Composition of Linear Mapping

Definition: If $\varphi : V \to W$ and $\Psi : W \to T$ are two linear mappings (i.e, transformations) where V, W and T are vector spaces, then the composite mapping $\Psi_0\phi$ from V to T is defined as $\Psi_0\phi(x) = \Psi(\phi(x))$, $x \in V$, called the product of the mappings ϕ and Ψ.

Note that the composite mapping $\Psi_0\phi : V \to T$ is linear.

To see this, we take $x, y \in V$, $\alpha, \beta \in F$, the common scalar field clearly,

$$\begin{aligned} \Psi_0\phi\,(\alpha x + \beta y) &= \Psi(\phi(\alpha x + \beta y)) \\ &= \Psi(\alpha\phi(x) + \beta\phi(y)), \text{ since } \phi \text{ is linear} \\ &= \alpha\Psi(\phi(x)) + \beta\Psi\,(\phi(y) \text{ since } \Psi \text{ is linear} \\ &= \alpha(\Psi_0\phi)(x) + \beta(\Psi_0\phi)\,(y) \end{aligned}$$

Hence $\Psi_0\phi$ is linear.

Remark 1: If $V = W = T$, then $\phi_0\Psi$ is also defined.

It is straightforward to prove $\phi_0\Psi \neq \Psi_0\phi$

We give an example below:

Let $\quad V = W = T = \mathbb{R}^2$ and $F = \mathbb{R}^2$.

Let $\quad \varphi(x_1, x_2) = (x_2, 2x_1)$ and $\Psi(x_1, x_2) = (x_1 + x_2, x_2)$

It is easy to check that ϕ and Ψ are linear.

Now $\quad \varphi_0\Psi\,(x_1, x_2) = \phi(\Psi(x_1, x_2) = \varphi(x_1 + x_2, x_2) = (x_2, 2x_1, 2x_2)$

$\quad \Psi_0\phi(x_1, x_2) = \Psi(\phi(x_1, x_2)) = \Psi(x_2, 2x) = (x_2 + 2x_1, 2x_1)$

clearly, $\phi_0\Psi \neq \Psi_0\phi$.

Remark 2: If $\varphi : V \to W, \Psi : W \to T, \chi : T \to S$ are linear mappings where V, W, T and S are vector spaces over the same field, then

$$(\chi_0\Psi)\,\phi = \chi_0(\Psi_0\phi) \qquad \text{[Associative Property]}$$

The proof of the above routine.

We can now prove the following.

Theorem: Let $\phi : V \to V$ be a linear mapping and $i_v : V \to V$ be the identity mapping $i_v(x) = x$ for all $x \in V$.

Than the following are true.

(*i*) If φ is bijective, then there is a unique linear mapping $\Psi : V \to V$ such that $\varphi_0\Psi = \Psi_0\phi = i_v$

(*ii*) If a linear mapping Ψ exists and $\varphi_0\Psi = \Psi_0\phi = i_v$, then φ is bijective.

Proof: (*i*) Since φ is onto, there exists $x \in V$ such that $\varphi(y) = x$.

As φ is one-one, so y is unique.

Clearly $\Psi(x) = y$ if and only if $\phi(y) = x \quad \forall x \in V$.

Then for each $x \in V$, there exists unique $y \in V$ such that $\Psi(x) = y$.

Hence Ψ is a mapping from V to V and

$$(\Psi_0\phi)(y) = \Psi(\varphi(y)) = \Psi(x) = y \quad \forall y \in V.$$

This means $\quad \Psi_0\phi = i_v$.

Similarly $\quad (\varphi_0\Psi)(x) = \varphi(\Psi(x)) = \varphi(y) = x \quad \forall x \in V.$

This means $\quad \varphi_0\Psi = i_v$.

(*ii*) Suppose φ is not one-one.

Then for some $x, y \in V, x \neq y, \varphi(x) = \varphi(y) = x \,\forall\, V$.

we have $\quad \Psi_0\varphi = i_v$

$\therefore \quad (\Psi_0\phi)(x) = i_v(x)$ implies $\Psi(\varphi(x)) = x_1$, i.e., $\Psi(x) = y$

and $\quad (\Psi_0\phi)(y) = i_v(y)$ implies $\Psi(\phi(y)) = y$, i.e., $\Psi(x) = y$

This means that Ψ is not a mapping which is a contradiction.

Hence φ is one-one.

Next suppose ϕ is not onto i.e., there exists an element $x \in V$ such that x is not the image of any element of V under φ.

We have $\varphi_0\Psi = i_v$, i.e., $(\varphi_0\Psi)(x) = i_v(x) = x \quad \forall x \in V$

i.e., $\quad \varphi(\Psi(x)) = x$

$\therefore \quad \Psi(x) \in V$. So x is the image of an element of V under ϕ. This is a contradiction. Hence φ is onto.

To prove uniqueness, we assume that there exists a mapping $\chi : V \to V$, such that $\chi_0\varphi = \varphi_0\chi = i_v$.

Let x be an arbitrary element of V.

Since φ is onto, these exists $y \in V$ such that $\varphi(y) = x$.

Now $\quad \Psi(x) = \Psi(\phi(y)) = (\Psi_0\phi)(y) = i_v(y) = y$

and $\quad \chi(x) = \chi(\phi(y)) = (\chi_0\phi)(y) = i_v(y) = y$

Thus $\quad \Psi(x) = \chi(x)$, x arbitrary. So $\Psi = \chi$. Hence Ψ is unique.

Inverse of a Linear Mapping

Definition: Let $\varphi : V \to W$ be a linear mapping where V and W vector spaces over F. A mapping $\Psi : W \to V$ is called an universe of ϕ if $\varphi_0\Psi = i_w$, $\Psi_0\varphi = i_v$. It is usually denoted by φ–1. If a linear mapping $\varphi : V \to W$ has an inverse, it is called invertible.

We can prove the following.

Theorem: If V and W are vector spaces over the same field F, a linear mapping $\varphi : V \to W$ is invertible, then

(*i*) φ^{-1} is linear

(*ii*) φ is bijective

(*iii*) φ is unique.

Proof: Let $\Psi : W \to V$ be a mapping such that $\varphi\Psi = i_w$, $\Psi\phi = i_v$.

Let $w_1, w_2 \in W$ and $\psi(w_1) = v_1$, $\Psi(w_2) = v_2$, $v_1, v_2 \in V$, $w_1, w_2 \in W$ clearly $\varphi(\Psi(w_1)) = \varphi(v_1)$ implies $i_w(w_1) = \phi(v)$, i.e., $w_1 = \phi(v_1)$

Similarly, $w_2 = \phi(v_2)$.

Since ϕ is linear.

$$\varphi(av_1 + bv_2) = a\,\varphi(v_1) + b\varphi(v_2), \quad a, b \text{ are scalars, implies}$$

$$\varphi(av_1 + bv_2) = a\,w_1 + bw_2$$

$\therefore \quad \Psi(\varphi(av_1 + bv_2)) = \Psi(aw_1 + bw_2)$

This implies $\quad i_v(av_1 + bv_2) = \Psi(aw_1 + bw_2)$

$$av_1 + bv_2 = \Psi\,(aw_1 + bw_2) \qquad [\because \; av_1 + bv_2 \in V]$$

i.e., $\quad a\,\Psi(w_1) + b\Psi\,(bw) = \Psi\,(aw_1 + bw_2)$

But this means that Ψ is linear i.e., φ^{-1} is linear.

(*ii*) Let $\phi : V \to W$ be invertible. Then there exists $\Psi : W \to V$ such that $\phi_0\Psi = i_w$ and $\Psi_0\phi = i_v$

Let $\quad v_1, v_2 \in V$ and $\phi(v_1) = \phi(v_2)$.

Then $\quad \Psi(\phi(v_1)) = \Psi(\phi(v_2))$

$\therefore \quad i_v(v_1) = i_v(v_2)$ or $v_1 = v_2$

Therefore ϕ is one-one.

Let $w \in W$. Then $\phi(\psi(w)) = i_w(w) = w$, $\Psi(w) \in V$.

$\therefore \quad w_1$ is the image of $\Psi(w)$ under ϕ. This implies ϕ is onto.

Thus ϕ is one-one and onto, i.e., bijective.

(*iii*) If possible, let there be two inverses $\Psi : W \to V$ and $\chi : W \to V$.

Then $\quad \varphi_0\Psi = i_w = \phi_0\Psi$ and $\Psi_0\varphi = i_v = \chi\phi$

Let $w \in W$ be arbitrary.

Since ϕ is onto, there exists $v \in V$ such that $\phi(v) = w$.

Now $\quad \Psi(w) = \Psi(\phi(v)) = (\Psi_0\phi)(v) = i_v(v_1) = v$

and $\quad \chi(w) = \chi(\varphi(v)) = (\chi_0\phi)(v) = i_v(v) = v$

$\therefore \quad \Psi(w) = \chi(w)$

As w is arbitrary, $\Psi = \chi$. This proves the uniqueness of ϕ.

Example 1: If $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\varphi(x, y, z) = (2x, 4x - y, 2x + 3yz)$ then show that φ is invertible.

Solution: Let $(x, y, z) \in \ker \phi$. Then $\phi(x, y, z) = (0, 0, 0)$. This gives $2x = 0$, $4x - y = 0$, $2x + 3y - z = 0$.

This system has only the trivial solution $(0, 0, 0)$, i.e., $\ker \phi = \{0\}$.

Hence ϕ is one-one. As ϕ is onto also, ϕ is invertible.

To find out the inverse of ϕ, we take $\phi(x, y, z) = (r, s, t)$, say

Then $\quad 2x = r,\ 4x - y = s \quad \text{and} \quad 2x + 3y - z = t$

or $\quad x = r/2,\ y = 2r - s,\ z = 7r - 3s - t$

Hence $\varphi^{-1}(r, s, t) = (r/2, 2r - s, 7r - 3s - t)$

Example 2: If S and T are linear operators on $\mathbb{R}^2$ defined as $S(x, y) = (y, x)$, $T(x, y) = (0, x)$, find ST, TS, S^2 and T^2.

Solution: We see

$$(ST)(x, y) = S(T(x, y)) = S(0, x) = (x, 0)$$
$$(TS)(x, y) = T(S(x, y)) = T(y, x) = (0, y)$$
$$S^2(x, y) = S(S(x, y)) = S(y, x) = (x, y)$$
$$T^2(x, y) = T(T(x, y)) = T(0, x) = (0, 0)$$

Note that $ST \neq TS$ and $S^2 = i$.