10.14 EIGEN VALUES AND EIGEN VECTORS

Let us consider a 2 × 2 matrix

$$A = \begin{pmatrix} x^2 + x + 1 & 3x^3 + 2x \\ 3x^3 + x & 4x^2 + 3 \end{pmatrix}$$

whose elements are real polynomials in x. A can be expressed as the polynomials in x.

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} x^3 + \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} x^2 + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

whose coefficients are real matrices of order 2 × 2.

Such a polynomial whose coefficients are matrices of the same order is called a *matric polynomial*. The degree of the matric polynomial is the degree of the constituent polynomial of highest degree appearing in the matrix A.

In general, if A be an $n \times n$ matrix whose elements are real (complex) polynomials in x, then A can be expressed as a matric polynomial whose coefficients are $n \times n$ real (complex) matrices.

Definition: Let A be an $n \times n$ matrix. Then det $(A - xI_n)$ is called the *characteristic polynomial* of A and is denoted by $\psi_A(x)$. The equation $\psi_A(x) = 0$ is called the *characteristic equation* of A.

Let $$A = (a_{ij})_{n+n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\text{Then} \qquad \psi_A(x) = \det(A - xI_n) = \begin{bmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{bmatrix}$$

The degree of the characteristic equation is same as the order of the matrix A.

We now state without proof a very useful theorem:

Definition: A root of the characteristic equation of a square matrix A is said to be a *characteristic root* or *latent root* or an *eigen value* of A.

Note 1: 0 is an eigen value of a singular matrix.

Note 2: The eigen values of a diagonal matrix are its diagonal elements.

Note 3: If λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

Proof: Since A is non-singular, $\lambda \neq 0$.

$\therefore$ λ^{-1} exists then A^{-1} also exists.

Let the order of A be n.

$$\text{Now } \det(A^{-1} - \lambda^{-1} I_n) = \frac{1}{\lambda^n} \det(\lambda A^{-1} - I_n)$$

$$= \frac{1}{\lambda^n \det A} \det[(\lambda A^{-1} - I_n)A]$$

$$= \frac{1}{\lambda^n \det A} \det[\lambda I_n - A]$$

$$= 0, \text{ since } \det(A - \lambda I_n) = 0$$

This proves that λ^{-1} is an eigen value of A^{-1}.

Example 1: Find the eigen values of the following matrices:

$$(i)\ A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \qquad (ii)\ A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}.$$

Solution: (*i*) The characteristic equation of A is given by

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)(5-\lambda)-12=0$ or $5-\lambda-5\lambda+\lambda^2-12=0$

or $\lambda^2-6\lambda-7=0$ or $\lambda^2-7\lambda+\lambda-7=0$

or $\lambda(\lambda-7)+1(\lambda-7)=0$ or $(\lambda-7)(\lambda+1)=0$ or $\lambda=7,-1$

The eigen values of A are $-1, 7$.

(*ii*) The characteristic equation of A is given by

$$\det(A-\lambda I_3)=0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & -1 \\ 3 & 2 & -2-\lambda \end{vmatrix}=0$$

or $(1-\lambda)[-(2-\lambda)(2+\lambda)+2]+1[-(2+\lambda)+3]+0[2-3(2-\lambda)]=0$

or $(1-\lambda)[\lambda^2-4+2]+(1-\lambda)=0$

or $(1-\lambda)[\lambda^2-2+1]=0$ or $(1-\lambda)(\lambda^2-1)=0$

$\therefore$ $\lambda=1, 1, -1$

$\therefore$ The eigen values of A are $1, 1, -1$

Definition: Let A be $n \times n$ matrix over $\mathbb{R}$. A non-null vector X belonging to $V_n(\mathbb{R})$ is said to be an eigen vector or a characteristic vector of A if there exists a $\lambda \in (\mathbb{R})$ such that

$AX=\lambda X$ holds, where λ is an eigen value of A.

Let A be an $n \times n$ matrix over $\mathbb{R}$ and λ be an eigen value belonging to $\mathbb{R}$. A result of importance is that to each $\mathbb{R}$ such eigen value of A there corresponds at least one eigen vector.

Example 2: Find the eigen vectors of the matrix $A=\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$.

Solution: The eigen value of A are $-1, 7$.

Let $X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector corresponding to -1, then

$AX=-X$ gives

$$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=-\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or $x_1+3x_2=-x_1$ or $2x_1+3x_2=0$

and $4x_1+5x_2=-x_2$ or $4x_1+6x_2=0$

The equivalent system is $x_1 + \frac{3}{2}x_2 = 0$

Let $x_2 = k$, then $x_1 = -\frac{3}{2}k$.

$\therefore$ The solution of the system is $\left(-\frac{3k}{2}, k\right)$ where k is an arbitrary real number.

$\therefore$ The eigen vectors are $k\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ or $c\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ where c is a non-zero-real number.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to 7, then

$$AX = 7X$$

or $$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 7\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or $$x_1 + 3x_2 = 7x_1 \quad \text{or} \quad -6x_1 + 3x_2 = 0$$

and $$4x_1 + 5x_2 = 7x_2 \quad \text{or} \quad 4x_1 - 2x_2 = 0$$

The system is equivalent to $x_1 - \frac{1}{2}x_2 = 0$.

Let $x_1 = c$, then $x_2 = 2c$ where c is an arbitrary real number.

$\therefore$ The eigen vectors are $c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ where c is a non-zero real number.

Example 3: Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: The characteristic equation of A is given by det $(A - \lambda I_3) = 0$

or $$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or $(2-\lambda)[(2-\lambda)(1-\lambda)-0]-1[(1-\lambda)-0]+1[0-0]=0$

or $(2-\lambda)^2(1-\lambda)-(1-\lambda)=0 \Rightarrow (1-\lambda)[4-4\lambda+\lambda^2-1]=0$

or $(1-\lambda)(\lambda^2-4\lambda+3)=0 \Rightarrow (1-\lambda)(\lambda^2-3\lambda-\lambda+3)=0$

or $(1-\lambda)\{\lambda(\lambda-3)-1(\lambda-3)\}=0 \Rightarrow (1-\lambda)(\lambda-1)(\lambda-3)=0$

$\therefore$ $\lambda = 1, 1, 3$

$\therefore$ The eigen values are 1, 1, 3.

Let $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 3, then

$$AX=3X \quad \text{or} \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or
$$2x_1+x_2+x_3=3x_1$$
$$x_1+2x_2+x_3=3x_2$$
$$x_3=3x_3$$

or
$$-x_1+x_2+x_3=0$$
$$x_1-x_2+x_3=0$$
$$-2x_3=0$$

which gives $x_3=0$,

$\therefore$ $$-x_1+x_2=0$$

and $$x_1-x_2=0 \Rightarrow x_1=x_2$$

Let $x_2=c$ be any arbitrary real number, then $x_1=c$

$\therefore$ The eigen vectors are $\begin{pmatrix} c \\ c \\ 0 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ where c is an arbitrary real number.

Let $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 1, then

$$AX=X \quad \text{or} \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{or} \quad 2x_1 + x_2 + x_3 = x_1$$
$$x_1 + 2x_2 + x_3 = x_2$$
$$x_3 = x_3$$

$$\text{or} \quad x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_3 = x_3$$

$$\therefore \quad x_1 = -(x_2 + x_3)$$

Let $x_3 = c$ and $x_2 = k$ be the arbitrary real numbers, then

$$x_1 = -(k + c)$$

$\therefore$ The eigen vectors are $\begin{pmatrix} -(k+c) \\ k \\ c \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ where k and c are arbitrary real numbers.

Example 4: Find the eigen values and the corresponding eigen vectors of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}.$$

Solution: The characteristic equation of A is given by $\det(A - \lambda I_3) = 0$

$$\text{or} \quad \begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & -2-\lambda & 4 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)[-(2+\lambda)(6-\lambda) + 12] + 1[2(6-\lambda) - 12] + 2[-6 + 3(2+\lambda)] = 0$

or $(1-\lambda)(-12 + 2\lambda - 6\lambda + \lambda^2 + 12) + 12 - 2\lambda - 12 + 2[-6 + 6 + 3\lambda] = 0$

or $\quad (1-\lambda)(\lambda^2 - 4\lambda) - 2\lambda + 6\lambda = 0$

or $\quad \lambda[(1-\lambda)(\lambda - 4) + 4] = 0$ or $\lambda[\lambda - 4 - \lambda^2 + 4\lambda + 4] = 0$

or $\quad \lambda[-\lambda^2 + 5\lambda] = 0$ or $\lambda^2(-\lambda + 5) = 0$

$$\therefore \quad \lambda = 0, 0, 5$$

$\therefore$ The eigen values are 0, 0, 5.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 0, then

$$AX = 0X \Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or
$$x_1 - x_2 + 2x_3 = 0$$
$$2x_1 - 2x_2 + 4x_3 = 0$$
$$3x_1 - 3x_2 + 6x_3 = 0$$

which gives
$$x_1 - x_2 + 2x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0$$

Let $x_3 = c$ and $x_2 = d$ be the arbitrary real numbers, then

$x_1 = x_2 - 2x_3 = d - 2c$

$\therefore$ The eigen vectors are $\begin{pmatrix} d-2c \\ d \\ c \end{pmatrix} = d\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ where c and d are arbitrary real numbers, i.e., $d(1, 1, 0)$ and $c(-2, 0, 1)^t$

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corresponding to the eigen value 5, then

Now $AX = 5X$ gives $\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

or
$$x_1 - x_2 + 2x_3 = 5x_1$$
$$2x_1 - 2x_2 + 4x_3 = 5x_2$$
$$3x_1 - 3x_2 + 6x_3 = 5x_3$$

or
$$-4x_1 - x_2 + 2x_3 = 0 \quad \text{...(1)}$$
$$2x_1 - 7x_2 + 4x_3 = 0 \quad \text{...(2)}$$
$$3x_1 - 3x_2 + x_3 = 0 \quad \text{...(3)}$$

Now eliminating x_3 from these equations, we get

$(1) \times 2 - (2) \quad \Rightarrow -10x_1 + 5x_2 = 0 \quad \Rightarrow x_2 = 2x_1$

$(1) - 2 \times (3) \quad \Rightarrow -10x_1 + 5x_2 = 0 \quad \Rightarrow x_2 = 2x_1$

Let $x_1 = c$ be any arbitrary real number, then $x_2 = 2c$

and $$x_3 = -3x_1 + 3x_2 = -3c + 6c = 3c$$

$\therefore$ The eigen vectors are $\begin{pmatrix} c \\ 2c \\ 3c \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ where c is an arbitrary real number.