## 10.12 ORTHOGONAL MATRIX

**Definition:** A matrix $A$ is called *orthogonal* if $AA^t = I$.

For example,

the matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is orthogonal since $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note that identity matrix is always orthogonal.

**Theorem:** (1) The determinant of an orthogonal matrix is either 1 or – 1, and hence $A$ is non-singular.

(2) If $A$ is orthogonal, then $A^{-1} = A^t$.

(3) If $A$ is orthogonal, then $A^{-1}$ is also orthogonal.

**Proof:** (1) Let $A$ be an orthogonal matrix, then

$$AA^t = I.$$

$\therefore \quad |AA^t| = |I| = 1.$

or $\quad |A||A^t| = 1. \qquad [\because |AB| = |A||B|]$

or $\quad |A||A| = 1 \qquad [\because |A^t| = |A|]$

or $\quad |A|^2 = 1 \;\; \therefore |A| = \pm 1.$

Hence, $A$ is non-singular.

(2) Since $A$ is orthogonal, $AA^t = I$.

Again, as $A$ is non-singular, $A^{-1}$ exists and $AA^{-1} = A^{-1}A = I$.

So premultiplying $AA^t = I$ by $A^{-1}$ we get

$$A^{-1}(AA^t) = A^{-1}I$$

or $\quad (A^{-1}A)A^t = A^{-1}I$

or $\quad IA^t = A^{-1} \quad$ or $\quad A^t = A^{-1}$.

(Note from $AA^{-1} = A^{-1}A = I$ it follows $A^tA = I$).

(3) Let $A$ be orthogonal. Then

$$(A^{-1})(A^{-1})^t = A^t\,(A^t)^{-1} = I$$

[Since $A^{-1} = A^t$ and $(A^{-1})^t = (A^t)^{-1}$.]

Hence, $A^{-1}$ is orthogonal.

## 10.16 DIAGONALISABLE MATRIX

A square matrix $A$ of order $n$ is said to be be *diagonalisable* if $A$ is similar to a diagonal matrix $D$ of order $n$. Then we say $A$ is diagonalisable to $D$.

**Necessary and sufficient condition for diagonalisability of $n \times n$ matrix:** A square matrix $A$ of order $n$ is diagonalisable if and only if there exists $n$ eigen vectors of $A$ which are linearly independent.

**Theorem:** (*i*) Let $A$ be a square matrix of order $n$. If the eigen values of $A$ be all distinct, then $A$ is diagonalisable.

(*ii*) If a square matrix $A$ of order $n$ has $n$ eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$, then $A$ is diagonalisable to the diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & & \lambda_n \end{pmatrix}$$

(*iii*) If $A = PDP^{-1}$ where $D = \begin{pmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n \end{pmatrix}$ then $i^{\text{th}}$ column of $P$ is an eigen vector of $A$ corresponding to the eigen value $d_i$ of $A$.

**Example 1:** Diagonalise the following matrix: $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

**Solution:** The characteristic equation of $A$ is $\det(A - \lambda I_2) = 0$

or $$\begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) = 0 \Rightarrow \lambda = 2, 3.$$

$\therefore$ The eigen values of $A$ are 2, 3.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 2, then

$$AX = 2X \quad \text{or} \quad \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{or} \quad \left.\begin{aligned} 2x_1 &= 2x_1 \\ x_1 + 3x_2 &= 2x_2 \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} x_1 &= x_1 \\ x_1 + x_2 &= 0 \end{aligned}$$

Let $x_1 = c$ be any arbitrary real number, then $x_2 = -c$.

Then the eigen vector is $\begin{pmatrix} c \\ -c \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen values 3, then

$$AX = 3X \Rightarrow \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or
$$\left.\begin{aligned} 2x_1 &= 3x_1 \\ x_1 + 3x_2 &= 3x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= 0 \\ x_1 &= 0 \end{aligned}$$

Let $x_2 = c$ be any arbitrary real number.

$\therefore$ The eigen vector is $\begin{pmatrix} 0 \\ c \end{pmatrix} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Since two eigen vectors of a square matrix $A$ corresponding two distinct eigen values are linearly independent.

Hence, $A$ is diagonalised and it is diagonalised to $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

**Example 2:** Show that the matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is not diagonalisable.

**Solution:** The characteristic equation of $A$ is det $(A - \lambda I_2) = 0$

or
$$\begin{vmatrix} 1-\lambda & 0 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 1, 1$$

The eigen values of $A$ are 1, 1.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen value $\lambda = 1$,

then
$$AX = X \Rightarrow \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 &= x_1 \\ 3x_1 + x_2 &= x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= x_1 \\ x_1 &= 0 \end{aligned}$$

Let $x_2 = c$ be any arbitrary real number.

$\therefore$ The eigen vector is $\begin{pmatrix} 0 \\ c \end{pmatrix} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $(c \neq 0)$

In this case, two linearly independent eigen vectors corresponding to the eigen value 1 cannot be found. Therefore, a non-singular matrix $P$ of order 2 having two linearly independent eigen vectors cannot be found and so $A$ cannot be similar to a diagonal matrix.

Hence, $A$ is not diagonalisable.

**Example 3:** Diagonalise, if possible; the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and find the matrix which diagonalise it.

**Solution:** The characteristic equation of $A$ is det $(A - \lambda I_2) = 0$

or $$\begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{pmatrix} = 0 \Rightarrow (\lambda - 1)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda + 1 - 1 = 0$$

or $$\lambda^2 - 2\lambda = 0$$

$\therefore$ $$\lambda = 0, 2$$

The eigen values are 0 and 2.

Since two eigen vectors of a square matrix $A$ corresponding two distinct eigen values of $A$ are linearly independent.

Hence, the matrix $A$ is diagonalisable.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 0,

then $$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow x_1 - x_2 = 0 \text{ and } -x_1 + x_2 = 0$$

Let $x_1 = c$ be any arbitrary real number, then $x_2 = c$

$\therefore$ The eigen vector is $\begin{pmatrix} c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 2,

then $$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{or} \quad \begin{matrix} x_1 - x_2 = 2x_1 \\ -x_1 + x_2 = 2x_2 \end{matrix}$$

or $$\begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix}$$

Let $x_1 = K$ be any arbitrary real number, then $x_2 = -K$

$\therefore$ The eigen vector is $\begin{pmatrix} K \\ -K \end{pmatrix} = K\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So the matrix which diagonalise the given matrix $A$ is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

where (1, 1), (1, –1) are linearly independent.