Gram-Schmidt Thorem: Let W be a non-null subspace of a finite dimensional Euclidean space V. Then it possesses an orthonormal basis.

Proof: Let $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_r\}$ be a basis of W where $\alpha_i \neq 0$ for all i of the new basis where $\beta_1 = \alpha_1$.

Let $\beta_2 = \alpha_2 - c_1\beta_1$ where $c_1\beta_1$ is the projection of α_2 upon β_i.

Then $\langle \alpha_2 - c_1\beta_1, \beta_1 \rangle = 0$ or $\langle \beta_2, \beta_1 \rangle = 0$ and $c_1 = \dfrac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}$

$\therefore$ β_2 is orthogonal to β_1 and $L\{\beta_1, \beta_2\} = L\{\beta_1, \alpha_2\} = L\{\alpha_1, \alpha_2\}$

$$\therefore \quad \beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_2 \rangle}$$

and $\alpha_3 \notin L\{\beta_1, \beta_2\}$

Let $\beta_3 = \alpha_3 - r_1\beta_1 - r_2\beta_2$ where $r_1\beta_1, r_2\beta_2$ are the projections of α_3 upon β_1, β_2 respectively and $r_1 = \dfrac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_2 \rangle}$, $r_2 = \dfrac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}$

Then β_3 is also orthogonal to β_1, β_2 and $L\{\beta_1, \beta_2, \beta_3\} = L\{\beta_1, \beta_2, \alpha_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

$$\therefore \quad \beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2$$

Since V is a finite dimensional Euclidean space, this process stops after finite number of steps and we get after rth step,

$$\beta_r = \alpha_r - \frac{\langle \alpha_r, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 - \frac{\langle \alpha_r, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2 - \ldots\ldots - \frac{\langle \alpha_r, \beta_{r-1} \rangle}{\langle \beta_{r-1}, \beta_{r-1} \rangle}\beta_{r-1}$$

and $\{\beta_1, \beta_2, \beta_r\}$ is an orthogonal basis of W and the corresponding orthonormal basis of W is

$$\left\{\frac{\beta_1}{\|\beta_1\|}, \frac{\beta_2}{\|\beta_2\|}, \frac{\beta_3}{\|\beta_3\|}, \ldots\ldots, \frac{\beta_r}{\|\beta_r\|}\right\}$$

Example: Apply Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product spanned by the vectors (1, 1, 0, 1), (1, –2, 0, 0), (1, 0, – 1, 2)

Solution: Let $\alpha_1 = (1, 1, 0, 1)$, $\alpha_2 = (1, -2, 0, 0)$ and $\alpha_3 = (1, 0, -1, 2)$ then the vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent and the basis of $\mathbb{R}^4$.

Let $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 - c_1\beta_1$ where $c_1\beta_1$ is the projection α_2 upon β_1. Then β_2 is orthogonal to β_1 and $L\{\beta_1, \beta_2\} = L\{\beta_1, \alpha_2\} = L\{\alpha_1, \alpha_2\}$.

$$\therefore \quad c_1 = \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \frac{-1}{3}$$

$$\therefore \quad \beta_2 = \alpha_2 + \frac{1}{3}\beta_1 = \alpha_2 + \frac{1}{3}\alpha_1 = (1, -2, 0, 0) + \frac{1}{3}(1, 1, 0, 1)$$

$$= \frac{1}{3}(4, -5, 0, 1)$$

Let $\beta_3 = \alpha_3 - r_1\beta_1 - r_2\beta_2$ where $r_1\beta_1$, $r_2\beta_2$ are the projections of α_3 upon β_1 and β_2 respectively. Then β_3 is orthogonal to β_1, β_2 and $L\{\beta_1, \beta_2, \beta_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$

$$\therefore \quad r_1 = \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \frac{3}{3} = 1$$

and $$r_2 = \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} = \frac{6/3}{14/3} = \frac{6}{14} = \frac{3}{7}$$

$$\therefore \quad \beta_3 = \alpha_3 - \beta_1 - \frac{1}{5}\beta_2 = (1, 0, -1, 2) - (1, 1, 0, 1)\ \frac{-3}{7}\cdot\frac{1}{3}(4, -5, 0, 1)$$

$$= (0, -1, -1, 1) - \frac{1}{7}(4, -5, 0, 1)$$

$$= \frac{1}{7}(-4, -2, -7, 6)$$

$\therefore$ The orthogonal basis of the subspace is

$$\{(1,1,0,1), \frac{1}{3}(4,-5,0,1), \frac{1}{7}(-4,-2,-7,6)\}$$

and the corresponding orthonormal basis is

$$\left\{\frac{1}{\sqrt{3}}(1,1,0,1), \frac{1}{\sqrt{42}}(4,-5,0,1), \frac{1}{\sqrt{105}}(-4,-2,-7,6)\right\}$$

Example: Use Gram-Schmidt process to obtain an orthogonal basis from the basis set $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ of the Euclidean space $\mathbb{R}^3$ with standard inner product.

Solution: Let $\alpha_1 = (1, 1, 0)$, $\alpha_2 = (0, 1, 1)$ and $\alpha_3 = (1, 0, 1)$

Let $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 - c_1\beta_1$ where $c_1\beta_1$ is the projection of α_2 upon β_1.

Then β_2 is orthogonal to β_1 and $L\{\beta_1, \beta_2\} = L\{\alpha_1, \alpha_2\}$.

$$\therefore \qquad c_1 = \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \frac{1}{2}$$

$$\therefore \qquad \beta_2 = \alpha_2 - \frac{1}{2}\beta_1 = (0,1,1) - \frac{1}{2}(1,1,0) = \frac{1}{2}(-1,1,2)$$

Let $\beta_3 = \alpha_3 - r_1\beta_1 - r_2\beta_2$ where $r_1\beta_1$ and $r_2\beta_2$ are the projections of α_3 upon β_1, β_2 respectively.

Then β_3 is orthogonal to β_1, β_2 and $L\{\beta_1, \beta_2, \beta_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$

$$\therefore \qquad r_1 = \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} = \frac{1}{2}$$

and $$r_2 = \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} = \frac{1/2}{3/2} = \frac{1}{3}$$

$$\therefore \qquad \beta_3 = \alpha_3 - \frac{1}{2}\beta_1 - \frac{1}{3}\beta_2 = (1,0,1) - \frac{1}{2}(1,1,0) - \frac{1}{3}\cdot\frac{1}{2}(-1,1,2)$$

$$= \frac{1}{2}(1,-1,2) - \frac{1}{6}(-1,1,2) = \frac{1}{6}(4,-4,3) = \frac{1}{3}(2,-2,2)$$

Hence, the orthogonal basis is $\left\{(1,1,0), \frac{1}{2}(-1,1,2), \frac{1}{3}(2,-2,2)\right\}$.