11.5 LINEAR MAPPING

A linear mapping is a mapping between two vector spaces which preserve the linearity structure of the vector spaces. The formal definition goes as follows:

Definition: A mapping $\phi : V \rightarrow W$ from a vector space V to another vector space W is a mapping from V to W which satisfies the condition

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y) \text{ for } x, y \in V, \alpha, \beta \in \mathbb{R}$$

or

(*i*) $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta) =$ for all $\alpha, \beta \in V$ and

(*ii*) $\phi(c\alpha) = c\phi(\alpha)$ for all $c \in \mathbb{R}$ and $\alpha \in V$.

Example: Prove that the following mappings are linear:

(*i*) $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\phi(x, y) = (2x + y, x - y)$.

(*ii*) $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $\phi(x, y) = (x + y, x - y, 2x)$.

(*iii*) $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\phi(x, y, z) = (2x + y - z, x + y + z)$.

(*iv*) $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is defined by $\phi(x, y, z) = x + 2y + 3z$.

Solution: (*i*) Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$.

Then $\phi(\alpha(x_1, y_1) + \beta(x_2, y_2) = \phi(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$$= (2(\alpha x_1 + \beta x_2) + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2)$$

$$= (2\alpha x_1 + 2\beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2)$$

$$= \alpha(2x_1 + y_1, x_1 - y_1) + \beta(2x_2 + y_2, x_2 - y_2)$$

$$= \alpha\phi(x_1, y_1) + \beta\phi(x_2, y_2)$$

Hence, ϕ is linear.

(*ii*) Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$.

Then $\phi(\alpha(x_1, y_1) + \beta(x_2, y_2) = \phi(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2, 2\alpha x_1 + 2\beta x_2)$$

$$= \alpha(x_1 + y_1, x_1 - y_1, 2x_1) + \beta(x_2 + y_2, x_2 - y_2, 2x_2)$$

$$= \alpha\phi(x_1, y_1) + \beta\phi(x_2, y_2)$$

Hence, ϕ is linear.

(*iii*) Let $(x_1, y_1\ z_1), (x_2, y_2\ z_2) \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$.

Then $\phi(\alpha(x_1, y_1, z_1), + \beta(x_2, y_2, z_2) = \phi(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$$= (2\alpha x_1 + 2\beta x_2 + \alpha y_1 + \beta y_2 - \alpha z_1 - \beta z_2, \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + \alpha z_1 + \beta z_2)$$

$$= \alpha(2x_1 + y_1 - z_1, x_1 + y_1 + z_1) + \beta(2x_2 + y_2 - z_2, x_2 + y_2 + z_2)$$

$$= \alpha\phi(x_1, y_1\ z_1) + \beta\phi(x_2, y_2, z_2)$$

Hence, ϕ is linear.

The following result about linear mapping plays a crucial role in many derivations.

Theorem: The following are true about a linear mapping $\phi : V \to W$:

(*i*) $\phi(\theta) = \theta'$ where θ and θ' are the null vectors of V and W respectively.

(*ii*) $\phi(x + y) = \phi(x) + \phi(y)$. $\quad \forall\, x, y \in V$

(*iii*) $\phi(\alpha x) = \alpha\phi(x)$. $\quad \forall\, x, \in V, \alpha \in \mathbb{R}$

(*iv*) $\phi(x - y) = \phi(x) - \phi(y)$.

Proof: (*i*) Taking $\alpha = 1, \beta = -1, y = x$, we get

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$$

or $\quad \phi(x - y) = 1\phi(x) - 1\phi(x) = \phi(x) - \phi(x) = \theta'$

or $\quad \phi(\theta) = \theta'$.

(*ii*) Taking $\alpha = \beta = 1$, we get

$$\phi(x + y) = \phi(1x + 1y) = 1\phi(x) + 1\phi(y) = \phi(x) + \phi(y)$$

(*iii*) Taking $\alpha = 1, \beta = 0$, we get

$$\phi(\alpha x + 0y) = \alpha\phi(x) + 0\,\phi(y)$$

or $\quad \phi(\alpha x) = \alpha\phi(x)$.

(*iv*) Taking $\alpha = 1, \beta = -1$, we get

$$\phi(x - y) = \phi(1x + (-1)y) = 1\phi(x) + (-1)\phi(y) = \phi(x) - \phi(y).$$

Definition. The kernel of a linear mapping $\phi : V \to W$ is defined as the set of all those elements of V whose images are the null vector of W. This is denoted by ker ϕ or $\phi^{-1}\{\theta'\}$.

Thus, $\ker \phi = \{x \in V; \phi(x) = \theta'\}$.

The image of a linear mapping $\phi : V \to W$, denoted by Im(ϕ) or $\phi(V)$ is defined as the set of images of all the elements of V.

Thus, $\text{Im}\phi = \{\phi(x); x \in V\}$.

Theorem: If $\phi : V \to W$ is a linear mapping, then

(*i*) ker ϕ is a subspace of V.

(*ii*) Imϕ is a subspace of V.

Proof: (*i*) Let $x, y \in \ker\phi, \alpha, \beta \in \mathbb{R}$.

Then, $\phi(x) = \theta', \phi(y) = \theta'$ and $\alpha x + \beta y \in V$.

Hence, $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$ since ϕ is linear

$$= \alpha\theta' + \beta\theta' = \theta'.$$

This implies $\alpha x + \beta y \in \ker\phi$. Thus ker ϕ is a subspace of V.

(*ii*) Let $u, v \in \text{Im}(\phi), \alpha, \beta \in \mathbb{R}$.

So, there exists $x, y \in V$ such that $u = \phi(x), v = \phi(y)$.

$\therefore\ \alpha x + \beta y \in V$ and $\alpha u + \beta v = \alpha\phi(x) + \beta\phi(y) = \phi(\alpha x + \beta y) \in \phi(V)$

Hence, $\phi(V)$ *i.e.*, Im(ϕ) is a subspace of W.

Remark: $\ker\phi \neq \phi$ since $\theta \in \ker\phi$.

Note: (*i*) ker ϕ is called the null space of ϕ and is denoted by $N(\phi)$.

(*ii*) Im(ϕ) is also called range of ϕ and is denoted by $R(\phi)$.

Theorem: Let $\phi : V \to W$ be a linear mapping, then, ϕ is injective if and only if $\ker \phi = \{\theta\}$.

Proof: Let ϕ be injective. Since $\phi(\theta) = \theta'$ where $\theta \in V$ and $\theta' \in W$, then $\phi(\alpha) \neq \theta'$ for non-zero α in V.

So $\quad \ker \phi = \{\theta\}$.

Conversely, let $\ker \phi = \{\theta\}$ and $\alpha, \beta \in V$ such that $\phi(\alpha) = \phi(\beta)$ in W.

Since $\phi(\alpha) = \phi(\beta) \Rightarrow \phi(\alpha) - \phi(\beta) = \theta'$

$$\Rightarrow \phi(\alpha - \beta) = \theta' \qquad (\because \phi \text{ is linear}).$$

This shows that $\alpha - \beta \in \ker \phi$ and since $\ker \phi = \{\theta\}$, then $\alpha = \beta$

$$\therefore \quad \phi(\alpha) = \phi(\beta) \Rightarrow \alpha = \beta$$

Hence, ϕ is injective.

Note: Let $\phi : V \to W$ be a linear mapping and $\{\alpha_1, \alpha_2, \dots \alpha_n)$ be a basis of V, then $\phi(\alpha_1), \phi(\alpha_2), \dots. \phi(\alpha_n)$ generate $I_m(\phi)$.

Example 1: Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\phi(x_1, x_2, x_3) = (x_1, x_2, 0)$ for $(x_1, x_2, x_3) \in \mathbb{R}^3$, then show that ϕ is linear mapping or linear transformation.

Solution: Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in \mathbb{R}^3$

$$\therefore \quad \alpha + \beta = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$\therefore \quad \phi(\alpha + \beta) = (x_1 + y_1, x_2 + y_2, 0) = (x_1, x_2, 0) + (y_1, y_2, 0)$$

$$= \phi(\alpha) + \phi(\beta) \qquad \dots (1)$$

Let $c \in \mathbb{R}$, then $c\alpha = (cx_1, cx_2, cx_3)$

$$\therefore \quad \phi(c\alpha) = (cx_1, cx_2, 0) = c(x_1, x_2, 0) = c\phi(\alpha) \qquad \dots(2)$$

From (1) and (2), we see that ϕ is linear mapping.

Example 2: Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\phi(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, then show that ϕ is not a linear transformation.

Solution: Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, z_3) \in \mathbb{R}^3$, then

$$\therefore \quad \alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$\therefore \quad \phi(\alpha + \beta) = (x_1 + y_1 + 1, x_2 + y_2 + 1, x_3 + y_3 + 1)$$

$$= (x_1 + 1, x_2 + 1, x_3 + 1) + (y_1, y_2, y_3)$$

$$\neq \phi(\alpha) + \phi(\beta)$$

Hence, ϕ is not a linear transformation.

Example 3: Show that the transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\phi(x, y) = (x - y, x + y, y)$ is a linear transformation.

Solution: Let $\alpha = (x_1, y_1), \beta = (x_2, y_2) \in \mathbb{R}^2$,

$$\left.\begin{aligned} \text{then} \quad \phi(\alpha) &= (x_1 - y_1, x_1 + y_1, y_1) \\ \text{and} \quad \phi(\beta) &= (x_2 - y_2, x_2 + y_2, y_2) \end{aligned}\right\} \quad \ldots(1)$$

Let $a, b \in \mathbb{R}$, then $a\alpha + b\beta \in \mathbb{R}^2$

$$\begin{aligned} \therefore \phi(a\alpha + b\beta) &= \phi[(ax_1, ay_1) + (bx_2, by_2)] = \phi(ax_1 + bx_2, ay_1 + by_2) \\ &= (ax_1 + bx_2 - ay_1 - by_2, ax_1 + bx_2 + ay_1 + by_2, ay_1 + by_2) \\ &= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 + y_1) + b(x_2 + y_2), ay_1 + by_2) \\ &= a(x_1 - y_1, x_1 + y_1, y_1) + b(x_2 - y_2, x_2 + y_2, y_2) \\ &= a\phi(\alpha) + b\phi(\beta), \text{ by } (1) \end{aligned}$$

Hence, ϕ is a linear transformation.

Example 4: If $\phi : V_3 \to V_1$ and $\phi(x_1, x_2, x_3), = x_1^2 + x_2^2 + x_3^2$ then show that ϕ is not a linear transformation.

Solution: Let $\alpha = (1, 0, 0), \beta = (-2, 0, 0) \in V_3$, then $\phi(\alpha) = 1$

and $\phi(\beta) = 4$

$$\therefore \quad \alpha + \beta = (1, 0, 0) + (-2, 0, 0) = (-1, 0, 0)$$

$$\therefore \quad \phi(\alpha + \beta) = (-1)^2 + 0^2 + 0^2 = 1 \neq \phi(\alpha) + \phi(\beta)$$

Hence, ϕ is not a linear transformation.

Example 5: Find ker ϕ and Im(ϕ) where

(*i*) $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\phi(x, y, z) = (x, y, 0)$.

(*ii*) $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\phi(x, y) = (x + y, x - y)$.

(*iii*) $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $\phi(x, y) = (y, x, x + y)$.

(*iv*) $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\phi(x, y, z) = (x + z, y - z)$.

(*v*) $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\phi(x, y, z) = (2x + y - z, x - y + 3z)$.

Solution: (*i*) To determine ker ϕ, we let $\varphi(x, y, z) = \theta'$

and get $\phi(x, y, z) = (0, 0, 0)$ $\therefore$ $x = 0, y = 0$

Hence, $\quad \ker \phi = \{(0, 0, z)\}$ *i.e.*, the z-axis in $\mathbb{R}^3$.

Since $x, y \in \mathbb{R}$, $\phi(\mathbb{R}^3) = \{(x, y, 0); x, y \in \mathbb{R}\}$ = The xy-plane.

(*ii*) Let $\phi(x, y) = \theta' = (0, 0)$. Then $x + y = 0$ and $x - y = 0$.

So $\quad x = y = 0$.

Hence, $\quad \ker \phi = \{\theta'\}$.

To determine Imϕ or $\phi(\mathbb{R}^2)$ here, we note if r_1 and r_2 are two arbitrary real numbers, then taking $\phi(x, y) = (r_1, r_2)$ we get

$$x + y = r_1, x - y = r_2$$

So $x = \frac{1}{2}(r_1 + r_2) \in \mathbb{R}, y = \frac{1}{2}(r_1 - r_2) \in \mathbb{R}$.

Hence, there exists real numbers x and y so that

$$\phi(x, y) = (r_1, r_2)$$

This implies $\phi(\mathbb{R}^2) = \mathbb{R}^2$.

(*iii*) To obtain ker ϕ, we let $\phi(x, y) = \theta'$ and get

$$y = 0, \ x = 0, x + y = 0.$$

Hence, solving, we get $x = 0, y = 0$.

Therefore, ker $\phi = \{\theta\}$ where $\theta = (0, 0)$.

Clearly, $\text{Im}(\phi) \subsetneq \mathbb{R}^2$.

(*iv*) To find ker ϕ, we write $\varphi(x, y) = \theta'$ and get

$$x + z = 0, y - z = 0$$

Hence, ker $\varphi = \{(x, y, z) \in \mathbb{R}^3; x + z = 0, y - z = 0\}$ which is a straight line in $\mathbb{R}^3$.

It is to be noted that all points on this straight line $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1} = t$ *i.e.*, $x = -t, y = t, z = t$, t parameters are mapped onto the null vector of $\mathbb{R}^2$ *e.g.*, $(-1, 1, 1), (2, 2, -2)$.

To obtain the Im(ϕ), we let $\phi(x, y, z) = (r_1, r_2)$ and get therefore $x + z = r_1$, $y - z = r_2$ which has infinitely many solutions.

Hence, for every point $(r_1, r_2) \in \mathbb{R}^2$ there is a point $(x, y, z) \in \mathbb{R}^3$ such that $\phi(x, y, z) = (r_1, r_2)$

This implies $\text{Im}(\phi) = \mathbb{R}^2$.

(*v*) As above, ker $\phi = \{(x, y, z) \in \mathbb{R}^3, 2x + y - z = 0 = x - y + 3z\}$ which is a straight line in $\mathbb{R}^3$ and $\text{Im}(\phi) = \mathbb{R}^2$.

Remark: A mapping of the type given in (*i*) is known as a projection. The mapping $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\phi(x, y, z) = (x, 0, 0)$ is also a projection. If P is a projection, it is easy to observe that $P^2 = P$. In the case of the mapping $\phi(x, y, z) = (x, y, 0)$, observe that $\phi\{\phi(x, y, z)\} = \phi(x, y, 0) = (x, y, 0) = \phi(x, y, z)$, *i.e.*, $\phi^2 = \phi$.

11.6 RANK AND NULLITY

If $\phi : V \to W$ is a linear mapping from a vector space V to the vector space W, then there are two important concepts associated with ϕ. These are rank and nullity defined as follows.

Definition: The *rank* of a linear mapping $\phi : V \to W$ is defined to be the dimension of Im(ϕ) and the *nullity* of ϕ is defined to be the demension of ker ϕ.