hence the stress vector

$\underset{\sim}{\overset{\nu}{T}}$ on the plane π at P_0 is directed along the normal to the stress quadric at P, P being the end point of the radius vector $A_i = \overline{p}0\overline{p}$

Remark 1: Equation (7.1.6) can be rewritten as

$$\underset{\sim}{\overset{\nu}{T}} = \frac{1}{2A} \nabla G \tag{7.1.7}$$

This relation gives an easy way of constructing the stress vector $\overset{\nu}{\underset{\sim}{T}}$ from the knowledge of the quadric surface G(x_1, x_2, x_3) = constant and the magnitude A of the radius vector $\overline{A}$.

Remark 2: Taking principal axes along the coordinate axes, the stress quadric of Cauchy assumes the form

$$\tau_1 x_1^2 + \tau_2 x_{21}^2 + \tau_3 x_3^2 = \pm k^2 \qquad (7.1.8)$$

Here the coefficients τ_1, τ_2, τ_3 are the principal stresses. Let the axes be so numbered that $\tau_1 \geq \tau_2 \geq \tau_3$

If $\tau_1 > \tau_2 > \tau_3 > 0$, then equation (7.1.8) represents an ellipsoid with plus sign. Then, the relation N = k^2/A^2 implies that the force acting on every surface element through P^0 is tensile (as N < 0).

If $0 > \tau_1 > \tau_2 > \tau_3$ then equation (7.1.8) represents an ellipsoid with a negative sign on the right and N = $-k^2/A^2$ indicates that the normal stress is compressive (N > 0).

lf $\tau_1 = \tau_2 \neq \tau_3$ or $\tau_1 \neq \tau_2 = \tau_3$ or $\tau_1 = \tau_3 \neq \tau_2$,Then the Cauchy's stress quadric is an ellipsoid of revolution. If $\tau_1 = \tau_2 = \tau_3$, then the stress quadric is a sphere.

7.2 PRINCIPAL STRESSES

In a general state of stress, the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ acting on a surface with outer normal $\hat{\nu}$ depends on the direction of $\hat{\nu}$.

Let us see in what direction $\hat{\nu}$ the stress vector $\overset{\hat{\nu}}{\underset{\sim}{T}}$ becomes normal to the surface, on which the shearing stress is zero. Such a surface shall be called a principal plane, its normal as **principal axis**, and the value of normal stress acting on the principal plane shall be called a **principal stress**.

Let $\hat{\nu}$ defines a principal axis at the point $P^0(x_i^0)$ and let τ be the corresponding principal stress and τ_{ij} be the stress tensor at that point. Let $\overset{\hat{\nu}}{\underset{\sim}{T}}$ be the stress vector. Then

$$\overset{\hat{\nu}}{\underset{\sim}{T}} = \tau \hat{\nu}$$

or $$\overset{\nu}{T_i} = \tau \nu_i \tag{7.2.1}$$

$$\overset{\nu}{T_i} = \tau_{ij} \nu_j \tag{7.2.2}$$

or $$(\tau_{ij} - \tau\delta_{ij})\nu_j = 0 \tag{7.2.3}$$

The three equations i =1, 2, 3 are to be solved for ν_1, ν_2, ν_3 since $\hat{\nu}$ is a unit vector, we must find a set of non- trivial solutions for which

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$$

Thus, equation (7.2.3) posses an eigenvalue problem, equation (7.2.3) has a set of non vanishing solutions, ν_1, ν_2, ν_3 iff the determinant of the coefficients vanishes

i.e., $$\left| \tau_{ij} - \tau\delta_{ij} \right| = 0$$

or $$\begin{bmatrix} \tau_{11} - \tau & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} - \tau & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} - \tau \end{bmatrix} = 0 \tag{7.2.3a}$$

On expanding (7.2.2), we find

$$-\tau^3 + \theta_1\tau^2 - \theta_2\tau + \theta_3 = 0 \tag{7.2.3b}$$

where

$$\theta_1 = \tau_{11} + \tau_{22} + \tau_{33} \qquad (7.2.4a)$$

$$\theta_2 = \begin{bmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} + \begin{bmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{bmatrix} \qquad (7.2.4b)$$

$$\theta_3 = \in_{ijk} \tau_{1i}\tau_{2j}\tau_{3k} = \det(\tau_{ij}) \qquad (7.2.4c)$$

Equation (7.2.3) is a cubic equation in τ. Let its roots be τ_1, τ_2, τ_3, since the matrix of stress, (τ_{ij}) is real and symmetric; the roots (τ_i) of (7.2.3) are all real. Thus τ_1, τ_2, τ_3 are the principal stresses. For each value of the principal stress, a unit normal vector $\hat{\nu}$ can be determined.

Case I: When $\tau_1 \neq \tau_2 \neq \tau_3$

let $\overset{1}{\nu}_i$, $\overset{2}{\nu}_i$, $\overset{3}{\nu}_i$, be the unit principal axes corresponding to the principal stresses τ_1, τ_2, τ_3, respectively. Then principal axes are mutually orthogonal to each other.

Case II: If $\tau_1 = \tau_2 \neq \tau_3$ are the principal stresses then the direction $\overset{3}{\nu}_i$, corresponding to principal stress τ_3 is a principal direction and any two mutually perpendicular lines in a plane with normal $\overset{3}{\nu}_i$, may be chosen as the other two principal direction of stress.

Case III: If $\tau_1 = \tau_2 = \tau_3$ then any set of orthogonal axes through P^0 may be taken as the principal axes.

Remark: Thus, for a symmetric real stress tensor, there are three principal stresses which are real and a set of three mutually orthogonal principal directions. If the reference axes x_1 , x_2 ,x_3 are chosen to coincide with the principal axes ,then the matrix of stress components becomes

$$\tau_{ij} = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \qquad (7.2.5)$$

Invariants of the stress tensor:

Equation (7.2.3) can be written as

$$(\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3) = 0, \quad (7.2.6)$$

and we find

$$\theta_1 = \tau_1 + \tau_2 + \tau_3$$

$$\theta_2 = \tau_1\tau_2 + \tau_2\tau_3 + \tau_1\tau_3$$

$$\theta_3 = \tau_1\tau_2\tau_3 \quad (7.2.7)$$

Since the principal stress τ_1, τ_2, τ_3 characterize the physical state of stress at point , they are independent of any coordinates of reference.

Hence, coefficients $\theta_1, \theta_2, \theta_3$ of equation (7.2.3) are invariant with respect to the coordinate transformation. Thus $\theta_1, \theta_2, \theta_3$ are the three scalar invariants of the stress tensor τ_{ij}. These scalar invariants are called the fundamental stress invariants.

Components of Stress τ_{ij} in terms of τ'_α

Let X_α be the principal axes. The transformation law for axes is

$$X_\alpha = \ell_{i\alpha} x_i$$

or

$$x_i = \ell_{i\alpha} X_\alpha \quad (7.2.8)$$

where

$$\ell_{i\alpha} = \cos(x_i, X_\alpha) \quad (7.2.9)$$

The stress matrix relative to axes X_α is

$$\tau'_{\alpha\beta} = diag(\tau_1, \tau_2, \tau_3) \quad (7.2.10)$$

Let τ_{ij} be the stress matrix relative to axes x_i axes .Then the transformation rule for second order tensor is

$$\tau_{i\alpha} = \ell_{i\alpha}\ell_{i\beta}\tau'_{\alpha\beta} = \sum_{\alpha=1}^{3} \ell_{i\alpha}(\ell_{j\alpha}\tau_\alpha)$$

This gives

$$\tau_{i\alpha} = \sum_{\alpha=1}^{3} a_{i\alpha}(a_{j\alpha}\tau_{\alpha}) \qquad (7.2.11)$$

Definition (Principal axes of stress)

A system of coordinate axes chosen along the principal directions of stress is referred to as principal axes of stress.

Question: Show that, as the orientation of a surface element at a point P varies the normal stress on the surface element assumes an extreme value when the element is a principal plane of stress at P and that this extremum value is a principal stress.

Solution: Let τ_{ij} be the stress tensor at the point P. Let τ be the normal stress on a surface element at P having normal in the direction of unit vector $\hat{\nu} = \nu_i$. Thus, we have to find $\tau = \tau_{ij}\nu_i\nu_j$ .We have to find $\hat{\nu} = \nu_i$ for which τ is an extremum. Since $\hat{\nu} = \nu_i$ is a unit vector, we have the restriction

$$\nu_k\nu_k - 1 = 0 \qquad (7.2.12)$$

We use the method of Lagrange multiplier to find the extreme values of τ. The extreme values are given by

$$\frac{\partial}{\partial \nu_i}\{\tau_{ij}\nu_i\nu_j - \lambda\{\nu_k\nu_k - 1)\} = 0 \qquad (7.2.13)$$

where λ is a Lagrange's multiplier. From (7.2.13), we find

$$\tau_{ij}\{\nu_i + \delta_{ij}\nu_j\} - \lambda\{2\nu_k\delta_{ik}\} = 0$$

$$2\tau_{ij}\nu_i - 2\lambda\nu_i = 0$$

$$\tau_{ij}\nu_i - \lambda\delta_{ij}\nu_j = 0$$

$$(\tau_{ij} - \lambda\delta_{ij})\nu_j = 0 \qquad (7.2.14)$$

These conditions are satisfied iff $\hat{\nu} = \nu_j$ is a principal direction of stress and τ=λ is the corresponding principal stress. Thus, τ assumes an extreme value on a principal plane of stress and principal stress is an extreme value of τ given by (7.2.12).

7.3 MAXIMUM NORMAL AND SHEAR STRESSES

Let the coordinate axes at a point P^0 be taken along the principle directions of stress τ=λ is the corresponding principal stress. Let τ_1, τ_2,τ_3 be the principal stresses at P^0.Then

$$\tau_{11} = \tau_1, \tau_{22} = \tau_2, \tau_{33} = \tau_3; \ \tau_{12} = \tau_{23} = \tau_{31} = 0$$

Let $\overset{\hat{\nu}}{\underset{\sim}{T}} = 0$ be the stress vector on a planar element at P^0 having the normals $\hat{\nu} = \nu_i$

Let N be the normal stress and S be the shearing stress. Then

$$\left|\overset{\nu}{\underset{\sim}{T}}\right| = N^2 + S^2 \tag{7.3.1}$$

The relation $\overset{\nu}{T}_i = \tau_{ij}\nu_j$ so that $N = \overset{\nu}{\underset{\sim}{T}}.\hat{\nu} = \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2$ (7.3.1a)

N is a function of three variables ν_1, ν_2, ν_3 connected by the relation

$$\nu_k\nu_k - 1 = 0 \tag{7.3.2}$$

From (7.3.l) and (7.3.2) we write

$$N = \tau_1(1 - \nu_2^2 - \nu_3^2) + \tau_2\nu_2^2 + \tau_3\nu_3^2$$

$$N = \tau_1 + (\tau_2 - \tau_1)\nu_2^2 + (\tau_3 - \tau_1)\nu_3^2 \tag{7.3.3}$$

The extreme value of N are given by

$$\frac{\partial N}{\partial \nu_2} = 0, \frac{\partial N}{\partial \nu_3} = 0$$

which yield

$\nu_2 = 0, \nu_3 = 0$ for $\tau_2 \neq \tau_1$ & $\tau_3 \neq \tau_1$

Hence $\nu_1 = \pm 1, \quad \nu_2 = \nu_3 = 0$ & $N = \tau_1$

Similarly, we can find other two directions

$$\nu_1 = 0, \quad \nu_2 = \pm 1, \nu_3 = 0 \;\&\; N = \tau_2$$

$$\nu_1 = 0 \quad \nu_2 = 0, \nu_3 = \pm 1 \;\&\; N = \tau_3$$

Thus, we find that the extreme values of the Normal stress N are along the principal directions of stress and the extreme values are themselves principal stresses. So, the absolute maximum normal stress is the maximum of the set $\{\tau_1, \tau_2, \tau_3\}$. Along the principal directions, the shearing stress is zero (i.e. the minimum)

Now $$S^2 = (\tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2) - (\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)^2 \qquad (7.3.3a)$$

To determine the directions associated with the maximum values of $N = |S|$ .We maximize the function S(ν_1, ν_2, ν_3) in (7.3.3) subject to the relation $\nu_i \nu_j = 1$

For this, we use the method of Lagrange multipliers to fin the free extremum of the functions

$$F(\nu_1, \nu_2, \nu_3) = S^2 - \lambda(\nu_i \nu_i - 1) \qquad (7.3.4)$$

For extreme values of F, we must have

$$\frac{\partial F}{\partial \nu_1} = \frac{\partial F}{\partial \nu_2} = \frac{\partial F}{\partial \nu_3} = 0 \qquad (7.3.5)$$

The equations $\frac{\partial F}{\partial \nu_i} = 0$, gives

$$2\tau_1^2\nu_1 - 4\tau_1\nu_1(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2) - 2\lambda\nu_1 = 0$$

or $$\lambda = \tau_1^2 - 2\tau_1(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2) \qquad (7.3.6)$$

Similarly from the equation, we obtain

$$\lambda = \tau_2^2 - 2\tau_2(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2) \qquad (7.3.7)$$

$$\lambda = \tau_3^2 - 2\tau_3(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2) \qquad (7.2.8)$$

Equations (7.3.6) & (7.3.7) yield

$$\tau_2^2 - \tau_1^2 = 2(\tau_2 - \tau_1)(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)$$

For $\tau_1 \neq \tau_2$, This leads to

$$\tau_2 + \tau_1 = 2(\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)$$

or
$$(2\nu_1^2 - 1)\tau_1 + (2\nu_2^2 - 1)\tau_2 + 2\nu_3^2\tau_3 = 0$$

This relation is identically satisfied if

$$\nu_1 = \pm\frac{1}{\sqrt{2}}, \nu_2 = \frac{1}{\sqrt{2}}, \nu_3 = 0 \qquad (7.3.9)$$

From equations (7.3.lb), (7.3.3a) and (7.3.9), the corresponding maximum value of $|S|$ is

$$|S|_{\max} = \frac{1}{2}|\tau_2 - \tau_1|$$

and
$$|N| = \frac{1}{2}|\tau_2 + \tau_1|$$

Also, for the direction

$$\nu_1 = 0, \nu_2 = \pm\frac{1}{\sqrt{2}}, \nu_3 = \pm\frac{1}{\sqrt{2}}$$

the corresponding values of and $|S|_{\max}$ and $|N|$ are , respectively ,

$$\frac{1}{2}|\tau_2 - \tau_1| \text{ and } \frac{1}{2}|\tau_2 + \tau_1|$$

The result can recorded in the following table

ν_1	ν_2	ν_3	$|S|_{\max/\min}$	$|N|$
0	0	± 1	Min S=0	$|\tau_3| = Max.$

0	±1	0	0(Min.)	$\lvert\tau_2\rvert = Max.$
±1	0	0	0(Min).	$\lvert\tau_1\rvert = Max.$
0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{2}\lvert\tau_2-\tau_3\rvert = Max.$	$\frac{1}{2}\lvert\tau_2+\tau_3\rvert = Min.$
$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{2}\lvert\tau_3-\tau_1\rvert = Max.$	$\frac{1}{2}\lvert\tau_3+\tau_1\rvert = Min.$
$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	$\frac{1}{2}\lvert\tau_1-\tau_2\rvert = Max.$	$\frac{1}{2}\lvert\tau_1+\tau_2\rvert = Min.$

If $\tau_1 > \tau_2 > \tau_3$, then τ_1 is the absolute maximum values of N and τ_3 is its minimum value, and the maximum value of $|S|$ is

$$|S|_{\max} = \frac{1}{2}|\tau_3 - \tau_1|$$

and the maximum shearing stress acts on the surface element containing the x_2 principal axis and bisecting the angle between the x_1 and x_3 axes. Hence the following theorem is proved.

Theorem: Show that the maximum shearing stress is equal to one half the differences between the greatest and least normal stress and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses.

7.4 MOHR'S CIRCLE OR MOHR’S DIAGRAM

(GEOMETRICAL PROOF OF THE THEOREM AS PROPOSED BY O. MOHR, 1882)

We know that

$$N = \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2 \qquad (7.4.1)$$

and

$$S^2 + N^2 = \tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2 \qquad (7.4.2)$$

Also $$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{7.4.3}$$

Solving equations (7.4.l) to (7.4.3), by Cramer's rule, for $\nu_1^2, \nu_2^2, \nu_3^2$ we find

$$\nu_1^2 = \frac{S^2 + (N-\tau_2)(N-\tau_3)}{(\tau_1-\tau_2)(\tau_1-\tau_3)} \tag{7.4.4}$$

$$\nu_2^2 = \frac{S^2 + (N-\tau_3)(N-\tau_1)}{(\tau_2-\tau_1)(\tau_2-\tau_3)} \tag{7.4.5}$$

$$\nu_3^2 = \frac{S^2 + (N-\tau_1)(N-\tau_2)}{(\tau_3-\tau_1)(\tau_3-\tau_2)} \tag{7.4.6}$$

Assume that $\tau_1 > \tau_2 > \tau_3$ so that $\tau_1 - \tau_2 > 0$ and $\tau_1 - \tau_3 > 0$. Since ν_1^2 is non - negative. We conclude from equation (7.4.4) that

$$S^2 + (N-\tau_1)(N-\tau_2) \geq 0$$

or $$S^2 + N^2 - N(\tau_2+\tau_3) + \tau_2\tau_3 \geq 0$$

$$S^2 + \left(N - \frac{(\tau_2+\tau_3)}{2}\right)^2 \geq \left(\frac{\tau_2-\tau_3}{2}\right)^2 \tag{7.4.7}$$

This represents a region outside the circle

$S^2 + \left(N - \frac{(\tau_2+\tau_3)}{2}\right)^2 = \left(\frac{\tau_2-\tau_3}{2}\right)^2$ in the (N, S) plane.

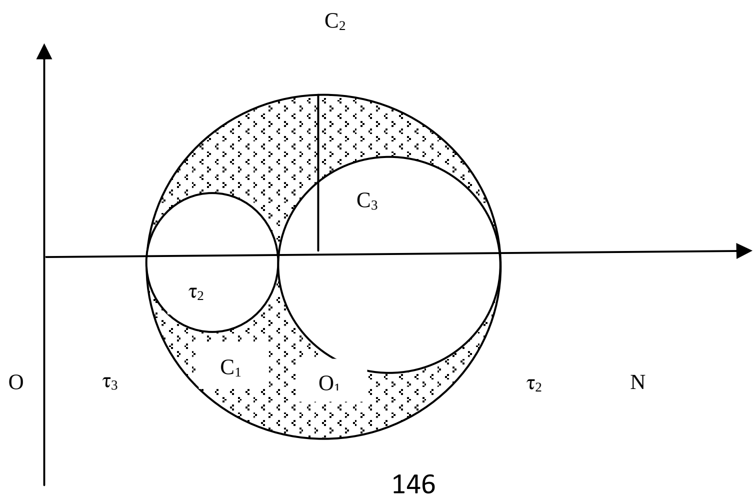


Figure (7.3) Mohr's Circles

This circle, say C_l, has centre $\left(\frac{\tau_1+\tau_3}{2},0\right)$ and radius $\left(\frac{\tau_2-\tau_3}{2}\right)$ in the Cartesian SN-plane with the values of N as abscissas and those of S as ordinates. Since $\tau_2-\tau_3>0$ and $\tau_2-\tau_1>0$, we conclude from (7.4.5) that

$$S^2+(N-\tau_3)(N-\tau_1)\leq 0 \qquad (7.4.8)$$

The region defined by (7.4.8) is a closed region, interior to the circle, whose equation is

$$S^2+(N-\tau_3)(N-\tau_1)=0 \qquad (7.4.8a)$$

The circle C_2 passed through the points (τ_3, 0), (τ_1, 0) have centre on the N — axis.

Finally, equation (7.4.6) yields

$$S^2+(N-\tau_1)(N-\tau_2)\geq 0 \qquad (7.4.9)$$

Since, $\tau_3-\tau_1<0$ and $\tau_3-\tau_2<0$. The region defined by (7.4.9) is exterior to the circle C_3, with centre on the N-axis and passing through the points (τ_1, 0), (τ_2, 0). It follows from inequalities (7.4.7) to (7.4.9) that the admissible values of S and N lie in the shaded region bounded by the circles as shown in the figure.

From figure, it is clear that the maximum value of shearing stress S is represented by the greatest ordinate $O'Q$ of the circle C_2.

Hence
$$S_{\max}=\frac{\tau_1-\tau_3}{2} \qquad (7.4.10a)$$

The value of N, corresponding to S_{max} is OO' where

$$OO'=\tau_3+\frac{\tau_1-\tau_3}{2}=\frac{\tau_1+\tau_3}{2} \qquad (7.4.10b)$$

Putting the values of S & N from equations (7.4.l0a, 7.4.10b) into equations (7.4.4) to (7.4.6) We find

$$\nu_1^2 = \nu_3^2 = \frac{1}{2}, \nu_2^2 = 0$$

or $$\nu_1 = \nu_3 = \pm\frac{1}{\sqrt{2}}, \nu_2 = 0 \qquad (7.4.11)$$

Equation (7.4.11) determines the direction of the maximum shearing stress and shows that the maximum shearing stress acts on the plane that bisects the directions of the largest and smallest principal stresses.

7.5 OCTAHEDERAL STRESSES

Consider a plane which is equally inclined to the principal directions of stress. Stresses acting on such a plane are known as octahedral stresses. Assume that coordinate axes coincide with the principal directions of stress, Let τ_1, τ_2, τ_3 be the principal stresses. Then the stress matrix is

$$\begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} \qquad (7.5.1)$$

A unit normal $\hat{\nu} = \nu_i$ to this plane is

$$\nu_1 = \nu_2, = \nu_3 = \frac{1}{\sqrt{3}}$$

Then the stress vector $\overset{\nu}{\underset{\sim}{T}}$ on a plane clement with normal $\hat{\nu}$ is given by

$$\overset{\nu}{T}_i = \tau_{ij}\nu_j$$

This gives

$$\overset{\nu}{T}_1 = \tau_1\nu_1 \ , \ \overset{\nu}{T}_2 = \tau_2\nu_2, \ \overset{\nu}{T}_3 = \tau_3\nu_3$$

Let N be the normal stress and S be the shear stress. Then

$$N = \overset{\nu}{\underset{\sim}{T}} . \hat{\nu} = \tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2 = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3), \; S^2 = \left|\overset{\nu}{\underset{\sim}{T}}\right| - N^2$$

$$= (\tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2) - \frac{1}{9}(\tau_1 + \tau_2 + \tau_3)^2$$

$$= \frac{1}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2) - \frac{1}{9}(\tau_1 + \tau_2 + \tau_3)^2$$

$$= \frac{1}{9}[3(\tau_1^2 + \tau_2^2 + \tau_3^2) - (\tau_1^2 + \tau_2^2 + \tau_3^2 + 2\tau_1\tau_2 + 2\tau_2\tau_3 + 2\tau_1\tau_3)]$$

$$= \frac{1}{9}[(\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2) + (\tau_2^2 + \tau_3^2 - 2\tau_2\tau_3) + (\tau_3^2 + \tau_1^2 - 2\tau_3\tau_1)]$$

$$= \frac{1}{9}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2] \qquad (7.5.2)$$

giving

$$S = \frac{1}{3}\sqrt{(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2}$$

Example: At a point P, the principal stresses are $\tau_1 = 4, \tau_2 = 1, \tau_3 = -2$. Find the stress vector, the normal stress and the shear stress on the octahedral plane at P.

[Hint: $N = 1, S = \sqrt{6}, \overset{\nu}{\underset{\sim}{T}} = \frac{1}{\sqrt{3}}(4\hat{e}_1 + \hat{e}_2 - 2\hat{e}_3)$]

7.6. STRESS DEVIATOR TENSOR

Let τ_{ij} be the stress tensor. Let

$$\sigma_0 = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)$$

Then the tensor

$$\tau_{ij}^{(d)} = \tau_{ij} - \sigma_0 \delta_{ij})$$

is called the stress deviator tensor. It specifies the deviation of the state of stress from the mean stress $\sigma_{0.}$

Books Recommended:

1. **Sokolnikoff, I. S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

2. **S. Timoshenko and N. Goodier,** Theory of Elasticity, McGraw Hill, New York, 1970.

CHAPTER-VIII

EQUATIONS OF ELASTICITY:GENERALIZED HOOKE'S LAW

8.1 INTRODUCTION

An ordinary solid body is constantly subjected to forces of gravitation, and, if it is in equilibrium, it is supported by other forces. We have no experience of a body which is free from the action of all external forces. From the equations of motion we know that the application of forces to a body necessitates the existence of stress within the body. Again, solid bodies are not absolutely rigid. By the application of suitable forces they can be made to change both in size and shape. When the induced changes of size and shape are considerable, the body does not, in general, return to its original size and shape after the forces which induced the change have ceased to act. On the other hand, when the changes are not too great the recovery may be apparently complete. The property of recovery of an original size and shape is the property that is termed ***elasticity***. The changes of size and shape are expressed by specifying ***strains***. The "unstrained state" with reference to which strains are specified, is as it were, an arbitrary zero of reckoning, and the choice of it is in our power. When the unstrained state is chosen, and the strain is specified, the internal configuration of the body is known.

We shall suppose that the differential coefficients of the displacement (u, v, w) by which the body could pass from the unstrained state to the strained state, are sufficiently small to admit of the calculation of the strain by the simplified method and we shall regard the configuration as specified by this displacement. The object of experimental investigations of the behavior of elastic bodies may be said to be the discovery of numerical relations between the quantities that can be measured, which shall be sufficiently varied and sufficiently numerous to serve as a basis for the inductive determination of the form of the intrinsic energy function. When such a function exists, and its form is known, we can deduce from it the relations between the components of stress and the components of strain and conversely, if, from any experimental results, we are able to infer such relations, we acquire thereby data which can serve for the construction of the function.

The components of stress or of strain within a solid body can never from the nature of the case be measured directly. If their values can be found it must always be by a process of inference from measurements of quantities that are not, in general components of stress or of strain. Instruments can be devised for measuring average strains in bodies of ordinary size, and others for measuring particular strains of small superficial parts. For example, the average cubical compression can be measured by means of a piezometer; the extension of a short length of a longitudinal filament on the outside of a bar can be measured by means of an extensometer. Sometimes, as for example in experiments on torsion and flexure, a displacement is measured. External forces applied to a body can often be measured with great exactness, e.g. when a bar is extended or bent by hanging a weight at one end. In such cases it is a resultant force that is measured directly, not the component tractions per unit of area that are applied to the surface of the body. In the case of a body under normal pressure, as in the experiments with the piezometer, the pressure per unit of area can be measured.

In any experiment designed to determine a relation between stress and strain, some displacement is brought about, in a body partially fixed, by the application of definite forces which can be varied in amount. We call these forces collectively "the load". It is a fact of experience that deformation of a solid body induces stresses within. The relationship between stress and deformation is expressed as a constitutive relation for the material and depends on the material properties and also on other physical observables like temperative and perhaps the electromagnetic field. An elastic deformation is defined to be one in which the stress is determined by the current value of the strain only, and not on rate of strain or strain history: $\tau = \tau(e)$.

An elastic solid that undergoes only an **infinitesimal** deformation and for which the governing material is **linear** is called a **linear elastic solid** or **Hooken**

solid. From experimental observations, it is known that under normal loading many structural materials such as metals, concrete, wood and rocks behave as linear elastic solids. The classical theory of elasticity (or linear theory) serves as an excellent model for studying the mechanical behavior of a wide variety of such solid materials.

8.2. Hook's Law

The first attempt at a scientific description of the strength of solids was made by Galileo. He treated bodies as inextensible, however, since at that time there existed neither experimental data nor physical hypotheses that would yield a relation between the deformation of a solid body and the forces responsible for the deformation. It was Robert Hooke who, some forty years after the appearance of Galileo's Discourses (1638), gave the first rough law of proportionality between the forces and displacements. Hooks published his law first in the form of an anagram "ceiinosssttuu" in (1676), and two years later gave the solution of the anagram: " ut tension sic vis," which can be translated freely as " the extension is proportional to the force."

Most hard solid show that same type of relation between load and measurable strain. It is found that, over a wide range of load, the measured strain is proportional to the load. This statement may be expressed more fully by saying that

1) When the load increases the measured strain increases in the same ratio,
2) When the load diminishes the measured strain diminishes in the same ratio,
3) When the load is reduced to zero no strain can be measured.

The most striking exception to this statement is found in the behavior of cast metals. It appears to be impossible to assign any finite range of load, within which the measurable strains of such metals increase and diminish in the sameproportion as the load.The experimental results which hold for most hard solids, other than cast metals. It appears to be impossible to assign any finite range of load, within which the measurable strains of such metals increase and diminish in the same proportion as the load.

The experimental results which hold for most hard solids, other than cast metals, lead by a process of inductive reasoning to the Generalized Hooke's Law of the proportionality of stress and strain. The general form of the law is expressed by the statement: ***Each of the six components of stress at any point of a body are linear functions of the six components of strain at the point***.

In 1678, Robert Hook, on experimental grounds, stated that the extension is proportional to the force. Cauchy in 1822 generalized Hook law for the deformation of elastic solids. According to Cauchy, "Each component of stress at any point of an elastic body is a linear function of the components of strain at the point".

In general, we write the following set of linear relations

$$\tau_{11} = c_{1111}e_{11} + c_{1112}e_{12} + \ldots\ldots\ldots + c_{1133}e_{33}$$

$$\tau_{12} = c_{1211}e_{11} + c_{1212}e_{12} + \ldots\ldots\ldots + c_{1233}e_{33}$$

……………………………………………..

……………………………………………..

$$\tau_{33} = c_{3311}e_{11} + c_{3312}e_{12} + \ldots\ldots\ldots + c_{3333}e_{33}$$

Or

$$\tau_{ij} = c_{ijkl}e_{kl},\ i, j, k, l = 1, 2, 3 \quad (8.2.1)$$

where τ_{ij} is the stress tensor and e_{kl} is the strain tensor. The coefficients, which are $81=3^4$ in number, are called elastic moduli. In general, these coefficients depend on the physical properties of the medium and are independent of the strain components e_{ij}. We suppose that relations (8.2.1) hold at every point of the medium and at every instant of time and are solvable for e_{ij} in terms of τ_{ij}. From (8.2.1), it follows that τ_{ij} are all zero whenever all e_{ij} are zero. It means that in the initial unstrained state the body is unstressed. From quotient law for tensors, relation (8.2.1) shows that c_{ijkl} are components of a fourth order tensor. This tensor is called ***elasticity tensor***. Since e_{ij} are dimensionless quantities, it follows that elastic moduli c_{ijkl} have the same dimensions as the stresses (force/Area). If, however, c_{ijkl} do not change throughout the medium for all time, we say that the **medium is (elastically) homogeneous.**Thus, for a homogeneous elastic solid, the elastic moduli are constants so that the mechanical properties remain the same throughout the solid for all times. The equation (8.2.1) represents the generalized Hooke's law in the x_i–system. These coefficients c_{ijkl} very from point to point of the medium and are called elastic constants. If c_{ijkl} are independent of position of point then the medium is called elastic homogeneous. These are 81 in numbers now we shall discussed onwards only those media which are homogeneous continuous and elastic. Also, components τ_{ij} are symmetric,

i.e., $\tau_{ij} = \tau_{ji}$ (8.2.2)

on interchanging the indices i and j in the formula will not change so that $c_{ijkl} = c_{jikl}$. Now let we denotes c_{ijkl} is also symmetric with respect to the last two indices k and l for this let we define

$$\tau_{ij} = \left\{\frac{1}{2}(c_{ijkl} + c_{ijlk}) + \frac{1}{2}(c_{ojkl} - c_{ijlk})\right\}e_{kl} \quad (8.2.3)$$

Let $$c'_{ijkl} = \frac{1}{2}\{c_{ijkl} + c_{ijlk}\},\ c''_{ijkl} = \frac{1}{2}\{c_{ijkl} - c_{ijlk}\}$$

$$\therefore \quad \tau_{ij} = \{(c'_{ijkl} + c''_{ijkl}\}e_{kl} \quad (8.2.4)$$

Now $$c''_{ijkl} = \frac{1}{2}\{c_{ijkl} - c_{ijlk}\} = -\frac{1}{2}\{c_{ijlk} - c_{ijkl}\} = -c''_{ijkl}$$

$$\Rightarrow \quad \tau_{ij} = \{(c'_{ijkl} + c''_{ijkl}\}e_{kl} \quad (8.2.5)$$

adding (8.2.4) and (8.2.5) we get

$$\tau_{ij} = \{(c'_{ijkl}\}e_{kl} \qquad (\because \quad i, j = 1,2,3)$$

where $c'_{ijkl} = c_{ijkl}$ then

$$\tau_{ij} = c_{ijkl}e_{kl} \quad (8.2.6)$$

where c_{ijkl} is symmetric with respect to first two indices and also with respect to last two indices. With the help of this symmetric property, the 81 constants in equations (8.2.6) are reduced into 45 constants. (out of these 81 constants 36 constants are decreased due to symmetric property of the constants). Introducing the notations (known as engineering notations)

$$\left.\begin{array}{l} \tau_{11} = \tau_1, \tau_{22} = \tau_2, \tau_{33} = \tau_3, \tau_{23} = \tau_4, \tau_{13} = \tau_5, \tau_{12} = \tau_6 \\ e_{11} = e_1,\ e_{22} = e_2,\ e_{33} = e_3, 2e_{23} = e_4, 2e_{13} = e_5, 2e_{12} = e_6 \end{array}\right\} (8.2.7)$$

Using the above into (8.2.1), the six equations becomes

$$\left.\begin{array}{l} \tau_1 = c_{11}e_1 + c_{12}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{16}e_6 \\ \tau_2 = c_{21}e_1 + c_{22}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{26}e_6 \\ \tau_3 = c_{31}e_1 + c_{32}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{36}e_6 \\ \tau_4 = c_{41}e_1 + c_{42}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{46}e_6 \\ \tau_5 = c_{51}e_1 + c_{52}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{56}e_6 \\ \tau_6 = c_{61}e_1 + c_{62}e_2 + \ldots\ldots\ldots\ldots\ldots\ldots + c_{66}e_6 \end{array}\right\} \qquad (8.2.8)$$

The equation (8.2.8) in tensor form can be given below:

$$\tau_i = c_{ij}e_j \ (i, j = 1, 2,\ 3, 4, 5, 6) \qquad (8.2.9)$$

For unique solution of equation (8.2.9), we must have $|c_{ij}| \neq 0$ then e_i can be expressed as

$$e_i = C_{ij}\tau_j \ (C \neq c \quad \& \quad i, j = 1, 2,\ 3, 4, 5, 6)\ (8.2.10)$$

Therefore 36 elastic constant are required to study the properties of elastic continuous medium. But the numbers of constants reduce to 21 in number,whenever there exists a function

$$W = \frac{1}{2} c_{ij} e_i e_j \ (i, j = 1, 2, 3, 4, 5, 6)$$

such that
$$\frac{\partial W}{\partial e_i} = \tau_i \tag{8.2.11}$$

This potential function W was first introduced by Green and W is called the strain energy density function and it exists when the process of deformation is Isothermal and Adiabatic. Also $W = \frac{1}{2} c_{ij} e_i e_j$ and taking its partial derivative with respect to e_k we get,

$$\frac{\partial W}{\partial e_k} = \frac{1}{2} c_{ij} \left(\frac{\partial e_i}{\partial e_k} e_j + e_i \frac{\partial e_j}{\partial e_k} \right) \tag{8.2.12}$$

Now
$$\frac{\partial e_i}{\partial e_k} = \delta_{ik} = \begin{cases} 0 & if\ i \neq k \\ 1 & if\ i = k \end{cases}$$

$\Rightarrow$
$$\frac{\partial W}{\partial e_k} = \frac{1}{2} c_{ij} \left(\delta_{ik} e_j + e_i \delta_{jk} \right) = \frac{1}{2} c_{ij} \delta_{ik} e_j + \frac{1}{2} c_{ij} \delta_{jk} e_i$$

but
$$\frac{\partial W}{\partial e_k} = \tau_k = \frac{1}{2} c_{kj} e_j + \frac{1}{2} e_j c_{jk} = \frac{1}{2} (c_{jk} + c_{kj}) e_j \tag{8.2.13}$$

now by the Hook's law

$$\tau_i = c_{ij} e_j \ (i, j = 1, 2, 3, 4, 5, 6)$$

$\Rightarrow$ or $\tau_k = c_{kj} e_j \ (k, j = 1, 2, 3, 4, 5, 6)$ (8.2.14)

using (8.2.13) and (8.2.14) we get

$$(\frac{1}{2} c_{jk} + \frac{1}{2} c_{kj}) e_j = c_{kj} e_j$$

$\Rightarrow$ $(c_{kj} + c_{jk}) = 2c_{ij}$ or $c_{jk} = c_{kj}$

Thus c_{jk} are symmetric.So we have, if process is isothermal or adiabatically, then $c_{ij} = c_{ji}$. Now our formula (8.2.8) in which elastic constants are 36 in number can be written in matrix form is

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \qquad (8.2.15)$$

So due to symmetric properties, these constants further reduce to 21. If the media is elastically symmetric in certain direction then the numbers of elastic constants c_{ij} are further reduced. We shall discuss two types of elastic symmetry

8.3 Case-1: Symmetry with respect to plane- Consider that medium is elastically symmetric with respect to the x_1x_2 $-$plane

$$\Rightarrow \qquad x_1 = x_1', \quad x_2 = x_2', \; x_3 = -x_3' \qquad (8.3.1)$$

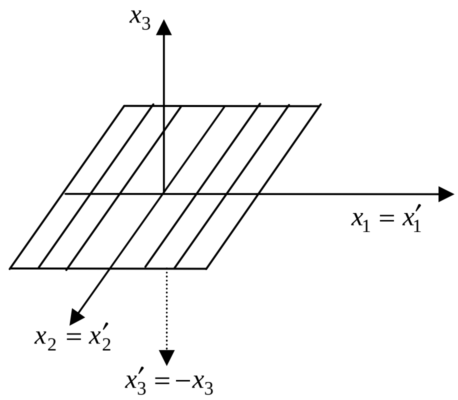


Figure 8.3.1

The elastic constants c_{ij} are invariant under the transformation, now we know that the law of transformation of tensor of order one is

$$x'_p = \ell_{pi} x_i \quad (\because \ell_{pi} = Cos(x'_p, x_i)) \tag{8.3.2}$$

where
$$\ell_{pi} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{8.3.3}$$

we know Hook's law is

$$\tau'_i = c_{ij} e'_j \text{ and } \tau_i = c_{ij} e_j \ (i, j = 1, 2, 3, 4, 5, 6)$$

$\Rightarrow$
$$\tau'_1 = c_{11}e'_1 + c_{12}e'_2 + \dots + c_{16}e'_6$$
$$\tau'_2 = c_{21}e'_1 + c_{22}e'_2 + \dots + c_{26}e'_6$$
$$\tau'_3 = c_{31}e'_1 + c_{32}e'_2 + \dots + c_{36}e'_6$$
$$\tau'_4 = c_{41}e'_1 + c_{42}e'_2 + \dots + c_{46}e'_6$$
$$\tau'_5 = c_{51}e'_1 + c_{52}e'_2 + \dots + c_{56}e'_6$$
$$\tau'_6 = c_{61}e'_1 + c_{62}e'_2 + \dots + c_{66}e'_6 \tag{8.3.4}$$

and

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + \dots + c_{16}e_6$$
$$\tau_2 = c_{21}e_1 + c_{22}e_2 + \dots + c_{26}e_6$$
$$\tau_3 = c_{31}e_1 + c_{32}e_2 + \dots + c_{36}e_6$$
$$\tau_4 = c_{41}e_1 + c_{42}e_2 + \dots + c_{46}e_6$$
$$\tau_5 = c_{51}e_1 + c_{52}e_2 + \dots + c_{56}e_6$$
$$\tau_6 = c_{61}e_1 + c_{62}e_2 + \dots + c_{66}e_6 \tag{8.3.5}$$

Law of transformation of tensor of order two is as given below

$$\tau'_{pq} = \ell_{pi}\ell_{qj}\tau_{ij} \qquad (8.3.6)$$

$$\begin{aligned}\tau'_{11} = \ell_{1i}\ell_{1j}\tau_{ij} &= \ell_{1i}\left(\ell_{11}\tau_{i1} + \ell_{12}\tau_{i2} + \ell_{13}\tau_{i3}\right) \\ &= \ell_{11}(\ell_{11}\tau_{11} + \ell_{12}\tau_{12} + \ell_{13}\tau_{13}) \\ &\quad + \ell_{12}(\ell_{11}\tau_{21} + \ell_{12}\tau_{22} + \ell_{13}\tau_{23}) \\ &\quad + \ell_{13}(\ell_{11}\tau_{31} + \ell_{12}\tau_{32} + \ell_{13}\tau_{33})\end{aligned}$$

using (8.3.3), we get $\tau'_{11} = 1(1\tau_{11} + 0\tau_{12} + 0\tau_{13}) + 0 + 0 = \tau_{11}$

$$\Rightarrow \qquad \tau'_{11} = \tau_{11} \text{ or } \tau'_1 = \tau_1 \qquad (8.3.7)$$

Similarly, $\tau'_2 = \tau_2, \tau'_3 = \tau_3, \tau'_6 = \tau_6, \tau'_4 = -\tau_4$ and $\tau'_5 = -\tau_5$;

$$e'_1 = e_1, e'_2 = e_2, e'_3 = e_3, e'_6 = e_6, e'_4 = -e_4 \text{ and } e'_5 = -e_5 \qquad (8.3.8)$$

from relations (8.3.5),(8.3.6),(8.3.7)and (8.3.8), we get

$$\begin{gathered}\tau'_1 = \tau_1 \\ c_{11}e'_1 + c_{12}e'_2 + c_{13}e'_3 + c_{14}e'_4 + c_{15}e'_5 + c_{16}e'_6 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{14}e_4 + c_{15}e_5 + c_{16}e_6 \\ c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{14}e_4 - c_{15}e_5 + c_{16}e_6 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{14}e_4 + c_{15}e_5 + c_{16}e_6 \\ or \quad 2(c_{14}e_4 + c_{15}e_5) = 0\end{gathered} \qquad (8.3.9)$$

$$\Rightarrow \qquad c_{14} = c_{15} = 0 \qquad (8.3.10)$$

Similarly

$$\begin{aligned}&\tau'_2 = \tau_2 \Rightarrow c_{24} = c_{25} = 0, \\ &\tau'_3 = \tau_3 \Rightarrow c_{34} = c_{35} = 0, \\ &\tau'_4 = -\tau_4 \Rightarrow c_{41} = c_{42} = c_{43} = c_{46} = 0, \\ &\tau'_5 = -\tau_5 \Rightarrow c_{51} = c_{52} = c_{53} = c_{56} = 0, \\ &\tau'_6 = \tau_6 \Rightarrow c_{64} = c_{65} = 0,\end{aligned} \qquad (8.3.11)$$

Then elastic constants reduces to 13, so matrix of coefficients is

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\\\tau_4\\\tau_5\\\tau_6\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&c_{16}\\c_{21}&c_{22}&c_{23}&0&0&c_{26}\\c_{31}&c_{32}&c_{33}&0&0&c_{36}\\0&0&0&c_{44}&c_{45}&0\\0&0&0&c_{54}&c_{55}&0\\c_{61}&c_{62}&c_{63}&0&0&c_{66}\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\\e_4\\e_5\\e_6\end{bmatrix} \qquad (8.3.12)$$

8.4 Case-II: Let us consider symmetrywith respect to another plane is- consider that medium is elastically symmetric with respect to the x_2x_3-plane

$$\Rightarrow \qquad x_1=-x_1', \quad x_2=x_2',\ x_3=x_3' \ (8.4.1)$$

again applying same transformation law as earlier, we get

$$c_{16}=c_{26}=c_{36}=c_{45}=c_{51}=c_{61}=c_{62}=c_{63}=0 \qquad (8.4.2)$$

Such materials which have three mutually orthogonal planes of symmetry are called **orthotropic.T**hus for orthotropic media matrix for c_{ij} takes the following form

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\\\tau_4\\\tau_5\\\tau_6\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\c_{21}&c_{22}&c_{23}&0&0&0\\c_{31}&c_{32}&c_{33}&0&0&0\\0&0&0&c_{44}&0&0\\0&0&0&0&c_{55}&0\\0&0&0&0&0&c_{66}\end{bmatrix}\begin{bmatrix}e_1\\e_2\\e_3\\e_4\\e_5\\e_6\end{bmatrix} \qquad (8.4.3)$$

From the relation (8.4.3) there are **nine**constants required to study the elastic property of the material.

Definition:-**Orthotropic Material:** A material is said to be orthotropic if it has three mutually orthogonal plane of elastic symmetry for example, wood, is a common example of an orthotropic material.

8.5 Case-III: Transversely Isotropic Media: If an orthotropic medium exists elastic symmetry with respect to arbitrary rotation about one of the axis, say x_3 − axis. Then it is called transversally isotropic. Let the system $ox_1'x_2'x_3'$ be obtains from the system $ox_1x_2x_3$ by a rotation about the x_3 − axis to an angle θ then direction cosine are given by

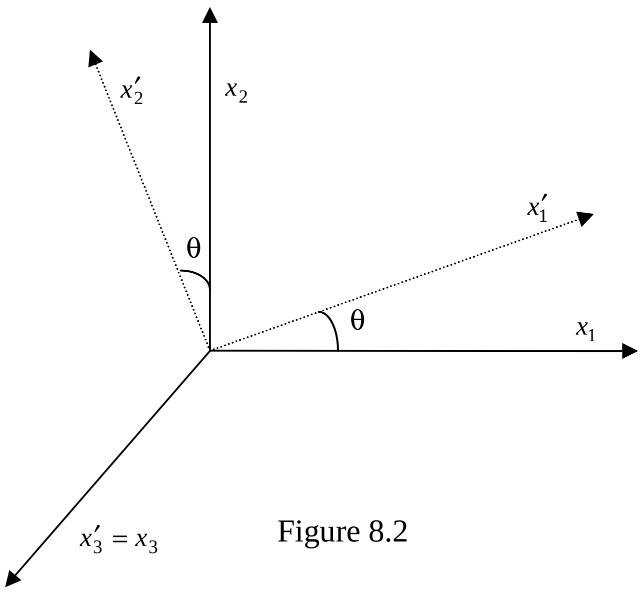


Figure 8.2

Law of transformation of tensor of order two is

$$\tau'_{pq} = \ell_{pi}\ell_{qj}\tau_{ij} \text{ where } \ell_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (8.5.1)$$

Then c_{ij} must be invariant under this rotation, using relation (8.5.1) we have

$$\tau'_{11} = \tau'_1 = \tau_1\cos^2\theta + \tau_2\sin^2\theta + \tau_6\cos\theta\sin\theta$$

$$\tau'_{22} = \tau'_2 = \tau_1\sin^2\theta + \tau_2\cos^2\theta - \tau_6\cos\theta\sin\theta$$

$$\tau'_{33} = \tau'_3 = \tau_3\,;\ \tau'_{23} = \tau'_4 = \tau_4 \cos\theta - \tau_5 \sin\theta\,;$$

$$\tau'_{13} = \tau'_5 = \tau_4 \sin\theta + \tau_5 \cos\theta$$

$$\tau'_{12} = \tau'_6 = -(\tau_1 - \tau_2)\cos\theta \sin\theta + \tau_6 (\cos^2\theta - \sin^2\theta) \qquad (8.5.2)$$

and

$$e'_{11} = e'_1 = e_1 \cos^2\theta + e_2 \sin^2\theta + e_6 \cos\theta \sin\theta$$

$$e'_{22} = e'_2 = e_1 \sin^2\theta + e_2 \cos^2\theta - e_6 \cos\theta \sin\theta$$

$$e'_{33} = e'_3 = \tau_3\,;\ \ e'_{23} = e'_4 = e_4 \cos\theta - e_5 \sin\theta\,;$$

$$e'_{13} = e_5 = e_4 \sin\theta + e_5 \cos\theta,\ 2e'_{23} = e'_4\,, 2e'_{13} = e'_5\,, 2e'_{12} = e'_6$$

$$e'_{12} = e'_6 = -(e_1 - e_2)\cos\theta \sin\theta + e_6 (\cos^2\theta - \sin^2\theta) \qquad (8.5.3)$$

For all possible value of θ.

Sub case:- (i) If we take $\theta = \dfrac{\pi}{2}$ then the relations (8.5.2) and (8.5.3) becomes

$$\left.\begin{aligned} \tau'_1 &= \tau_2 \\ \tau'_2 &= \tau_1 \\ \tau'_3 &= \tau_3 \\ \tau'_4 &= -\tau_5 \\ \tau'_5 &= \tau_4 \\ \tau'_6 &= -\tau_6 \end{aligned}\right\} \text{and} \quad \left.\begin{aligned} e'_1 &= e_2 \\ e'_2 &= e_1 \\ e'_3 &= e_3 \\ e'_4 &= -e_5 \\ e'_5 &= e_4 \\ e'_6 &= -e_6 \end{aligned}\right\} \qquad (8.5.4)$$

using the relation (8.4.3) and (8.5.4) we have

$$\tau'_1 = \tau_2$$
$$c_{11}e'_1 + c_{12}e'_2 + c_{13}e'_3 = c_{21}e_1 + c_{22}e_2 + c_{23}e_3$$
$$c_{11}e_2 + c_{12}e_1 + c_{13}e_3 = c_{21}e_1 + c_{22}e_2 + c_{23}e_3$$
$$(c_{21} - c_{12})e_1 + (c_{11} - c_{22})e_2 + (c_{13} - c_{23})e_3 = 0$$

$$\Rightarrow\ c_{12} = c_{21},\ c_{11} = c_{22},\ c_{13} = c_{23}$$

Similarly, by comparison of other relations of (8.5.4), we get the constants as follows

$$c_{12} = c_{21}, c_{22} = c_{11}, c_{23} = c_{13}, c_{44} = c_{55}, c_{32} = c_{31} \tag{8.5.5}$$

Sub case:- (ii) If we take $\theta = \dfrac{\pi}{4}$ then the relations (8.5.2) and (8.5.3) gives a new relation between that stresses as follows

$$\begin{aligned} \tau'_{12} &= \frac{1}{2}(\tau_{22} - \tau_{11}) \\ \Rightarrow \quad \tau'_6 &= \frac{1}{2}(\tau_2 - \tau_1) \end{aligned} \quad \text{and } e'_6 = (e_2 - e_1) \tag{8.5.6}$$

after comparing the coefficient on both sides of (8.5.6) we get

$$c_{66} = \frac{1}{2}(c_{22} - c_{12}) = \frac{1}{2}(c_{11} - c_{12}) \tag{8.5.7}$$

Thus, the matrix of elastic moduli given in relation (8.4.3) becomes

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix} \tag{8.5.8}$$

This matrix has **five**independent elastic constants.

8.6 Case-IV: Homogenous Isotropic Medium: In the case of an isotropic material the elastic coefficients c_{ij} are independent of orientation of coordinate axes.In particular, every plane is the plane of isotropic elastic symmetry and is a particular case of a transversely isotropic elastic symmetry in addition to elastic symmetric about the x_3-axis. Let there is an elastic symmetry about the x_1-axis, i.e. a rotation of axis through a right angle about the x_1-axis is given by the transformation.

$$x_1 = x_1', \quad x_3 = x_2', \; x_2 = -x_3' \qquad (8.6.1)$$

this transformation leads to the relation

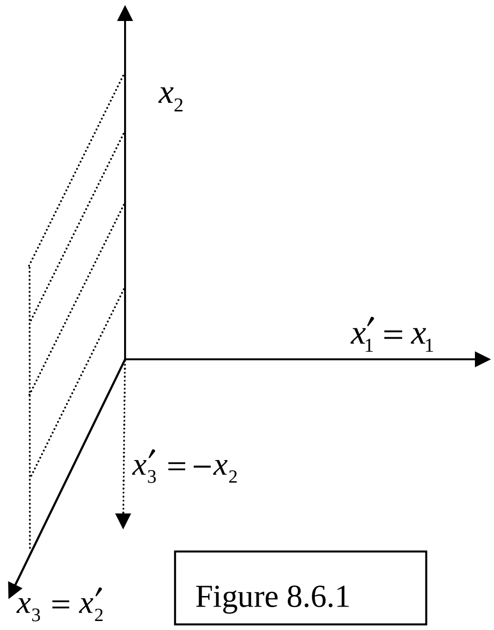


Figure 8.6.1

$$c_{12} = c_{13}, c_{33} = c_{11}, c_{66} = c_{44}, c_{31} = c_{13} \qquad (8.6.2)$$

If we define $\quad c_{12} = c_{13} = \lambda, c_{66} = c_{44} = \mu \qquad (8.6.3)$

then $\quad c_{11} = c_{22} = c_{33} = \lambda + 2\mu,$

Therefore the number of independent elastic coefficients for an isotropic medium are **two,**i.e., λ and μ, these coefficients are known as Lame's constants. Thus the generalized Hooke's law becomes

$$\begin{bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}e_{11}\\ e_{22}\\ e_{33}\\ 2e_{23}\\ 2e_{13}\\ 2e_{12}\end{bmatrix} \quad (8.6.4)$$

This can be written as

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \quad (8.6.5)$$

The results in relation (8.6.5) are known as stress-strain relation. Putting $i = j$, we find

$$\tau_{ii} = 3\lambda e_{kk} + 2\mu e_{ii}$$

$$\tau_{11}+\tau_{22}+\tau_{33} = 3\lambda(e_{11}+e_{22}+e_{33}) + 2\mu(e_{11}+e_{22}+e_{33})$$

$$\Theta = (3\lambda+2\mu)\vartheta \quad (8.6.6)$$

from relation (8.6.5)

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu}\delta_{ij}e_{kk} \quad (8.6.7)$$

using (8.6.6) into (8.6.7) we get

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu}\frac{1}{(3\lambda+2\mu)}\delta_{ij}\tau_{kk}$$

$$\Rightarrow \qquad e_{ij} = -\frac{\lambda}{2\mu}\frac{1}{(3\lambda+2\mu)}\delta_{ij}\tau_{kk} + \frac{1}{2\mu}\tau_{ij} \quad (8.6.8)$$

This is possible when $\mu \neq 0$ and $3\lambda+2\mu \neq 0$. So this relation express as strains as a linear combination of stress components.

8.7. The generalized Hooke’s law for anisotropic linear elastic medium.

Solution: Let e_{kl} and τ_{ij} be the components of strain and stress tensors, respectively. According to the generalized Hooke's law for an elastic media

$$\tau_{ij} = c_{ijkl} e_{kl} \tag{8.7.1}$$

where c_{ijkl} is a tensor of order four since the media is isotropic therefore the tensor c_{ijkl} is an isotropic tensor. Hence c_{ijkl} can be represented in the form

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \tag{8.7.2}$$

where α , β and γ are scalars from (8.7.1) and (8.7.2) we obtains

$$\tau_{ij} = \alpha\delta_{ij}(\delta_{kl}e_{kl}) + \beta\delta_{ik}(\delta_{jl}e_{kl}) + \gamma\delta_{il}(\delta_{jk}e_{kl})$$

$$\tau_{ij} = \alpha\delta_{ij}e_{kk} + \beta\delta_{ik}e_{jk} + \gamma\delta_{il}e_{jl}$$

$$\tau_{ij} = \alpha\delta_{ij}e_{kk} + \beta e_{ij} + \gamma e_{ji}$$

$$\tau_{ij} = \alpha\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{8.7.3}$$

where $\alpha = \lambda$ and $2\mu = \beta + \gamma$

Hence, $\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}$ is known as Hooke's law for **anisotropic** linear elastic medium.

Question: Show that if the medium is isotropic, the principal axes of stress are coincident with the principal axes of strain.

Solution: Let the $x_i -$axes be directed along the principal axes of strain. Then

$$e_{12} = e_{13} = e_{23} = 0 \tag{8.7.4}$$

The stress-strain relations for an isotropic medium are

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{8.7.5}$$

Combining (8.7.4) and (8.7.5), we find

$$\tau_{12} = \tau_{13} = \tau_{23} = 0 \tag{8.7.6}$$

This shows that the coordinates axes x_i are also the principal axes of stress. This proves the result. Thus, there is no distinction between the principal axes of stress and of strain for isotropic media.

Books Recommended:

1. **Y.C. Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey, 1965.
2. **Sokolnikoff, I.S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977

CHAPTER-IX

ELASTIC MODULI FOR ISOTROPIC MEDIA

9.1 INTRODUCTION

To study the statement "**the extension is proportional to the force**", we discuss the deformation of a thin rod subjected to a tensile stress. Consider a thin rod (of a low-carbon steel, for example), of initial cross sectional area a, which is subjected to a variable tensile force F. If the stress is assumed to be distributed uniformly over the area of the cross section, then the *nominal stress* $T = F/a$ can be calculated for any applied load F. The actual stress is obtained, under the assumption of a uniform stress distribution, by dividing the load at any stage of the test by the actual area of the cross section of the rod at that stage. The difference between the nominal and the actual stress is negligible, however, through-out the elastic range of the material.

If the nominal stress T is plotted as a function of the extension e (change in length per unit length of the specimen), then for some ductile metals a graph is very nearly a straight line with the equation

$$T = Ee \tag{9.1.1}$$

until the stress reached the proportional limit. The position of this point, however, depends on a considerable extent upon the sensitivity of the testing apparatus. The constant of proportionality E is known as Young's modulus. In most metals, especially in soft and ductile materials, careful observation will reveal very small permanent elongations which are the results of very small tensile forces. In many metals, however (for example, steel and wrought iron), if these very small permanent elongations are neglected (less that $1/100{,}000$ of the length of a bar under tension), then the graph of stress against extension is a straight line, as noted above, and practically all the deformation disappear after the force has been removed. The greatest stress that can be applied without producing a permanent deformation is called the elastic limit of the material. When the applied force is increased beyond this fairly sharply defined limit, the material exhibits both elastic and plastic properties. The determination of this limit requires successive loading and unloading by ever larger forces until a permanent set is recorded. For many materials the proportional limit is very nearly equal to the elastic limit, and the distinction between the two is sometimes dropped, particularly since the former is more easily obtained. When the stress increases beyond the elastic limit, a point is reached

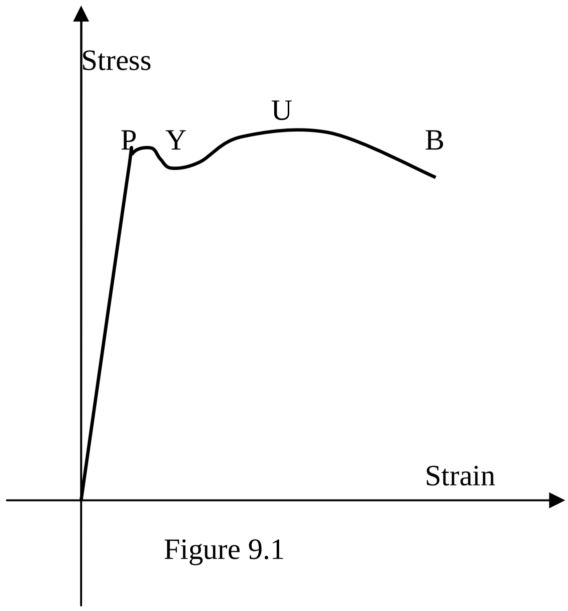


Figure 9.1

(Y on the graph) at which the rod suddenly stretches with little or no increase in the load. The stress at point Y is called the yield-point stress.

The nominal stress T may be increased beyond the yield point until the ultimate (point U) is reached. The corresponding force $F = Ta$ is the greatest load that the rod will bear. When the ultimate stress is reached, a brittle material (such as highcarbon steel) breaks suddenly, while a rod of some ductile metal begins to "neck"; that is, its cross sectional area is greatly reduced over a small portion of the length of the rod. Further elongation is accompanied by an increase in actual stress but by a decrease in total load, in cross-sectional area, and in nominal stress until the rod breaks (point B).

We shall consider only the behavior of elastic materials subjected to stresses below the proportional limit; that is, we shall be concerned only with those materials and situations in which Hooke's law, expressed by or a generalization of it, is valid.

9.2 PHYSICAL MEANINGS OF ELASTIC MODULI

We have already introduced two elastic moduli λ and μ in the generalized Hooke's law for an isotropic medium. In order to gain some insight into the physical significance of elastic constants entering in generalized Hooke's law, we consider the behavior of elastic bodies subjected to simple tension, pure shear and hydrostatic pressure.

Sub Case-I:- Simple Tension

Consider a right cylinder with its axis parallel to the x_1 – axis which is subjected to longitudinal forces applied to the ends of the cylinder. These applied

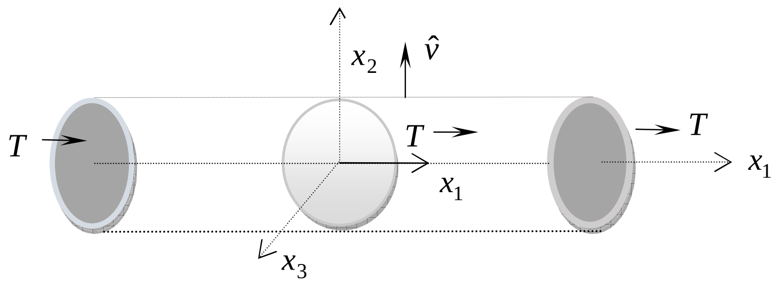


Figure 9.2

forces give rise to a uniform tension T in every cross-section of the cylinder so that the stress tensor τ_{ij} has only one non-zero component $\tau_{11} = T$, i.e.,

$$\tau_{11} = T,\ \tau_{12} = \tau_{13} = \tau_{22} = \tau_{33} = \tau_{23} = 0 \tag{9.2.1}$$

Since the body forces are absent $F_i = 0$, the state of stress given by (9.2.1) satisfies the equilibrium equations $\tau_{ij,j} = 0$ in the interior of the cylinder. A normal $\hat{v}$ to the lateral surface lies in the plane parallel to $x_2 x_3 -$ plane, so $\hat{v} = (0, v_2, v_3)$.

The relation $\overset{v}{T}_i = \tau_{ij} v_j$ implies that $\overset{v}{T}_1 = \overset{v}{T}_2 = \overset{v}{T}_3 = 0$.

Hence $\overset{v}{T}_i = 0$. (9.2.2)

This shows that the lateral surface of the cylinder is free from tractions. The generalized Hooke's law giving stains in terms of stresses is

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \delta_{ij} \tau_{kk} + \frac{1}{2\mu} \tau_{ij} \qquad (9.2.3)$$

We find from equations (9.2.1) and (9.2.3) that

$$e_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} T$$

$$e_{22} = e_{33} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} T, \; e_{12} = e_{23} = e_{31} = 0 \qquad (9.2.4)$$

Since $$\frac{1}{E} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \text{ and } \frac{\sigma}{E} = \frac{\lambda}{2\mu(3\lambda + 2\mu)} \qquad (9.2.5)$$

Therefore $$e_{11} = \frac{T}{E}, \; e_{22} = e_{33} = -\frac{\sigma}{E} T = -\sigma e_{11}; \; e_{12} = e_{13} = e_{23} = 0 \qquad (9.2.6)$$

These strain components obviously satisfy the compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad (9.2.7)$$

and therefore, the state of stress given in (9.2.1) actually corresponds to one which can exist in a deformed elastic body. From equation (9.2.6), we write

$$\frac{\tau_{11}}{e_{11}} = E, \quad \frac{e_{22}}{e_{11}} = \frac{e_{33}}{e_{11}} = -\sigma \qquad (9.2.8)$$

Experiments conducted on most naturally occurring elastic media show that a tensile longitudinal stress produces a longitudinal extension together with a contraction in a transverse directions. According to $\tau_{11} = T > 0$, we take

$$e_{11} > 0 \text{ and } e_{22} < 0, e_{33} < 0.$$

It then follows from (9.2.8) that

$$E > 0 \text{ and } \sigma > 0 \qquad (9.2.9)$$

From equation (9.2.8), we see that E represents the ratio of the longitudinal stress τ_{11} to the corresponding longitudinal stain e_{11} produced by the stress τ_{11}. From equation (9.2.8), we get

$$\left|\frac{e_{22}}{e_{11}}\right| = \left|\frac{e_{33}}{e_{11}}\right| = \sigma \qquad (9.2.10)$$

Thus, the Poisson's ratio σ represents the numerical value of the ratio of the **contraction** e_{22} (or e_{33}) **in a transverse direction** to the corresponding **extension** e_{11} **in the longitudinal direction**.

Sub Case-II:- Pure Shear

From generalized Hooke's law for an isotropic medium, we write

$$2\mu = \frac{\tau_{12}}{e_{12}} = \frac{\tau_{13}}{e_{13}} = \frac{\tau_{23}}{e_{23}} \qquad (9.2.11)$$

The constant 2μ is thus the ratio of a shear stress component to the corresponding shear strain component. It is, therefore, related to the rigidity of the elastic material. For this reason, the coefficient μ is called the **modulus of rigidity** or the **shear modulus**.

The other Lame's constant λ has no direct physical meaning. The value of μ in terms of Young's modulus E and Poisson ratio σ is given by

$$\mu = \frac{E}{2(1+\sigma)} \tag{9.2.12}$$

Since $E > 0$, $\sigma > 0$, it follows that $\mu > 0$ (9.2.13)

Sub Case-III:- Hydrostatic Pressure

Consider an elastic body of arbitrary shape which is put in a large vessel containing a liquid. A hydrostatic pressure p is exerted on it by the liquid and the elastic body experience all around pressure. The stress tensor is given by $\tau_{ij} = -p\delta_{ij}$. That is,

$$\tau_{11} = \tau_{22} = \tau_{33} = -p, \qquad \tau_{12} = \tau_{23} = \tau_{31} = 0 \tag{9.2.14}$$

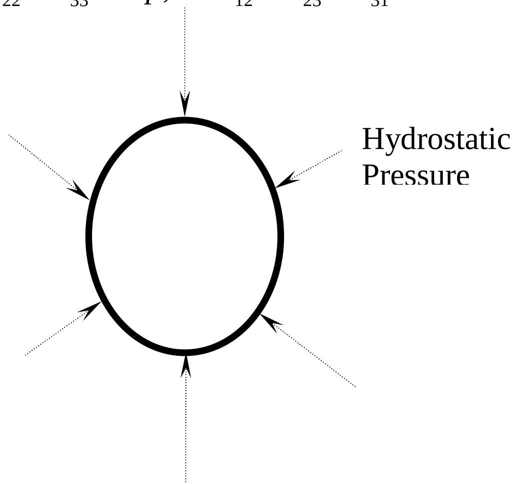


Figure 9.3

These stress components satisfy the equilibrium equations for the zero body force. We find

$$\tau_{kk} = -3p$$

And the generalized Hooke's law giving strains in terms of stresses

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\tau_{kk} + \frac{1}{2\mu}\tau_{ij} \qquad (9.2.15)$$

using (9.2.14) in to (9.2.15) we get

$$e_{12} = e_{23} = e_{31} = 0$$

$$e_{11} = e_{22} = e_{33} = \frac{1}{2\mu}\left[\frac{3\lambda p}{3\lambda + 2\mu} - p\right] = \frac{-p}{3\lambda + 2\mu} \qquad (9.2.16)$$

which obviously satisfy the compatibility equations. We find

$$e_{kk} = \frac{-3p}{3\lambda + 2\mu} = \frac{-p}{\lambda + \frac{2}{3}\mu} = \frac{-p}{k} \qquad (9.2.17)$$

That is,

$$\vartheta(cubical\ dilatation) = \frac{-p}{k} \qquad (9.2.18)$$

From experiments, it has been found that **a hydrostatic pressure tends to reduce the volume of the elastic material**. That is, if $p > 0$, then

$$e_{kk} = v < 0. \qquad (9.2.19)$$

Consequently, it follows from (9.2.19) that $k > 0$. Relation (9.2.18) also shows that the constant k represents the **numerical value** of the **ratio of the compressive stress** to the dilatation.

Substituting the value of λ and μ in terms of E and σ, we find

$$k = \frac{E}{3(1 - 2\sigma)} \qquad (9.2.20)$$

Since $k > 0$ and $E > 0$, if follows that $0 < \sigma < \frac{1}{2}$ for all physical substances. Since

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \qquad (9.2.21)$$

and $E > 0$, $0 < \sigma < \frac{1}{2}$, it follows that $\lambda > 0$.

Remark: The solutions of many problems in elasticity are either exactly or approximately independent of the value chosen for Poisson's ratio. This fact suggests that approximate solutions may be found by so choosing Poisson's ratio as to simplify the problem.

Question: Show that, if $\sigma = 0$ then $\lambda = 0, \mu = \frac{E}{2}, k = \frac{E}{3}$ and Hooke's law is expressed by

$$\tau_{ij} = Ee_{ij} = \frac{1}{2}E(u_{i,j} + u_{j,i}) \qquad (9.2.22)$$

Note 1: The elastic constants μ, E, σ, k have physical meanings. These constants are called engineering elastic modulus.

Note 2: The material such as steel, brass, copper, lead, glass, etc. are isotropic elastic materials.

Note 3: We find

$$e_{kk} = \frac{\tau_{kk}}{3k} = \frac{1-2\sigma}{E}\tau_{kk} \qquad (9.2.23)$$

Thus $e_{kk} = 0$ iff $\sigma = \frac{1}{2}$, provided E and τ_{kk} remain finite.

when

$$\sigma \to \frac{1}{2},\ \lambda \to \infty,\ k \to \infty, \mu = \frac{E}{3},\ v = e_{ii} = u_{i,i} = 0 \qquad (9.2.24)$$

This limiting case corresponds to which is called an incompressible elastic body.

Question: In an elastic beam placed along the x_3 – axis and bent by a couple about the x_2 – axis, the stresses are found to be

$$\tau_{33} = -\frac{E}{R}x_1,\ \tau_{11} = \tau_{22} = \tau_{12} = \tau_{13} = \tau_{23} = 0,\ R = \text{constant}$$

Find the corresponding strains.

Solution: The strains in terms of stresses and elastic moduli E and σ are given by the Hooke's law

$$e_{ij} = \frac{-\sigma}{E}\delta_{ij}\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij} \tag{9.2.25}$$

Here $\tau_{kk} = -\frac{E}{R}x_1$

Hence, (9.2.25) becomes

$$e_{ij} = \frac{-\sigma}{R}x_1\delta_{ij} + \frac{1+\sigma}{E}\tau_{ij} \tag{9.2.26}$$

This gives $e_{11} = e_{22} = \frac{\sigma}{R}x_1$, $e_{33} = -\frac{1}{R}x_1, e_{12} = e_{23} = e_{13} = 0$

Question: A beam placed along the x_1-axis and subjected to a longitudinal stress τ_{11} at every point is so constrained that $e_{11} = e_{33} = 0$ at every point. Show that $\tau_{22} = \sigma\tau_{11}, e_{11} = \frac{1-\sigma^2}{E}\tau_{11},\ e_{33} = \frac{-\sigma(1+\sigma)}{E}\tau_{11}$.

Solution: The Hooke's law giving the strain in terms of stresses is

$$e_{ij} = \frac{-\sigma}{E}\delta_{ij}\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij} \tag{9.2.27}$$

It gives $e_{22} = \frac{-\sigma}{E}(\tau_{11} + \tau_{22} + \tau_{33}) + \frac{1+\sigma}{E}\tau_{22}$

$$e_{22} = \frac{-1}{E}\tau_{22} - \frac{\sigma}{E}(\tau_{11} + \tau_{33}) \tag{9.2.28}$$

Putting $e_{22} = e_{33} = 0$ in (9.2.28), we get

$$\tau_{22} = \sigma\tau_{11} \tag{9.2.29}$$

Also, from (9.2.27) we find

$$e_{11} = \frac{-\sigma}{E}(\tau_{11} + \tau_{22} + \tau_{33}) + \frac{1+\sigma}{E}\tau_{11}$$

$$= \frac{-\sigma}{E}(\tau_{11} + \sigma\tau_{11}) + \frac{1+\sigma}{E}\tau_{11}$$

$$= \frac{1}{E}[-\sigma - \sigma^2 + 1 + \sigma]\tau_{11} \qquad (9.2.30)$$

Also, from (9.2.27), we get

$$e_{33} = \frac{-\sigma}{E}(\tau_{11} + \tau_{22}) + \frac{1+\sigma}{E}\tau_{33}$$

$$= \frac{-\sigma}{E}(\tau_{11} + \sigma\tau_{11})$$

$$= \frac{-\sigma}{E}(1+\sigma)\tau_{11} \qquad (9.2.31)$$

Practice:1 Find the stresses with the following displacement fields:-

(i) $u = kyz\,, v = kzx\,,\ w = kxy$

(ii) $u = kyz\,, v = kzx\,,\ w = k(x^2 - y^2)$

(iii) $u = k(y^2 + z^2)\,, v = kzx\,,\ w = kxy$

(iv) $u = kyz\,, v = k(z^2 + x^2)\,,\ w = kxy$

(v) $u = kyz\,, v = kzx\,,\ w = kxy$

(vi) $u = ky^2z^3\,, v = k(z^2 - x^3)\,,\ w = kxy$

(vii) $u = kyz\,, v = k(z^3 + x^2)\,,\ w = kx^2y$

Practice: 2 (i) A rod placed along the x_1 − axis and subjected to a longitudinal stress τ_{11} is so constrained that there is no lateral contraction. Show that

$$\tau_{11} = \frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)} e_{11}$$

(ii) A rod placed along the x_3 – axis and subjected to a longitudinal stress τ_{33} is so constrained that there is no lateral contraction. Show that

$$\tau_{33} = \frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)} e_{33}$$

(iii) A rod placed along the x_2 – axis and subjected to a longitudinal stress τ_{22} is so constrained that there is no lateral contraction. Show that

$$\tau_{22} = \frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)} e_{22}$$

Practice: 3 Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.

Practice: 2 Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the displacements of the points on the surface of the body are prescribed functions.

Practice: 3 Are the principal axes of strain coincident with those of stress for an anisotropic medium with Hooke's law expressed? For a medium with one plane elastic symmetry? For an orthotropic medium?

Practice: 4 Show directly from the generalized Hooke's law that in an isotropic body the principal axes of strain coincide with those of stress.

9.3 RELATIONSHIP BETWEEN YOUNG MODULUS OF ELASTICITY AND LAME'S CONSTANTS

We have already introduced two elastic moduli λ and μ in the generalized Hooke's law for an isotropic medium. We introduce three more elastic moduli defined below

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \sigma = \frac{\lambda}{2(\lambda + \mu)}, \; k = \lambda + \frac{2}{3}\mu \tag{9.3.1}$$

The quantity σ is dimensionless and is called the Poisson ratio. It was introduced by Simon D. Poisson in 1829.

The quantity E is called **Young's modulus** after Thomas Young who introduced it in the early 19th century, probably in 1807. Its dimension is that of a stress (force/area).

The elastic modulus k is called the modulus of compression or the bulk modulus.

Solving the first two equations for λ and μ (in terms σ and E), we find

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)} \tag{9.3.2}$$

from relation (9.3.2), we find the following relations

$$\left.\begin{aligned} \lambda + 2\mu = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}, \quad \frac{\lambda+\mu}{\mu} = \frac{1}{1-2\sigma} \\ \frac{\lambda+2\mu}{\mu} = \frac{2(1-\sigma)}{(1-2\sigma)}, \quad \frac{\lambda}{\lambda+2\mu} = \frac{\sigma}{1-\sigma} \end{aligned}\right\} \tag{9.3.3}$$

Practice: Derive the following relations between the Lame coefficients λ and μ, Poisson's ratio σ, Young's modulus E, and the bulk modulus k:

$$\lambda = \frac{2\mu\sigma}{1-2\sigma} = \frac{\mu(E-2\mu)}{3\mu - E} = k - \frac{2}{3}\mu = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} = \frac{3kE}{1+\sigma} = \frac{3k(3k-E)}{9k-E} \tag{9.3.4}$$

$$\mu = \frac{\lambda(1-2\sigma)}{2\sigma} = \frac{3}{2}(k-\lambda) = \frac{E}{2(1+\sigma)} = \frac{3k(1-2\sigma)}{2(1+\sigma)} = \frac{3kE}{9k-E} \tag{9.3.5}$$

$$\sigma = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3k-\lambda} = \frac{E}{2\mu} - 1 = \frac{3k-2\mu}{2(3k+\mu)} = \frac{3k-E}{6k} \tag{9.3.6}$$

$$\begin{aligned} E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma} \\ &= \frac{9k(k-\lambda)}{3k-\lambda} = 2\mu(1+\sigma) = \frac{9k\mu}{3k+\mu} = 3k(1-2\sigma) \end{aligned} \tag{9.3.7}$$

$$k = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\sigma)}{3\sigma} = \frac{2\mu(1+\sigma)}{3(1-2\sigma)} = \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1-2\sigma)} \tag{9.3.8}$$

9.4 EQUILIBRIUM EQUATIONS FOR ISOTROPIC ELASTIC SOLID

We know that Cauchy's equation's of equilibrium in term of stress components are

$$\tau_{ij,j} + F_i = 0 \tag{9.4.1}$$

where F_i is the body force per unit volume and $i, j = 1, 2, 3.$ The generalized Hooke's law for a homogeneous isotropic elastic body is

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \tag{9.4.2}$$

$$= \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) \tag{9.4.3}$$

where λ and μ are Lame constants. Putting the value of τ_{ij} form (9.4.3) into equation (9.4.1), we find

$$\lambda\delta_{ij}u_{k,kj} + \mu(u_{i,jj} + u_{j,ij}) + F_i = 0$$

$$\lambda u_{k,ki} + \mu\nabla^2 u_i + \mu u_{k,ki} + F_i = 0$$

$$(\lambda+\mu)\frac{\partial\vartheta}{\partial x_i} + \mu\nabla^2 u_i + F_i = 0 \tag{9.4.4}$$

where $\vartheta = u_{k,k} = div\ \bar{u} = cubical\ \ dilatation$ and $i = 1, 2, 3$.

Equations in (9.4.4) form a synthesis of the analysis of strain, analysis of stress and the stress-strain relation.

These fundamental partial differential equations of the elasticity theory are known a Navier's equations of equilibrium, after **Navier (1821).** Equation (9.4.4.) can be put in several different forms.

(I): In vector form, equation (9.4.4) can be written as

$$(\lambda+\mu) grad\ div\ \overline{u} + \mu \nabla^2 \overline{u} + \overline{F} = \overline{0} \quad (9.4.5)$$

(II): We know that the following vector identity

$$\nabla \times \nabla \times \overline{u} = grad\ div\ \overline{u} - \nabla^2 \overline{u} \quad (9.4.6)$$

Putting the value of $\nabla^2 \overline{u}$ from (9.4.6) into (9.4.5), we obtain

$$(\lambda+\mu) grad\ div\ \overline{u} + \mu[grad\ div \overline{u} - curl\ curl\ \overline{u}] + \overline{F} = \overline{0} \quad (9.4.7)$$

or

$$(\lambda+2\mu) grad\ div\ \overline{u} - \mu\ curl\ curl\ \overline{u} + \overline{F} = \overline{0}$$

(III): Putting the value of $grad\ div \overline{u}$ from (9.4.6) into (9.4.5), we get

$$(\lambda+\mu)[\nabla^2 \overline{u} + curl\ curl\ \overline{u}] + \mu \nabla^2 \overline{u} + \overline{F} = \overline{0}$$

or

$$(\lambda+2\mu)\nabla^2 \overline{u} + (\lambda+\mu) curl\ curl\ \overline{u} + \overline{F} = \overline{0} \quad (9.4.8)$$

(IV): We know that

$$\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\sigma} \quad (9.4.9)$$

Form (9.4.9) and (9.4.5), we find

$$\nabla^2 \overline{u} + \frac{1}{1-2\sigma} grad\ div \overline{u} + \frac{1}{\mu} \overline{F} = \overline{0} \quad (9.4.10)$$

9.5 DYNAMIC EQUATIONS FOR ISOTROPIC ELASTIC SOLID

Let ρ be the density of the medium. The components of the force (mass× acceleration /volume) per unit volume are $\rho \frac{\partial^2 u_i}{\partial t^2}$. Hence, the dynamical equations in terms of the displacements u_i become

$$(\lambda+\mu)\frac{\partial \vartheta}{\partial x_i} + \mu \nabla^2 u_i + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \text{ for } i = 1, 2, 3. \quad (9.5.1)$$

Various form of it can be obtained as above for equilibrium equations.

Practice: In an isotropic elastic body in equilibrium under the body force $\vec{f} = ax_1x_2\hat{e}_3$, where

a is constant, the displacements are of the form $u_1 = Ax_1^2x_2x_3$, $u_2 = Bx_1x_2^2x_3$, $u_3 = Cx_1x_2x_3^2$ where A, B, C are constants. Find A, B, C. Evaluate the corresponding stresses.

Practice: In an isotropic elastic body in equilibrium under the body force $\vec{f} = ax_2x_3\hat{e}_1$, where

a is constant, the displacements are of the form $u_1 = Ax_1^3x_2^2x_3$, $u_2 = Bx_1^2x_2^2x_3$, $u_3 = Cx_1x_2^3x_3^2$ where A, B, C are constants. Find A, B, C. Evaluate the corresponding stresses.

9.6 BELTRAMI-MICHELL COMPATIBILITY EQUATIONS IN TERMS OF THE STRESSES FOR ISOTROPIC SOLID

The strain-stress relations for an isotropic elastic solid are

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta, \qquad \Theta = \tau_{ij} \tag{9.6.1}$$

In which σ is the Poisson's ration and E is the Young's modulus. The Saint-Venant's compatibility equations in terms of strain components are

$$e_{ij},_{kl} + e_{kl},_{ij} - e_{ik},_{jl} - e_{jl},_{ik} = 0 \tag{9.6.2}$$

Which impose restrictions on the strain components to ensure that given e_{ij} yield single-valued continuous displacements u_i.

When the region τ is simply connected, using (9.6.1) in (9.6.2), we find

$$\frac{1+\sigma}{E}\{\tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik},_{jl} - \tau_{jl},_{ki}\} = \frac{\sigma}{E}\{\delta_{ij}\Theta,_{kl} + \delta_{kl}\Theta,_{ij} - \delta_{ik}\Theta,_{jl} - \delta_{jl}\Theta,_{ik}\}$$

$$\{\tau_{ij,kl}+\tau_{kl,ij}-\tau_{ik},_{jl}-\tau_{jl},_{ki}\}=\frac{\sigma}{1+\sigma}\{\delta_{ij}\Theta,_{kl}+\delta_{kl}\Theta,_{ij}-\delta_{ik}\Theta,_{jl}-\delta_{jl}\Theta,_{ik}\} \quad (9.6.3)$$

with $$\tau_{ij,kl}=\frac{\partial^2\tau_{ij}}{\partial x_k\partial x_l},\quad \Theta,_{ij}=\frac{\partial^2\Theta}{\partial x_i\partial x_j}.$$

These are equations of compatibility in stress components. These are 81 (3^4) in number but all of them are not independent. If i and j or k and l are interchanged, we get same equations. Similarly for $i=j=k=l$, equations are identically satisfied. Actually, the set of equations (9.6.3) contains only six independent equations obtained by setting

$$\begin{array}{ll}
k=l=1, & i=j=2 \\
k=l=2, & i=j=3 \\
k=l=3, & i=j=1 \\
k=l=1, & i=2,\quad j=3 \\
k=l=2, & i=3,\quad j=1 \\
k=l=3, & i=1,\quad j=2
\end{array}$$

Setting $k=l$ in (9.6.3) and then taking summation over the common index, we get

$$\tau_{ij,kk}+\tau_{kk,ij}-\tau_{ik},_{jk}-\tau_{jk},_{ik}=\frac{\sigma}{1+\sigma}\{\delta_{ij}\Theta,_{kk}+\delta_{kk}\Theta,_{ij}-\delta_{ik}\Theta,_{jk}-\delta_{jk}\Theta,_{ik}\}$$

Since $\Theta,_{kk}=\nabla^2\Theta$, $\tau_{ij,kk}=\nabla^2\tau_{ij}$, $\tau_{kk,ij}=\Theta,_{ij}$ and $\delta_{kk}=3$ (9.6.4)

Therefore, above equations become

$$\nabla^2\tau_{ij}+\Theta,_{ij}-\tau_{ik,jk}-\tau_{jk,ik}=\frac{\sigma}{1+\sigma}[\delta_{ij}\nabla^2\Theta+3\Theta,_{ij}-2\Theta,_{ij}]$$

or $$\nabla^2\tau_{ij}+\frac{1}{1+\sigma}\Theta,_{ij}-\tau_{ik,jk}-\tau_{jk,ik}=\frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta$$

(9.6.5) This is a set of **nine** equations and out of which only **six** are independent due to the symmetry of i and j. In combining equations (9.6.3) linearly, the number of independent equations is not reduced.

Hence the resultant set of equations in (9.6.5) is equivalent to the original equations in (9.6.3). Equilibrium equations are

$$\tau_{ik,k} + F_i = 0 \tag{9.6.6}$$

where F_i is the body force per unit volume. Differentiating these (9.6.6) equations with respect to x_j, we get

$$\tau_{ik,kj} = -F_{i,j} \tag{9.6.7}$$

Using (9.6.7), equation (9.6.5) can be rewritten in the form

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} + \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = -(F_i,_j + F_j,_i)\,. \tag{9.6.8}$$

Setting $j = i$ in relation (9.6.8) and adding accordingly, we write

$$\nabla^2\Theta + \frac{1}{1+\sigma}\nabla^2\Theta - \frac{3\sigma}{1+\sigma}\nabla^2\Theta = -2F_i,_i$$

$$(1 + \frac{1}{1+\sigma} - \frac{3\sigma}{1+\sigma})\nabla^2\Theta = -2F_i,_i$$

$$\frac{2(1-\sigma)}{1+\sigma}\nabla^2\Theta = -2F_i,_i = -2\; div\vec{F}\,,$$

giving

$$\nabla^2\Theta = -\frac{1+\sigma}{1-\sigma}div\,\vec{F} \tag{9.6.9}$$

Using relation (9.6.9) in the relations (9.6.8) , we find the final form of the compatibility equations in terms of stresses.

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} = -\frac{\sigma}{1+\sigma}\delta_{ij} div\,\vec{F} - (F_i,_j + F_j,_i) \tag{9.6.10}$$

These equations in Cartesian coordinates (x, y, z) can be written as

$$\nabla^2\tau_{xx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\sigma}{1-\sigma}div\,\vec{F} - 2\frac{\partial F_x}{\partial x}$$

$$\nabla^2\tau_{yy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} = -\frac{\sigma}{1-\sigma}div\,\vec{F} - 2\frac{\partial F_y}{\partial y}$$

$$\nabla^2\tau_{zz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = -\frac{\sigma}{1-\sigma}div\,\vec{F} - 2\frac{\partial F_z}{\partial z}$$

$$\nabla^2\tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\partial z} = -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right)$$

$$\nabla^2\tau_{zx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z\partial x} = -\left(\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z}\right)$$

$$\nabla^2\tau_{xy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\partial y} = -\left(\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}\right) \qquad (9.6.11)$$

In **1892,** Beltrami obtained these equations for $\vec{F} = \vec{0}$ and in 1900 Michell obtained them in the form as given in (9.6.11). The equations (9.6.11) are called the **Beltrami-Michell** compatibility equations.

9.7 HARMONIC AND BIHARMONIC FUNCTIONS

Definition: A function V of class C^4 is called a **biharmonic** function when

$$\nabla^2\nabla^2 V = 0$$

Theorem 1: When the components of the body force $\vec{F}$ are constants, show that the stress and strain invariants Θ and ϑ are harmonic functions and the stress components τ_{ij} and strain components e_{ij} are biharmonic functions.

Proof: The Beltrami-Michal compatibility equations in terms of stress are

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} = -\frac{\sigma}{1+\sigma}\delta_{ij}div\,\vec{F} - (F_i,_j + F_j,_i) \qquad (9.7.1)$$

In which $\vec{F}$ is the body force per unit volume.

It is given that the vector $\vec{F}$ is constant. In this case, equations in (9.7.1) reduce to

$$\nabla^2 \tau_{ij} + \frac{1}{1+\sigma} \Theta,_{ij} = 0 \qquad (9.7.2)$$

Setting $i = j$ in (9.7.2) and taking summation accordingly, we get

$$\nabla^2 \tau_{ii} + \frac{1}{1+\sigma} \Theta,_{ii} = 0$$

$$\nabla^2 \Theta + \frac{1}{1+\sigma} \nabla^2 \Theta = 0$$

$$(1 + \frac{1}{1+\sigma}) \nabla^2 \Theta = 0$$

$$\nabla^2 \Theta = 0 \qquad (9.7.3)$$

This shows that the stress invariant $\Theta = \tau_{kk}$ is harmonic function.

The standard relation between the invariants Θ and ϑ is

$$\Theta = (3\lambda + 2\mu)\vartheta \qquad (9.7.4)$$

and the equation (9.7.3) implies that

$$\nabla^2 \vartheta = 0 \qquad (9.7.5)$$

showing that the strain invariant $\vartheta = e_{kk}$ is also a harmonic function. Again

$$\nabla^2 \nabla^2 \tau_{ij} = \nabla^2 \left(-\frac{1}{1+\sigma} \Theta,_{ij} \right)$$

$$= -\frac{1}{1+\sigma} \nabla^2 (\Theta,_{ij})$$

$$= -\frac{1}{1+\sigma} (\nabla^2 \Theta),_{ij} \qquad (9.7.6)$$

Using (9.7.3) in the relations (9.7.6), we get

$$\nabla^2\nabla^2\tau_{ij} = 0 \quad (9.7.7)$$

This shows that the stress components τ_{ij} are biharmonic functions.

The following stain-stress relations

$$e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\Theta + \frac{1}{2\mu}\tau_{ij}$$

give

$$\nabla^2\nabla^2 e_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\nabla^2\nabla^2\Theta + \frac{1}{2\mu}\nabla^2\nabla^2\tau_{ij}$$

$$\nabla^2\nabla^2 e_{ij} = 0 \quad (9.7.8)$$

Equation (9.7.8) shows that the strain components e_{ij} are also biharmonic functions.

Theorem 2: If the body force $\vec{F}$ is derived from a harmonic potential function, show that the strain and stress invariants ϑ and Θ are harmonic functions and the **strain and stress** components are biharmonic function.

Proof: Let ϕ be the potential function and $\vec{F}$ is derived from ϕ so that

$$\vec{F} = \vec{\nabla}\phi \text{ or } F_j = \phi,_j \quad (9.7.9)$$

Then

$$div\,\vec{F} = \phi,_{jj} = \nabla^2\phi = 0 \quad (9.7.10)$$

Since ϕ is a harmonic function (given). Further

$$F_i,_j = F_j,_i = \phi,_{ij} \quad (9.7.11)$$

The Beltrami-Michell compatibility equations in term of stresses, in this case, reduce to

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta,_{ij} = -2\phi,_{ij} \quad (9.7.12)$$

Putting $j = i$ in relation (9.7.12) and taking the summation accordingly, we obtain

$$\nabla^2\tau_{ii} + \frac{1}{1+\sigma}\Theta,_{ii} = -2\phi,_{ii}$$

$$\nabla^2\Theta + \frac{1}{1+\sigma}\nabla^2\Theta = -2\nabla^2\phi$$

Using relations (9.7.10) we get

$$\nabla^2\Theta = 0 \qquad (9.7.13)$$

This shows that Θ is harmonic. And, the relation $\Theta = (3\lambda + 2\mu)\vartheta$ immediately shows that ϑ is also harmonic.

From relation (9.7.12), we write

$$\nabla^2\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\nabla^2\Theta,_{ij} = -2\nabla^2\phi,_{ij}$$

This gives $\nabla^2\nabla^2\tau_{ij} = 0$ as $\nabla^2\Theta = \nabla^2\phi = 0$ (9.7.14)

It shows that the components τ_{ij} are biharmonic. The strain-stress relations yield that the strain components are also biharmonic function.

9.8 APPLICATION OF THE BELTRAMI-MICHELL EQUATIONS

Example 1: Find whether the following stress system can be a solution of an elastostatic problem in the absence of body forces:

$$\tau_{11} = x_2 x_3,\ \tau_{22} = x_3 x_1,\ \tau_{12} = x_3^2,\ \tau_{13} = \tau_{33} = \tau_{32} = 0. \qquad (9.8.1)$$

Solution: In order that the given stress system can be a solution of an elastostatic problem in the absence of body forces, the following equations are to be satisfied:

(i) Cauchy's equations of equilibrium with $F_i = 0$, i.e.

$$\tau_{11},_1 + \tau_{12},_2 + \tau_{13},_3 = 0$$

$$\tau_{12},_1 + \tau_{22},_2 + \tau_{23},_3 = 0$$

$$\tau_{13},_1 + \tau_{23},_2 + \tau_{33},_3 = 0 \qquad (9.8.2)$$

(ii) Beltrami-Michell equations with $F_i = 0$, i.e.

$$\nabla^2\tau_{11} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{11} = 0$$

$$\nabla^2\tau_{22} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{22} = 0$$

$$\nabla^2\tau_{33} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{33} = 0$$

$$\nabla^2\tau_{12} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{12} = 0$$

$$\nabla^2\tau_{13} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{13} = 0$$

$$\nabla^2\tau_{23} + \frac{1}{1+\sigma}(\tau_{11}+\tau_{22}+\tau_{33}),_{23} = 0 \qquad (9.8.3)$$

It is easy to check that all the equilibrium equations in (9.8.2) are satisfied. Moreover, all except the fourth one in (9.8.3) are satisfied by the given stress system.

Since the given system does not satisfy the Beltrami-Michell equations fully, it cannot form a solution of an elastostatic problem.

Remark: The example illustrates the important fact that a stress system may not be a solution of an elasticity problem even though it satisfies Cauchy's equilibrium equations.

Practice 1: Show that the stress-system $\tau_{11} = \tau_{22} = \tau_{13} = \tau_{23} = \tau_{12} = 0$, $\tau_{33} = \rho g x_3$, where ρ and g are constants, satisfies that equations of equilibrium and the equations of compatibility for a suitable body force.

Practice 2: Show that the following stress system cannot be a solution of an elastostatic problem although it satisfies cauchy's equations of equilibrium with zero body forces: $\tau_{11} = x_2^2 + \sigma(x_1^2 - x_2^2)$, $\tau_{22} = x_1^2 + \sigma(x_2^2 - x_1^2)$, $\tau_{33} = \sigma(x_1^2 + x_2^2)$, $\tau_{12} = -2\sigma x_1 x_2$, $\tau_{23} = \tau_{31} = 0$ where σ is a constant of elasticity.

Practice 3: Determine whether or not the following stress components are a possible solution in elastostatics in the absence of body forces: $\tau_{11} = ax_2x_3$, $\tau_{22} = bx_3x_1$, $\tau_{33} = cx_1x_2$, $\tau_{12} = d\,x_3^2$, $\tau_{13} = e\,x_2^2$, $\tau_{23} = f\,x_1^2$ where a, b, c, d, e and f all are constants.

Practice 4: In an elastic body in equilibrium under the body force $\vec{f} = ax_1x_2\hat{e}_3$, where a is constant, the stresses are of the form: $\tau_{11} = ax_1x_2x_3$, $\tau_{22} = bx_1x_2x_3$, $\tau_{33} = cx_1x_2x_3$, $\tau_{12} = (a\,x_1^2 + bx_2^2)x_3$, $\tau_{23} = (b\,x_2^2 + cx_3^2)x_1$, $\tau_{13} = (c\,x_3^2 + ax_1^2)x_2$; where a, b, c are constants. Find these constants

Practice 5: Define the stress function S by $\tau_{ij} = S,_{ij} = \dfrac{\partial^2 S}{\partial x_i \partial x_j}$ and consider the case of zero body force. Show that, if $\sigma = 0$, then the equilibrium and compatibility equations reduce to

$\nabla^2 S = Cons\tan t$.

Books Recommended:

1. **Y.C.Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey, 1965.
2. **Sokolnikoff, I.S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
3. **A. E. H. Love** A Treatise on the Mathematical Theory of Elasticity, Combridge University Press, London.

CHAPTER-X

STRAIN ENERGY DENSITY FUNCTION

10.1 INTRODUCTION

The energy stored in an elastic body by virtue of its deformation is called the strain energy. This energy is acquired by the body when the body force and surface traction do same work. This is also termed as internal energy. It depends upon the shape and temperature of the body.

10.2 STRAIN-ENERGY FUNCTION

Let τ_{ij} be the tensor and e_{ij} be the strain tensor for an infinitesimal affine deformation of an elastic body. We write

$$\begin{aligned} &\tau_{11} = \tau_1, \tau_{22} = \tau_2, \tau_{33} = \tau_3 \\ &\tau_{23} = \tau_4, \tau_{31} = \tau_5, \tau_{12} = \tau_6 \end{aligned} \qquad (10.2.1)$$

and

$$\begin{aligned} &e_{11} = e_1, e_{22} = e_2, e_{33} = e_3 \\ &2e_{23} = e_4, 2e_{13} = e_5, 2e_{12} = e_6 \end{aligned} \qquad (10.2.2)$$

In terms of engineering notations.

We assume that the deformation of the elastic body is isothermal or adiabatic. Love (1944) has proved that, under this assumption there exist a function of strains

$$W = W(e_1, e_2, e_3, e_4, e_5, e_6) \qquad (10.2.3)$$

with the property

$$\frac{\partial W}{\partial e_i} = \tau_i \qquad \text{for i=1,2,....,6.} \qquad (10.2.4)$$

This function W is called the strain energy function.

W represents strain energy, per unit of undeformed volume, stored up in the body by the strains e_i.

The units of W are $\frac{force \times L}{L^3} = \frac{force}{L^2}$ **that of a stress**.

The existence of W was first introduced by George Green (1839).Expanding the strain energy function W, given by (10.2.3) in a power series in terms of strains e_i, we write

$$2W = d_0 + 2d_i e_i + d_{ij} e_i e_j \quad \text{i,j=1,2,....,6} \qquad (10.2.5)$$

After discarding all terms of order 3 and higher in the strain e_i as strains e_i are assumed to be small. In second terms, summation of i is to be taken and in 3^{rd} term, summation over dummy suffices i & j are to be taken.

In the **natural state,** $e_i = 0$**, consequently** W =0 for $e_i = 0$.

This gives

$$d_0 = 0 \qquad (10.2.6)$$

Even otherwise, the constant term in (10.2.5) can be neglected since we are interested only in the partial derivatives of W. therefore, equation (10.2.5) and (10.2.6) yield

$$2W = 2d_i e_i + d_{ij} e_i e_j \qquad (10.2.7)$$

This gives

$$\frac{\partial W}{\partial e_k} = d_i \delta_{ik} + \frac{1}{2} \frac{\partial}{\partial e_k} \{d_{ij} e_i e_j\}$$

$$= d_k + \frac{1}{2} \{d_{ij} \delta_{ki} e_j + d_{ij} e_i \delta_{kj}\}$$

$$= d_k + \frac{1}{2} [d_{kj} e_j + d_{ki} e_i]$$

$$= d_k + \frac{1}{2} (d_{kj} + d_{kj}) e_j$$

$$= d_k + (d_{kj} e_j) e_j$$

This gives

$$\tau_i = d_i + c_{ij} e_j \qquad (10.2.8)$$

where

$$c_{ij} = \frac{1}{2}(d_{ij} + d_{ji}) = c_{ji} \qquad (10.2.9)$$

we observe that the c_{ij} is symmetric.

we further assume that the stress $\tau_i = 0$ in the undeformed state, when e_i =0.

This assumption, using equation (10.2.8), gives

$$d_i = 0 \qquad \text{i=1, 2,….,6} \qquad (10.2.10)$$

Equations (10.2.7), (10.2.8) and (10.2.10) give

$$\tau_i = c_{ij} e_j \qquad 10.2.11)$$

And

$$W = \frac{1}{2} c_{ij} e_i e_j = \frac{1}{2} e_i \tau_i \qquad (10.2.12)$$

Since, two quadric homogeneous forms for W are equal as

$$d_{ij} e_i e_j = c_{ij} e_i e_j \qquad (10.2.13)$$

Equation (10.2.12) shows that the strain energy function W **is a homogeneous function of degree 2** in strains e_i**,** $i = 1, 2, 3, 4, 5, 6$**,** and coefficients c_{ij} are symmetric.

The generalized Hooke’s law under the conditions of existence of strain energy function is given in equations (10.2.9) and (10.2.11).

The matrix form, it can be expressed as

$$\begin{bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{33}\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix}\begin{bmatrix}e_{11}\\ e_{22}\\ e_{33}\\ 2e_{23}\\ 2e_{13}\\ 2e_{12}\end{bmatrix} \qquad (10.2.14)$$

This law contains 21 independent elastic constants.

Result 1: From equation (10.2.2); we write

$$\begin{aligned} W &= \frac{1}{2}\left[\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3 + \tau_4 e_4 + \tau_5 e_5 + \tau_6 e_6\right] \\ &= \frac{1}{2}\left[\tau_{11}e_{11} + \tau_{22}e_{22} + \tau_{33}e_{33} + 2\tau_{23}e_{23} + 2\tau_{13}e_{13} + 2\tau_{12}e_{12}\right] \qquad (10.2.15) \\ &== \frac{1}{2}\tau_{ij}e_{ij} \qquad i, j = 1,2,3 \end{aligned}$$

The result in (10.2.14) **is called Claperon formula.**

Result II: For an isotropic elastic medium, the Hooke's law gives

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},\; i, j = 1,2,3 \quad (10.2.16)$$

This gives

$$\begin{aligned} W &= \frac{1}{2}e_{ij}\left[\lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\right] \\ &= \frac{1}{2}\lambda e_{kk}e_{kk} + \mu e_{ij}\, e_{ij} \\ &= \frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}^2 \qquad (10.2.17) \\ &= \frac{1}{2}\lambda(e_{11} + e_{22} + e_{33})^2 + \mu(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{13}^2 + 2e_{23}^2) \end{aligned}$$

Result 3: Also, we have

$$e_{ij} = -\frac{\sigma}{E}\delta_{ij}\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij} \qquad (10.2.18)$$

Hence,

$$W = \frac{1}{2}\tau_{ij}\left(-\frac{\sigma}{E}\delta_{ij}\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij}\right)$$

$$W = -\frac{\sigma}{2E}\tau_{ij}\tau_{kk} + \frac{1+\sigma}{E}\tau_{ij}\tau_{ij} \qquad (10.2.19)$$

Result 4: From (10.2.12), we note that in the value of W**,** we may interchange e_i and τ_i**.** Consequently, interchanging e_i and τ_i in equation (10.2.4), we obtain

$$\frac{\partial W}{\partial \tau_i} = e_i, \text{ for } i = 1, 2, 3, 4, 5, 6 \qquad (10.2.20)$$

This result is due to **Castigliano (1847-1884).**

It follows form the assumed linear stress-strain relations.

Result 5: We know that the elastic moduli λ and μ are both positive for all physical elastic solids. The quadratic form on the right side of (10.2.17) takes only positive values for every set of values for every set of values of the strains. This shows that the strain energy function W is a positive definite form in the strain components e_{ij}**,** for an isotropic elastic solid.

10.3 Application of Strain Energy Function

Example 1: Show that the strain-energy function W for an isotropic solid is **independent** of the **choice of coordinate** axes.

Solution: We know that the strain energy function W is given by

$$W = \frac{1}{2}\tau_{ij}e_{ij}$$

$$= \frac{1}{2}e_{ij}(\lambda\delta_{ij}e_{kk} + 2\mu e_{ij}) \qquad (10.3.1)$$

$$= \frac{1}{2}\lambda(e_{11} + e_{22} + e_{33})^2 + \mu(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{13}^2 + 2e_{23}^2)$$

Let

$$I_1 = e_{ii} = e_{11} + e_{22} + e_{33}) \qquad (10.3.2)$$

$$I_2 = e_{ii}e_{jj} - e_{ij}e_{ji} \qquad (10.3.3)$$

be the first and second invariants of the strain tensor e_{ij}. As the given medium is isotropic, the elastic moduli λ and μ are also independent of the choice of coordinate axes. We write

$$W = \frac{1}{2}\lambda I_1^2 + \mu\left[(e_{11}+e_{22}+e_{33})^2 - 2e_{11}e_{22} - 2e_{22}e_{33} - 2e_{33}e_{11} + 2e_{12}^2 + 2e_{13}^2 + 2e_{23}^2\right]$$

$$= \frac{1}{2}\lambda I_1^2 + \mu\left[I_1^2 - 2\{(e_{11}e_{22} - e_{12})^2 + (e_{22}e_{22} - e_{23})^2 + (e_{11}e_{33} - e_{13})^2\}\right]$$

$$= \frac{1}{2}\lambda I_1^2 + \mu I_1^2 - 2\mu I_2$$

$$= \left(\frac{\lambda}{2} + \mu\right) I_1^2 - 2\mu I_2 \qquad (10.3.4)$$

Hence, equation (10.3.4) shows that the strain energy function W is invariant relative to all rotations of Cartesian axes.

Example 2: Evaluate W for the stress field (for isotropic solid) $\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$ $\tau_{13} = -\mu\alpha x_2, \tau_{23} = \mu\alpha x_1$, $\alpha \neq 0$ is constant and μ is the Lame's constant.

Solution: We find $\tau_{kk} = \tau_{11} + \tau_{22} + \tau_{33} = 0$

Hence, the relation $e_{ij} = \frac{1}{2\mu}\left[\tau_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\tau_{kk}\right];\ i, j = 1, 2, 3$

gives $$e_{ij} = \frac{1}{2\mu}\tau_{ij}$$

That is $$e_{11} = e_{22} = e_{33} = e_{12} = 0 \qquad (10.3.5)$$

$$e_{13} = -\frac{1}{2}\alpha x_2, e_{12} = \frac{1}{2}\alpha x_1 \qquad (10.3.6)$$

The energy function W is given by

$$W = \frac{1}{2}\tau_{ij}e_{ij}$$

$$= \frac{1}{4\mu}\tau_{ij}\tau_{ij} = \frac{1}{4\mu}(\tau_{13}^2 + \tau_{23}^2)$$

$$= \frac{1}{4}\mu\alpha^2(x_1^2 + x_2^2) \quad (10.3.7)$$

Example 3: Show that the strain energy W is given by

$$W = W_1 + W_2$$

where $W_1 = \frac{1}{2}k e_{ii}\, e_{ii} = \frac{1}{18k}\tau_{ii}\tau_{ii}$, $k = bulk \bmod ulus$, and

$$W_2 = \frac{1}{3}\mu\left[(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2 + 6(e_{12}^2 + e_{23}^2 + e_{31}^2)\right]$$

$$= \frac{1}{12\mu}\left[(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)\right] \quad (10.3.8)$$

Example 4: If $W = \frac{1}{2}\left[\lambda e_{kk}^2 + 2\mu\, e_{ij} e_{ij}\right]$, Prove the following,

(i) $\frac{\partial W}{\partial e_{ij}} = \tau_{ij}$,

(ii) $W = \frac{1}{2}\tau_{ij} e_{ij}$

(iii) W is a scalar invariant.

(iv) $W \geq 0$ and $W = 0$ iff $e_{ij} = 0$

(v) $\frac{\partial W}{\partial \tau_{ij}} = e_{ij}$

Solution: (i) We note that W is a function to e_{ij}. Partial differentiation of this function with respect to e_{ij} gives

$$\frac{\partial W}{\partial e_{ij}} = \frac{1}{2}\left[\lambda 2 e_{kk}\frac{\partial e_{kk}}{\partial e_{ij}} + 4\mu\, e_{ij}\right] = \left[\lambda e_{kk}\delta_{ij} + 2\mu\, e_{ij}\right] = \tau_{ij} \quad (10.3.9)$$

(ii) $$W = \frac{1}{2}\left[\lambda e_{kk} e_{kk} + 2\mu\, e_{ij} e_{ij}\right]$$

$$= \frac{1}{2}\left[\lambda e_{kk}(\delta_{ij} e_{ij}) + 2\mu\, e_{ij} e_{ij}\right]$$

$$= \frac{1}{2}\left[\lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\right]e_{ij}$$

$$W = \frac{1}{2}e_{ij}\tau_{ij} \qquad (10.3.10)$$

(iii) Since τ_{ij} and e_{ij} are components of tensors, each of order 2, respectively. So by contraction rule, $W = \frac{1}{2}\tau_{ij}\, e_{ij}$ is a scalar invariant.

(iv) Since $\lambda > 0$, $\mu > 0$, $e_{kk}^2 \geq 0$ and $e_{ij}.e_{ij} > 0$, if follows that $W > 0$ Moreover $W = 0$ iff $e_{kk} = 0$. Since $e_{ij} = 0$ automatically implies that $e_{kk} = 0$. Hence $W = 0$ hold iff $e_{ij} = 0$

(v)Putting (10.3.11)

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}\delta_{ij} \qquad (10.3.11)$$

Into (10.3.10) we find

$$W = \frac{1}{2}\left[\frac{1+\sigma}{E}\tau_{ij}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}\delta_{ij}\tau_{ij}\right] = \frac{1}{2}\left[\frac{1+\sigma}{E}\tau_{ij}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}^2\right]$$

This implies
$$\frac{\partial W}{\partial \tau_{ij}} = \left[\frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}\frac{\partial \tau_{kk}}{\partial \tau_{ij}}\right]$$

$$\Rightarrow \quad \frac{\partial W}{\partial \tau_{ij}} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\tau_{kk}\delta_{ij} = e_{ij} \qquad (10.3.12)$$

10.4 Theorem: Show that the total work done by the external forces in altering (changing) the configuration of the **natural state** to the state at time '**t**' is equal to the sum of the **kinetic energy** and the **strain energy**.

Proof: the natural/ unstrained state of an elastic body is one in which there is a uniform temperature and zero displacement with reference to which all strains will be specified.

Let the body be in the natural state when t=0. Let (x_1, x_2, x_3) denote the coordinate of an arbitrary material point of the elastic body in the undeformed/unstrained state.

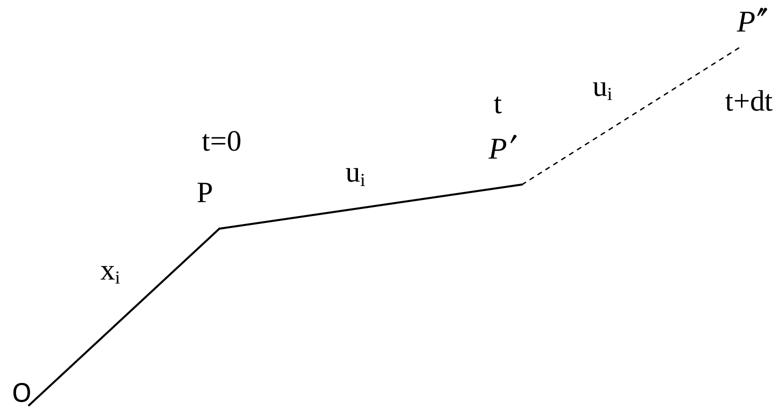


Figure 10.1

If the elastic body is subjected to the action of external forces, then it may produce a deformation of the body and at any time ‘t’, the coordinate of the same material point will be $x_i + u_i(x_1, x_2, x_3)$.

The displacement of the point P in the interval of time (t, t+dt) is given by

$$\frac{\partial u_i}{\partial t} dt = \dot{u}_i dt \, , \tag{10.4.1}$$

where $$\dot{u}_i = \frac{\partial u_i}{\partial t}$$

The work done by the body force F_i acting on the volume element $d\tau$, in time dt sec, located at the material point P is

$$(F_i d\tau)(\dot{u}_i dt) = F_i \dot{u}_i d\tau \, dt \, ,$$

and the work performed by the external surface forces $\overset{v}{T}_i$ in time interval (t, t+dt) is $\overset{v}{T}_i \, \dot{u}_i dt \, d\tau$, where $d\sigma$ is the element of surface.

Let E denote the work done by the body and surface forces acting on the elastic body. Then the rate of doing work on the body originally occupying some region τ (by external forces) is

$$\frac{dE}{dt} = \int_{\tau} F_i \dot{u}_i d\tau + \int_{\Sigma} \overset{v}{T}_i \dot{u}_i d\sigma \qquad (10.4.2)$$

Where $\sum$ denote the original surface of the elastic body.

Now $$\int_{\Sigma} \overset{v}{T}_i \dot{u}_i d\sigma = \int_{\Sigma} (\tau_{ij} v_j) \dot{u}_i d\sigma$$

$$= \int_{\Sigma} (\tau_{ij} \dot{u}_i) v_j d\sigma$$

$$= \int_{\tau} (\tau_{ij} \dot{u}_i),_j d\tau$$

$$= \int_{\tau} [\tau_{ij\,j} \dot{u}_i + \tau_{ij} \dot{u}_i,_j] d\tau$$

$$= \int_{\tau} \tau_{ij},_j \dot{u}_i d\tau + \int_{\tau} \tau_{ij} \dot{e}_{ij} d\tau + \int_{\tau} \tau_{ij} \dot{w}_{ij} d\tau \qquad (10.4.3)$$

Where

$$\dot{e}_{ij} = (\dot{u}_i,_j + \dot{u}_j,_i)/2 \text{ and } \dot{w}_{ij} = (\dot{u}_i,_j - \dot{u}_j,_i)/2 \qquad (10.4.4)$$

Since $\dot{w}_{ij} = -\dot{w}_{ji}$ and $\tau_{ij} = \tau_{ji}$,

So $$\tau_{ij} \dot{w}_{ij} = 0 \qquad (10.4.5)$$

Form dynamical equations of motion for an isotropic body, we write

$$\tau_{ij,j} = \rho \ddot{u}_i - F_i$$

Therefore, $$\tau_{ij,j} \dot{u}_i = \rho \dot{u}_i \ddot{u}_i - \dot{u}_i F_i \qquad (10.4.6)$$

Using results (10.4.5) and (10.4.6); we write form equations (10.4.4) and (10.4.2)

$$\begin{aligned} \frac{dE}{dt} &= \int_{\tau} F_i \dot{u}_i d\tau + \int_{\tau} [\rho \dot{u}_i \ddot{u}_i - F_i \dot{u}_i] d\tau + \int_{\tau} \tau_{ij} \dot{e}_{ij} d\tau \\ &= \int_{\tau} \rho \dot{u}_i \ddot{u}_i d\tau + \int_{\tau} \tau_{ij} \dot{e}_{ij} d\tau \end{aligned} \qquad (10.4.7)$$

The kinetic energy K of the body in motion is given by

$$K = \frac{1}{2}\int_\tau \rho \dot{u}_i \dot{u}_i \, d\tau \tag{10.4.8}$$

Then
$$\frac{dK}{dt} = \frac{1}{2}\int_\tau \rho \ddot{u}_i \dot{u}_i \, d\tau \tag{10.4.9}$$

We define the engineering notation

$$\left.\begin{aligned} &\tau_{11} = \tau_1, \tau_{22} = \tau_2, \tau_{33} = \tau_3, \tau_{23} = \tau_4, \tau_{13} = \tau_5, \tau_{12} = \tau_6 \\ &e_{11} = e_1, e_{22} = e_2, e_{33} = e_3, 2e_{23} = e_4, 2e_{13} = e_5, 2e_{12} = e_6 \end{aligned}\right\} \tag{10.4.10}$$

Then
$$\int_\tau \tau_{ij} \dot{e}_{ij} \, d\tau = \int_\tau \tau_i \frac{\partial e_i}{\partial t} \, d\tau \tag{10.4.11}$$

for $i = 1, 2, 3, \ldots\ldots\ldots\ldots, 6$ and under isothermal condition, there exists a energy function

$$W = W(e_1, e_2, e_3 \ldots\ldots\ldots e_6)$$

We the property that
$$\frac{\partial W}{\partial e_i} = \tau_i \tag{10.4.12}$$

$1 \le i \le 6$. From equations (10.4.11) and (10.4.12), we write

$$\int_\tau \tau_{ij} \dot{e}_{ij} \, d\tau = \int_\tau \left(\frac{\partial W}{\partial e_i} \frac{\partial e_i}{\partial t} \right) d\tau = \frac{d}{dt}\int_\tau W d\tau = \frac{dU}{dt} \tag{10.4.13}$$

where
$$U = \int_\tau W d\tau \tag{10.4.14}$$

from equations (10.4.7) , (10.4.9) and (10.4.14), we write

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} \tag{10.4.15}$$

Integrating equation (10.4.15) with respect to 't' between the limits $t = 0$ and $t = t$, we obtain

$$E = K + U \tag{10.4.16}$$

Since both E and K are zero at $t = 0$. The equation (10.4.16) proves the required result.

Note 1: If the elastic body is in equilibrium instead of in motion, then $K = 0$ and consequently $E = U$.

Note 2: U is called the total strain energy of the deformation.

10.5 CLAPEYRON'S THEOREM

Statement: If an elastic body is in equilibrium under a given system of body forces F_i and surface forces $\overset{v}{T}_i$, then the strain energy of deformation is equal to one-half the work that would be done by the external forces (of the equilibrium state) acting through the displacements u_i from the unstressed state to the state of equilibrium.

Proof. We are required to prove that

$$\int_\tau F_i u_i d\tau + \int_\Sigma \overset{v}{T}_i u_i d\sigma = 2\int_\tau W d\tau \qquad (10.5.1)$$

where Σ denotes the original surface of the unstressed region τ of the body and W is the energy density function representing the strain every per unit volume. Now

$$\int_\Sigma \overset{v}{T}_i u_i d\sigma = \int_\sigma \tau_{ij} u_i v_j d\sigma$$

$$= \int_\sigma (\tau_{ij} u_i),_j \, d\sigma \qquad \text{(using Gauss divergence theorem)}$$

$$= \int_\tau \{\tau_{ij},_j u_i + \tau_{ij} u_i,_j\} d\tau$$

$$= \int_\tau \{\tau_{ij},_j u_i + \tau_{ij} \left\{ \frac{u_i,_j + u_j,_i}{2} + \frac{u_i,_j - u_j,_i}{2} \right\}\} d\tau$$

$$= \int_\tau \tau_{ij},_j u_i d\tau + \int_\tau \tau_{ij}(e_{ij} + w_{ij}) d\tau$$

$$= \int_\tau \tau_{ij},_j u_i d\tau + \int_\tau d\tau$$

$$= \int_\tau (\tau_{ij},_j u_i + \tau_{ij} e_{ij}) d\tau \qquad (10.5.2)$$

Since

$$w_{ij} = -w_{ji} \text{ and } \tau_{ij} = \tau_{ji}$$

again from (10.5.2)

$$\int_{\Sigma} \overset{v}{T}_i u_i d\sigma = \int_{\tau} (-F_i u_i + 2W) d\tau , \quad (10.5.3)$$

Since $\tau_{ij},_j + F_i = 0$

Being the equilibrium equations and

$$W = \frac{1}{2} \tau_{ij} e_{ij} ,$$

From (10.5.4), we can write

$$\int_{\tau} F_i u_i d\tau + \int_{\Sigma} \overset{v}{T}_i u_i d\sigma = 2 \int_{\tau} W d\tau , \quad (10.5.4)$$

proving the theorem.

Books Recommended:

1. **Y.C.Fung:** Foundation of Solid Mechanics, Prentice Hall, Inc., New Jersey,1965.
2. **Sokolnikoff, I.S.** Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company, Ltd., New Delhi, 1977
3. **A. E. H. Love** A Treatise on the Mathematical Theory of Elasticity, Combridge University Press, London.
4. **MDU, Rohtak** Handbook of Directorate of Distance Education, MDU, Rohtak.