11.8 INNER PRODUCT SPACE

Definition: Let V be a real vector space. A real inner product on V is a mapping $f: V \times V \to \mathbb{R}$ that assigns to each ordered pair of vectors (α, β) of V a real number $f(\alpha, \beta)$, denoted by $\langle \alpha, \beta \rangle$, satisfying the following properties:

(*i*) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in V$ (*symmetry*).

(*ii*) $\langle \alpha, \beta + \gamma \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle$ for all $\alpha, \beta, \gamma \in V$ (*linearity*).

(*iii*) $\langle a\alpha, \beta \rangle = a\langle \alpha, \beta \rangle = \langle \alpha, a\beta \rangle$ for all $\alpha, \beta \in V$ and all $a \in \mathbb{R}$ (*homogeneity*).

(*iv*) $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq \theta$ (positivity) where θ is the null vector.

If $\alpha = \theta$, then $\langle \alpha, \alpha \rangle = 0$.

Definition: A real vector space V endowed with a real inner product defined on it is said to be an *Euclidean space*.

Complex Inner Product: Let V be a complex vector space. A complex inner product is a mapping $f: V \times V \to C$ that assigns to each ordered pair of vectors (α, β) of V a complex number $f(\alpha, \beta)$, denoted by $\langle \alpha, \beta \rangle$, satisfying the following properties:

(*i*) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ where $\overline{\langle \beta, \alpha \rangle}$ is the conjugate of the complex number $\langle \beta, \alpha \rangle$.

(*ii*) $\langle c\alpha + d\beta, \gamma \rangle = c\langle \alpha, \gamma \rangle + d\langle \beta, \gamma \rangle$

(*iii*) $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq \theta$ and $\langle \theta, \theta \rangle = 0$

From (*i*) it follows that $\langle \alpha, \alpha \rangle = \overline{(\alpha, \alpha)}$ showing that (α, α) is a real number and (*iii*) says that the complex inner product satisfies the positivity condition as in the case of a real inner product.

Definition: A complex vector space V together with a complex inner product defined on it is said to be a *Unitary space*.

Example 1: In the real vector space $\mathbb{R}^n$, let $\alpha = (a_1, a_2, ..., a_n)$, $\beta = (b_1, b_2, ..., b_n)$ be the two vectors and we define

$$\langle \alpha, \beta \rangle = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

The (α, β) satisfies all the conditions for a real inner product. This inner product is called the standard inner product and is often called the dot product of α, β and is denoted by $\alpha \cdot \beta$. The vector space R^n with this inner product becomes a Euclidean space.

Example 2: In $\mathbb{R}^2$ the standard inner product is defined by $\langle \alpha, \beta \rangle = a_1 b_1 + a_2 b_2$ where $\alpha = (a_1, a_2) \in \mathbb{R}^2$ and $\beta = (b_1, b_2) \in \mathbb{R}^2$.

We define $\langle \alpha, \beta \rangle = 2a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$. Then $\langle \alpha, \beta \rangle$ satisfies the conditions (*i*), (*ii*), (*iii*) of real inner product space.

Now $\langle \alpha, \alpha \rangle = 2a_1^2 + 2a_1 a_2 + a_2^2 = a_1^2 + (a_1 + a_2)^2 > 0$ for $a_1, a_2 \neq 0$

$\therefore \quad \langle \alpha, \alpha \rangle > 0$ for $\alpha \neq \theta$.

Hence, the positivity condition is also satisfied by $\langle \alpha, \beta \rangle$.

Therefore, the vector space $\mathbb{R}^2$ becomes a Euclidean space under this inner product.

This example shows that a real vector space can be made a Euclidean space in many ways.