11.7 MATRIX OF LINEAR MAPPING

With every linear mapping from a finite dimensional vector space to another finite dimensional vector space is associated a matrix, called its matrix representation.

In this section, we shall discuss only how to determine the matrix associated with a linear mapping. This matrix depends essentially on the basis of the two vector spaces and also on the order on which the vectors of the bases are taken. Unless otherwise mentioned we shall always take the standard basis for the vector space $\mathbb{R}^n$, given by $\{1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 1)\}$.

Matrix representation of a linear mapping: Let $\phi : V \to W$ be a linear mapping where V and W are both finite dimensional vector space over $\mathbb{R}$ and the dimension of V, W are n, m respectively.

Let $B_1 = \{\alpha_1, \alpha_2, \alpha_3; ... \alpha_n\}$ be an ordered basis of V and $B_2 = \{\beta_1, \beta_2, ...\beta_3\}$ be an ordered basis of W.

Since ϕ is linear transformation from V to W, then each of the n vectors $\phi(\alpha_1), \phi(\alpha_2) ... \phi(\alpha_n)$, can be expressed uniquely as a linear combination of $B_2 = \{\beta_1, \beta_2 ...\beta_m\}$.

Let $\phi(\alpha_1) = a_{11}\,\beta_1 + a_{21}\,\beta_2 + \ldots + a_{m1}\,\beta_m$

$\phi(\alpha_2) = a_{12}\,\beta_1 + a_{22}\,\beta_2 + \ldots + a_{m2}\,\beta_m$

$\vdots$ ....

$\vdots$ ....

$(\alpha_n) = a_{1n}\,\beta_1 + a_{2n}\,\beta_2 + \ldots + a_{mn}\,\beta_m$ where $a_{ij} \in \mathbb{R}$

$\therefore$ The scalars $a_{ij}, a_{2j}, \ldots, a_{mj}$ are the coordinates of $\phi(\alpha_j), j = 1, 2, \ldots n$.

This is the coordinate vector of $\phi(\alpha_j)$ relative to the order basis B_2.

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the matrix of ϕ relative to the ordered basis of B_1 and B_2.

Let $\gamma = x_1\,\alpha_1 + x_2\,\alpha_2 + \ldots + x_n\,\alpha_n$ be an arbitrary vector of V and

let $\phi(\gamma) = y_1\,\beta_1 + y_2\,\beta_2 + \ldots + y_m\,\beta_m, x_i, y_i \in \mathbb{R}$

Now, $\phi(\gamma) = \phi(x_1\,\alpha_1 + x_2\,\alpha_2 + \ldots + x_n\,\alpha_n)$

$= x_1\,\phi(\alpha_1) + x_2\,\phi(\alpha_2) + \ldots + x_n\,\phi(\alpha_n)$ ($\because \phi$ is linear)

$= x_1\,(a_{11}\,\beta_1 + a_{21}\,\beta_2 + \ldots + a_{m1}\beta_m) + x_2\,(a_{12}\,\beta_1 + a_{22}\,\beta_2 + \ldots + a_{m2}\,\beta_m) + \ldots + x_n\,(a_{1n}\,\beta_1 + a_{2n}\,\beta_2 + \ldots + a_{mn}\beta_m)$

$= (x_1\,a_{11} + x_2\,a_{12} + \ldots + x_n\,a_{1n})\,\beta_1 + (x_1 a_{21} + x_2\,a_{22} + \ldots + a_{2n}\,x_n)\beta_2 + \ldots + (x_1\,a_{m1} + x_2\,a_{m2} + \ldots + x_n\,a_{mn})\beta_m$

Since $\{\beta_1, \beta_2, \ldots \beta_m)$ is linearly independent set,

$y_1 = a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1n}\,x_n$

$y_2 = a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2n}\,x_n$

....

....

$y_m = a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{mn}\,x_n$

The above system of equation can be written as $Y = AX$ ...(1)

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

and X is the co-ordinate vector of an arbitrary element γ in V relative to the ordered basis $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ and Y is the co-ordinate vector of $\phi(\gamma)$ in W relative to the ordered basis $\{\beta_1, \beta_2, \dots \beta_n\}$.

Then $Y = AX$ is called the matrix representation of the linear mapping ϕ relative to the chosen ordered basis of V and W.

Example 1: A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is defind by $\phi(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of ϕ relative to the ordered bases

(*i*) (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$ and (1, 0), (0, 1) of $\mathbb{R}^2$

(*ii*) (0, 1, 0), (1, 0, 0) (0, 0, 1) of $\mathbb{R}^3$ and (0, 1) (1, 0) of $\mathbb{R}^2$

(*iii*) (0, 1, 1), (1, 0, 1) (1, 1, 0) of $\mathbb{R}^3$ and (1, 0) (0, 1) of $\mathbb{R}^2$.

Solution: (*i*) Now $\phi(1, 0, 0) = (3, 1) = 3(1, 0) + 1(0, 1)$

$$\phi(0, 1, 0) = (-2, -3) = -2(1, 0) + (-3)(0, 1)$$

$$\phi(0, 0, 1) = (1, -2) = 1(1, 0) + (-2)(0, 1)$$

$$\therefore \text{ The matrix of } \phi = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

(*ii*) Now, $\phi(0, 1, 0) = (-2, -3) = -3(0, 1) + (-2)(1, 0)$

$$\phi(1, 0, 0) = (3, 1) = 1(0, 1) + 3(1, 0)$$

$$\phi(0, 0, 1) = (1, -2) = -2(0, 1) + 1(1, 0)$$

$$\therefore \quad \text{The matrix of } \phi = \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

(*iii*) Now, $\phi(0, 1, 1) = (-1, -5) = -1(1, 0) - 5(0, 1)$

$$\phi(1, 0, 1) = (4, -1) = 4(1, 0) - 1(0, 1)$$

$$\phi(1, 1, 0) = (1, -2) = 1(1, 0) - 2(0, 1)$$

$$\therefore \text{ The matrix of } \phi = \begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$$

Example 2: A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\phi(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_2 + 4x_3, x_1 - x_2 + 3x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of ϕ relative to the ordered bases

(*i*) (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$;

(*ii*) (0, 0, 1), (1, 0, 0), (0, 1, 0) of $\mathbb{R}^3$;

(*iii*) (0, 1, 1), (1, 0, 1), (1, 1, 0) of $\mathbb{R}^3$.

Solution: (*i*) Now, $\phi(1, 0, 0) = (2, 0, 1) = 2(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$

$$\phi(0,1,0) = (1,1,-1) = 1(1,0,0) + 1(0,1,0) - 1(0,0,1)$$

$$\phi(0,0,1) = (-1,4,3) = -(1,0,0) + 4(0,1,0) + 3(0,0,1)$$

$$\therefore \text{ The matrix of } \phi = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 4 \\ 1 & -1 & 3 \end{pmatrix}$$

(*ii*) Now, $\phi(0,0,1) = (-1,4,3) = 3(0,0,1) - 1(1,0,0) + 4(0,1,0)$

$$\phi(1,0,0) = (2,0,1) = 1(0,0,1) + 2(1,0,0) + 0(0,1,0)$$

$$\phi(0,1,0) = (1,1,-1) = -1(0,0,1) + 1(1,0,0) + 1(0,1,0)$$

$$\therefore \text{ The matrix of } \phi = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 1 \\ 4 & 0 & 1 \end{pmatrix}$$

(*iii*) Now, $\phi(0, 1, 1) = (0, 5, 2) = \frac{7}{2}(0, 1, 1) - \frac{3}{2}(1, 0, 1) + \frac{3}{2}(1, 1, 0)$

$$\phi(1, 0, 1) = (1, 4, 4) = \frac{7}{2}(0, 1, 1) + \frac{1}{2}(1, 0, 1) + \frac{1}{2}(1, 1, 0)$$

$$\phi(1, 1, 0) = (3, 1, 0) = -(0, 1, 1) + 1(1, 0, 1) + 2(1, 1, 0)$$

$$\therefore \text{ The matrix of } \phi = \begin{pmatrix} \frac{7}{2} & \frac{7}{2} & -1 \\ -\frac{3}{2} & \frac{1}{2} & 1 \\ \frac{3}{2} & \frac{1}{2} & 2 \end{pmatrix}$$

[$\because$ $(0, 5, 2) = c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$

or $(0, 5, 2) = (c_2 + c_3, c_1 + c_3, c_1 + c_2)$

$\Rightarrow$ $0 = c_2 + c_3, c_1 + c_3 = 5, c_1 + c_2 = 2$

$\therefore$ $c_2 = -c_3$ and $(c_1 + c_3) - (c_1 + c_2) = 5 - 2 \Rightarrow c_3 - c_2 = 3 \Rightarrow 2c_3 = 3$

$\therefore$ $c_3 = \frac{3}{2}$ and $c_2 = -\frac{3}{2}$

$\therefore$ $c_1 = 2 - c_2 = 2 + \frac{3}{2} = \frac{7}{2}$

$\therefore$ $c_1 = \frac{7}{2}, c_2 = -\frac{3}{2}, c_3 = \frac{3}{2}$]

Example 3: The matrix of a linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ of $\mathbb{R}^3$ and $(1, 0), (1, 1)$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find ϕ.

Solution: Now

$$\phi(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$$
$$\phi(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$$
$$\phi(1,1,0) = 4(1,0) + 0(1,1) = (4,0)$$

Let $(x, y, z) \in \mathbb{R}^3$ and let $(x, y, z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

or
$$(x, y, z) = (c_2 + c_3, c_1 + c_3, c_1 + c_2)$$

Then $x = c_2 + c_3, y = c_1 + c_3, z = c_1 + c_2$ which gives the solution

$$c_1 = \frac{1}{2}(y + z - x), c_2 = \frac{1}{2}(z + x - y) \text{ and } c_3 = \frac{1}{2}(x + y - z)$$

$$\therefore \quad \phi(x,y,z) = c_1\phi(0,1,1) + c_2\phi(1,0,1) + c_3\phi(1,1,0)$$
$$= c_1(3,2) + c_2(3,1) + c_3(4,0)$$
$$= (3c_1 + 3c_2 + 4c_3, 2c_1 + c_2)$$
$$= \left(\frac{3}{2}(y+z-x) + \frac{3}{2}(z+x-y) + 2(x+y-z), (y+z-x) + \frac{1}{2}(z+x-y)\right)$$
$$= \left(2x + 2y + z, \frac{1}{2}(-x + y + 3z)\right)$$

$\therefore$ ϕ is given by $\phi(x, y, z) = \left(2x + 2y + z, \frac{1}{2}(-x + y + 3z)\right), (x, y, z) \in \mathbb{R}^3$.

Example 4: If $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ then verify by means of the transformation $X = AY$ that $(x_1^2 + x_2^2)$ is transformed to $(y_1^2 + y_2^2)$.

Solution: We have $X = AY$

or
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \quad x_1 = \frac{1}{\sqrt{2}}(y_1 - y_2)$$

$$x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2)$$

$$\therefore \quad x_1^2 + x_2^2 = \frac{1}{2}[(y_1 - y_2)^2 + (y_1 + y_2)^2] = \frac{1}{2}(2y_1^2 + 2y_2^2)$$

$$= (y_1^2 - y_2^2).$$

Example 5: Obtain the matrix of the linear mapping ϕ where

(*i*) $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\phi(x, y, z) = (x + 2y - z, x + z, y + 2z)$.

(*ii*) $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\phi(x, y, z) = (x + y + 2z, 3y - 2z)$.

(*iii*) $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $\phi(x, y) = (2x + y,\ x - y, y, x + 3y)$.

Solution: Since the bases have not been mentioned, we work with the standard bases.

(*i*) We first note how the basis vectors are transformed.

$$\phi(1, 0, 0) = (1 + 2\,.\,0 - 0,\ 1 + 0,\ 0 + 2\,.\,0)$$
$$= (1, 1, 0) = 1(1, 0, 0) + 1\,(0, 1, 0) + 0\,(0, 0, 1)$$
$$\phi(0, 1, 0) = (0 + 2\,.\,1 - 0,\ 0 + 0,\ 1 + 2\,.\,0) = (2, 0, 1)$$
$$= 2\,(1, 0, 0) + 0\,(0, 1, 0) + 1(0, 0, 1)$$
$$\phi(0, 0, 1) = (1 + 2\,.\,0 - 1,\ 0 + 1,\ 0 + 2\,.\,1) = (-1, 1, 2)$$
$$= -1(1, 0, 0) + 1(0, 1, 0) + 2(0, 0, 1)$$

Hence, the matrix of the linear mapping ϕ is given by

$$M(\phi) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Note that the coefficients in the linear expression of the transformed vectors have constituted the columns of the matrix.

(*ii*) Here $\phi(1, 0, 0) = (1 + 0 + 2\,.\,0,\ 3\,.\,0 - 2\,.\,0) = (1, 0)$

$$= 1(1, 0) + 0(0, 1)$$
$$\phi(0, 1, 0) = (0 + 1 + 2\,.\,0,\ 3\,.\,1 - 2\,.\,0) = (1, 3)$$
$$= 1(1, 0) + 3(0, 1)$$
$$\phi(0, 0, 1) = (0 + 0 + 2.1,\ 3.0 - 2.1)$$
$$= 2(1, 0) - 2(0, 1)$$

Hence, $M(\phi) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & -2 \end{pmatrix}$.

(*iii*) Here $\phi(1, 0) = (2 \,.\, 1 + 0, 1 - 0, 1 + 3 \,.\, 0) = (2, 1, 1)$

$= 2(1, 0, 0) + 1\,(0, 1, 0) + 1(0, 0, 1)$

$\phi(1, 0) = (2 \,.\, 0 + 1, 0 - 1, 0 + 3 \,.\, 1) = (1, -1, 3)$

$= 1(1, 0, 0) - 1(0, 1, 0) + 3(0, 0, 1)$

Hence, $M(\phi) = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 3 \end{pmatrix}$.