11.6 RANK AND NULLITY

If $\phi : V \rightarrow W$ is a linear mapping from a vector space V to the vector space W, then there are two important concepts associated with ϕ. These are rank and nullity defined as follows.

Definition: The *rank* of a linear mapping $\phi : V \rightarrow W$ is defined to be the dimension of $\mathrm{Im}(\phi)$ and the *nullity* of ϕ is defined to be the demension of $\ker \phi$.

Note these are defined as Im(ϕ) and ker ϕ are vector spaces.

If $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\phi(x, y, z) = (2x + y - z, x + y + z)$ then evidently rank (ϕ) = 2, nullity (ϕ) = 1.

A result of prime importance is the following:

Theorem: If $\phi : V \to W$ be a linear mapping from a finite dimensional space V to another space W, then

$$\dim (V) = \text{rank} (\phi) + \text{nullity} (\phi)$$

The proof of this theorem is outside the scope of this book. The reader is advised to verifiy the theorem for all the linear mappings of the above example.

Example 1: A mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\phi(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Show that ϕ is a linear mapping. Find ker ϕ and the dimenision of ker ϕ, Im(ϕ) and dimension of Im(ϕ).

Solution: Let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in \mathbb{R}^3$, then

$$\left.\begin{aligned} \phi(\alpha) &= (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) \\ \phi(\beta) &= (y_1 + y_2 + y_3, 2y_1 + y_2 + 2y_3, y_1 + 2y_2 + y_3) \end{aligned}\right\} \quad ...(1)$$

$\therefore\ \alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$$\begin{aligned} \therefore\ \phi(\alpha+\beta) &= ((x_1+y_1)+(x_2+y_2)+(x_3+y_3), 2(x_1+y_1)+(x_2+y_2)+2(x_3+y_3), \\ &\qquad (x_1+y_1)+2(x_2+y_2)+(x_3+y_3)) \\ &= ((x_1+x_2+x_3)+(y_1+y_2+y_3), (2x_1+x_2+2x_3)+(2y_1+y_2+2y_3), \\ &\qquad (x_1+2x_2+x_3)+(y_1+2y_2+y_3)) \\ &= (x_1+x_2+x_3, 2x_1+x_2+2x_3, x_1+2x_2+x_3)+(y_1+y_2+y_3, 2y_1+y_2+2y_3, \\ &\qquad y_1+2y_2+y_3) \\ &= T(\alpha) + T(\beta) \quad \text{by (1)} \end{aligned}$$

Let $c \in \mathbb{R}$, then $c\alpha = (cx_1, cx_2, cx_3)$

$$\begin{aligned} \therefore \quad \phi(c\,\alpha) &= (cx_1 + cx_2 + cx_3, 2cx_1 + cx_2 + 2cx_3, cx_1 + 2cx_2 + cx_3) \\ &= c(x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) \\ &= c\phi(\alpha) \quad \text{(by (1))} \end{aligned}$$

$\therefore\ \phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ for all $\alpha, \beta \in \mathbb{R}^3$.

and $\phi(c\alpha) = c\,\phi(\alpha)$ for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}^3$.

Hence, ϕ is a linear mapping.

Now, ker $\phi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \phi(x_1, x_2, x_3) = (0, 0, 0)\}$

Let $(x_1, x_2, x_3) \in$ ker ϕ, then

$$(x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) = (0, 0, 0)$$

This gives
$$x_1 + x_2 + x_3 = 0$$
$$2x_1 + x_2 + 2x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
From the first two equations, we get
$$\frac{x_1}{2-1} = \frac{x_2}{2-2} = \frac{x_3}{1-2} \text{ i.e., } \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k \text{ (say)}$$
$\therefore$ $x_1 = k, x_2 = 0, x_3 = -k$ and the last equation is satisfied.

$\therefore$ $(x_1, x_2, x_3) = k(1, 0, -1), k \in \mathbb{R}$

Let $\alpha = (1, 0, -1)$, then $\ker \phi = L\{\alpha\}$ and $\dim \ker \phi = 1$

Let ξ be an arbitrary vector in $\text{Im}(\phi)$.

Then $\xi = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3)$
$$= x_1 (1, 2, 1) + x_2 (1, 1, 2) + x_3 (1, 2, 1)$$
Thus, ξ is a linear combination of the vectors (1, 2, 1), (1, 1, 2).

Hence, $\text{Im}(\phi) = L\{(1, 2, 1), (1, 1, 2)\}$

Then we shall prove that the vectors (1, 2, 1), (1, 1, 2) are linearly independent.

Let us consider the relation $c_1 (1, 2, 1) + c_2 (1, 1, 2) = (0, 0, 0)$, where $c_1, c_2 \in \mathbb{R}$

$\Rightarrow$ $(c_1 + c_2, 2c_1 + c_2, c_1 + 2c_2) = (0, 0, 0)$

$\Rightarrow$ $c_1 + c_2 = 0, 2c_1 + c_2 = 0, c_1 + 2c_2 = 0$. This gives $c_1 = c_2 = 0$.

Hence, the set $\{(1, 2, 1), (1, 1, 2)\}$ is linearly independent and the dimension of $\text{Im}(\phi)$ is 2.

Example 2: A linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by

$\phi(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3), (x_1, x_2, x_3) \in \mathbb{R}^3$.

Find $\ker \phi$ and verify that $\{\phi(\epsilon_1), \phi(\epsilon_2), \phi(\epsilon_3)\}$ is linearly independent set in $\mathbb{R}^4$ where $\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0), \epsilon_3 = (0, 0, 1)$ and also find $\text{Im}(\phi)$ and the dimension of $\text{Im}(\phi)$.

Solution: Now $\ker \phi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \phi(x_1, x_2, x_3) = (0, 0, 0, 0)\}$

Let $(x_1, x_2, x_3) \in \ker \phi$, then $\phi(x_1, x_2, x_3) = (0, 0, 0, 0)$

$\therefore$ $(x_1 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3) = (0, 0, 0, 0)$

This gives $x_2 + x_3 = 0, x_3 + x_1 = 0, x_1 + x_2 = 0, x_1 + x_2 + x_3 = 0$

The solution is $x_1 = x_2 = x_3 = 0$

$\therefore$ $\ker \phi = \{\theta\}$

Now $\phi(\epsilon_1) = \phi(1, 0, 0) = (0 + 0, 0 + 1, 1 + 0, 1 + 0 + 0) = (0, 1, 1, 1)$
$$\phi(\epsilon_2) = \phi(0, 1, 0) = (1 + 0, 0 + 0, 0 + 1, 0 + 1 + 0) = (1, 0, 1, 1)$$

$$\phi(\in_3) = \phi(0, 0, 1) = (0 + 1, 1 + 0, 0 + 0, 0 + 0 + 1) = (1, 1, 0, 1)$$

Let us consider the relation $c_1\ \phi(\in_1) + c_2\ \phi(\in_2) + c_3\ \phi(\in_3) = \theta$ where $c_1, c_2, c_3 \in \mathbb{R}$

or $c_1\ (0, 1, 1, 1) + c_2\ (1, 0, 1, 1) + c_3(1, 1, 0, 1) = (0, 0, 0, 0)$

or $(c_2 + c_3, c_1 + c_3, c_1 + c_2, c_1 + c_2 + c_3) = (0, 0, 0, 0)$

This gives $c_2 + c_3 = 0, c_1 + c_3 = 0, c_1 + c_3 = 0$ and $c_1 + c_2 + c_3 = 0$

The solution is $c_1 = c_2 = c_3 = 0$.

This proves that $\{\phi(\varepsilon_1), \phi(\varepsilon_2), \{\phi(\varepsilon_3)\}$ is linearly independent set. $\text{Im}(\phi)$ is a linear span of the vectors $\phi(\varepsilon_1), \phi(\varepsilon_2), \phi(\varepsilon_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the basis of $\mathbb{R}^3$.

Since $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis of $\mathbb{R}^3$.

$$\therefore \quad \text{Im}(\phi) = L\{\phi(\varepsilon_1), \phi(\varepsilon_2), \phi(\varepsilon_3)\}$$
$$= L\ \{(0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1)$$

Since the set $\{\phi(\varepsilon_1), \phi(\varepsilon_2), \phi(\varepsilon_3)$ is linearly independent set.

The dimension of $\text{Im}\phi$ is 3.

Example 3: Prove that the linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\phi(x, y, z) = (x + y, y + z, z + x) \in \mathbb{R}^3 \text{ is one-to-one and onto.}$$

Solution: Now, $\ker \phi = \{(x, y, z) \in \mathbb{R}^3 : \phi\ (x, y, z) = (0, 0, 0)\}$

Let $(x, y, z) \in \ker \phi$, then $\phi\ (x, y, z) = (0, 0, 0)$

or $(x + y, y + z, z + x) = (0, 0, 0)$

This gives $x + y = 0, y + z = 0, z + x = 0$

The solution is $x = y = z = 0$.

$\therefore \quad \ker \phi = \{\theta\}$ and hence ϕ is one-to-one.

The standard basis of $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Now, $\phi(1, 0, 0) = (1 + 0, 0 + 0, 0 + 1) = (1, 0, 1)$

$\phi(0, 1, 0) = (0 + 1, 1 + 0, 0 + 0) = (1, 1, 0)$

$\phi(0, 0, 1) = (0 + 0, 0 + 1, 1 + 0) = (0, 1, 1)$

$\text{Im}(\phi)$is the linear span of the vectors $\phi(1, 0, 0), \phi(0, 1, 0), \phi(0, 0, 1)$

$\therefore \quad \text{Im}(\phi) = L\{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$

Then, we shall prove that the set $\{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$ is linearly independent.

Let us consider the relation $c_1\ (1, 0, 1\ ) + c_2\ (1, 1, 0) + c_3\ (0, 1, 1) = (0, 0, 0)$ where $c_1, c_2, c_3 \in \mathbb{R}$.

$\therefore \quad (c_1 + c_2, c_2 + c_3, c_1 + c_3) = (0, 0, 0)$

This gives $c_1 + c_2 = 0, c_2 + c_3 = 0, c_1 + c_3 = 0$.

The solution is $c_1 = c_2 = c_3 = 0$.

Hence, the set $\{(1, 0, 1), (1, 1, 0)\ (0, 1, 1)\}$ is linearly independent set.

$\therefore \quad \text{Im}(\phi) = \mathbb{R}^3$.

and therefore, ϕ is an onto mapping.

Example 4: Determine the linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1), (2, 3), (-1, 2)$ respectively. Find $\phi(1, 2, 0)$.

Solution: Let $\xi = (x, y, z)$ be an arbitrary vector of $\mathbb{R}^3$

$$\therefore \quad \xi = (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

Since ϕ is a linear, $\phi(\xi) = x\,\phi(1, 0, 0) + y\,\phi(0, 1, 0) + z\,\phi(0, 0, 1)$

$$= x\,(1, 1) + y\,(2, 3\,) + z\,(-1, 2)$$

$$= (x + 2y - z, x + 3y + 2z)$$

$\therefore$ ϕ is defined by $\phi\,(x, y, z) = (x + 2y - z, x + 3y + 2z), (x, y, z) \in \mathbb{R}^3$

$$\therefore \quad \phi(1, 2, 0) = (1 + 4 - 0, 1 + 6 + 0) = (5, 7).$$

Example 5: Determine the linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0), (0, 1, 0)\ (0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1), (2, 3), (3, 2)$ respectively. Find $\ker\phi$ and $\text{Im}(\phi)$.

Solution: Let $\xi = (x, y, z)$ be an arbitrary vector of $\mathbb{R}^3$

$$\therefore \quad \xi = (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z\,(0, 0, 1)$$

Since ϕ is linear, then $\phi(\xi) = x\phi(1, 0, 0) + y\phi(0, 1, 0) + z\,\phi(0, 0, 1)$

$$= x\,(1, 1) + y(2, 3) + z(3, 2)$$

$$= (x + 2y + 3z, x + 3y + 2z)$$

$\therefore$ ϕ is defined by $\phi(x, y, z) = (x + 2y + 3z, x + 3y + 2z), (x, y, z) \in \mathbb{R}^3$.

Now, $\quad \ker\phi = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = (0, 0)\}$

Let $\quad \alpha = (x, y, z) \in \ker\phi$, then $\phi(\alpha) = \theta$

$$\therefore \quad \phi(x, y, z) = (0, 0) \Rightarrow (x + 2y + 3z, x + 3y + 2z) = (0, 0)$$

This gives $\quad x + 2y + 3z = 0$

$$x + 3y + 2z = 0$$

By the cross-multiplication, we get $\dfrac{x}{4-9} = \dfrac{y}{3-2} = \dfrac{z}{3-2}$ *i.e.* $\dfrac{x}{-5} = \dfrac{y}{1} = \dfrac{z}{1}$ $= k$ (say)

$$\therefore \quad x = -5k, y = k, z = k$$

$\therefore \quad (x, y, z) = k(-5, 1, 1)$ where $k \in \mathbb{R}$.

$\therefore \quad \ker\phi = L(\alpha)$ where $\alpha = (-5, 1, 1)$

Im(ϕ) is linear span of the vectors $\phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of $\mathbb{R}^3$.

Since $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is a basis of $\text{Im}(\phi) = L\{(1, 1), (2, 3), (3, 2)\}$.

Example 6: Determine the linear mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ of $\mathbb{R}^3$ to $(1, 1, 1), (1, 1, 1), (1, 1, 1)$ respectively. Verify that $\dim(\ker\phi) + \dim(\text{im}\,\phi) = 3$

Solution: Let $\xi = (x, y, z)$ be an arbitrary vector of the domain set $\mathbb{R}^3$.

Let $\quad \xi = c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$ where $c_1, c_2, c_3 \in \mathbb{R}$

or $\quad (x, y, z) = (c_2 + c_3, c_1 + c_3, c_1 + c_2)$

$\therefore \; c_2 + c_3 = x, c_1 + c_3 = y, c_1 + c_2 = z$

$\therefore \; (2c_1 + c_2 + c_3) - (c_2 + c_3) = y + z - x \Rightarrow c_1 = \dfrac{y + z - x}{2}$

$\therefore \; c_2 = \dfrac{z + x - y}{2}, c_3 = \dfrac{x + y - z}{2}$

Since ϕ is linear, then

$$\begin{aligned} \therefore \quad \phi(\xi) &= c_1\phi(0, 1, 1) + c_2\phi(1, 0, 1) + c_3\phi(1, 1, 0) \\ &= c_1(1, 1, 1) + c_2(1, 1, 1) + c_3(1, 1, 1) \\ &= (c_1 + c_2 + c_3, c_1 + c_2 + c_3, c_1 + c_2 + c_3) \\ &= \left(\frac{x + y + z}{2}, \frac{x + y + z}{2}, \frac{x + y + z}{2}\right) \end{aligned}$$

Hence, ϕ is defined by $\phi(x, y, z) = \left(\dfrac{x + y + z}{2}, \dfrac{x + y + z}{2}, \dfrac{x + y + z}{2}\right)$, $(x, y, z) \in \mathbb{R}$.

Now, $\ker\phi = \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z) = \theta\}$

Let $(x, y, z) \in \ker\phi$, then $\phi(x, y, z) = \theta$

i.e., $\quad \left(\dfrac{x + y + z}{2}, \dfrac{x + y + z}{2}, \dfrac{x + y + z}{2}\right) = (0, 0, 0)$

$\therefore \quad x + y + z = 0$

Let $\quad y = c, z = d$, then $x = -c - d$

$\therefore \quad (x, y, z) = (c, d, -c - d) = c(1, 0, -1) + d(0, 1, -1)$ where $c, d \in \mathbb{R}$.

Hence, $\quad \ker\phi = L\{(1, 0, -1), (0, 1, -1)\}$

Let us consider the relation $c_1(1,0,-1)+c_2(0,1,-1)=(0,0,0), c_1, c_2 \in \mathbb{R}$.

or $\quad (c_1, c_2, -c_1-c_2)=(0,0,0) \Rightarrow c_1=c_2=0$

Hence, $\{(1,0,-1),(0,1,-1)\}$ is a linearly independent set.

So $\quad \dim \ker \phi = 2$

$\text{Im}(\phi)$ is a linear span of the vectors $\phi(\alpha), \phi(\beta), \phi(\gamma)$ where $\{\alpha, \beta, \gamma\}$ is any basis of the domain space $\mathbb{R}^3$.

Since $(0,1,1),(1,0,1),(1,1,0)$ is a basis of $\mathbb{R}^3$, $\text{Im}(\phi)=L\{(1,1,1)\}$

$\therefore \quad \dim \text{Im}(\phi)=1$

$\therefore \quad \dim \ker \phi + \dim \text{Im}(\phi)=2+1=3$ (verified).

Example 7: Determine the linear mapping $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1),(1,0,1),(1,1;0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0),(0,2,0),(0,0,2)$ respectively. Find $\ker \phi$ and $\text{Im}(\phi)$. Verify that $\dim(\ker \phi)+\dim \text{im } \phi=3$.

Solution: Let $\xi=(x,y,z)$ be an arbitrary vector in $\mathbb{R}^3$.

Let $\xi = c_1(0,1,1)+c_2(1,0,1)+c_3(1,1,0)$ where $c_1, c_2, c_3 \in \mathbb{R}$

or $\quad (x,y,z)=(c_2+c_3, c_1+c_3, c_1+c_2)$

$\therefore \quad c_2+c_3=x, c_1+c_3=y, c_1+c_2=z$

$$\therefore \quad c_1=\frac{x+z-x}{2}, c_2=\frac{x+z-y}{2} \quad c_3=\frac{x+z-z}{2}$$

Since ϕ is linear, then

$$\begin{aligned}\phi(\xi) &= c_1\phi(0,1,1)+c_2\phi(1,0,1)+c_3\phi(1,1,0)\\ &= c_1(2,0,0)+c_2(0,2,0)+c_3(0,0,2)\\ &= (2c_1, 2c_2, 2c_3)\\ &= (y+z-x, x+z-y, x+y-z)\end{aligned}$$

$\therefore \phi$ is defined by $\phi(x,y,z)=(y+z-x, x+z-y, x+y-z), (x,y,z) \in \mathbb{R}^3$

Now $\quad \ker \phi = \{(x,y,z) \in \mathbb{R}^3 : \phi(x,y,z)=\theta\}$

Let $\quad (x,y,z) \in \ker \phi$, then $\phi(x,y,z)=\theta$

or $\quad (y+z-x, x+z-y, x+y-z)=(0,0,0)$

$\therefore \quad y+z-x=0, x+z-y=0, x+y-z=0$

This gives the solution $x=y=z=0$

Hence, $\quad \ker \phi = \{\theta\}$ and $\dim(\ker \phi)=0$

$\text{Im}\phi$ is a linear span of the vectors $\phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3)$ where $\{\alpha_1, \alpha_2, \alpha_3\}$ is any basis of the domain $\mathbb{R}^3$.

Since $\{(0,1,1),(1,0,1),(1,1,0)\}$ is a basis of $\mathbb{R}^3$, then

$$\text{Im}\phi = L\{(2,0,0),(0,2,0)(0,0,2)\}$$

Since the set of vectors $\{(2,0,0),(0,2,0),(0,0,2)\}$ is linearly independent.

$$\therefore \quad \dim \text{Im}\phi = 3$$

Hence, $\dim(\ker\phi) + \dim(\text{Im}\phi) = 0 + 3 = 3$.