11.4 GENERATOR SET, BASIS AND DIMENSION

We begin with the definition of a set of generators.

Definition: A set G of vectors in a vector space V is called a set of generators or a spanning set of V if every vector V can be expressed as a linear combination of the vectors of G. The fact that G is a set of generators for V is expressed symbolically by $V = \text{sp}(G)$. Note that a set of generators need not be linearly independent.

Example 1: Show that the following sets are spanning sets for $\mathbb{R}^3$.

(*i*) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

(*ii*) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$

(*iii*) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

(*iv*) $\{(2, 1, 0), (1, 1, 2), (-1, 0, 1)\}$.

Solution: (*i*) Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ be arbitrary.

Then evidently

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1), x_1, x_2, x_3 \in \mathbb{R}$$

Thus, every vector of $\mathbb{R}^3$ is expressible as a linear combination of the vectors of the given set.

Hence, the given set is a set of generators for $\mathbb{R}^3$.

(*ii*) Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ be arbitrary.

Then, if possible, let

$$(x_1, x_2, x_3) = \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1).$$

$$\therefore \quad x_1 = \alpha + \beta, x_2 = \gamma + \alpha, x_3 = \beta + \gamma.$$

This system of equations in α, β, γ has a solution *viz.*

$$\alpha = \frac{1}{2}(x_1 + x_2 - x_3), \beta = \frac{1}{2}(x_1 - x_2 + x_3), \gamma = \frac{1}{2}(x_2 + x_3 - x_1).$$

Clearly, (x_1, x_2, x_3) is expressible as a linear combination of the vectors of the given set as $\alpha, \beta, \gamma \in \mathbb{R}$.

Hence, the given set is a spanning set.

(*iii*) Since any arbitrary vector (c_1, c_2, c_3) of $\mathbb{R}^3$ can be expressed as a linear combination of the given vectors as

$$(c_1, c_2, c_3) = c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) + 0(1, 1, 1)$$

We conclude that the given set of vectors is a spanning set for $\mathbb{R}^3$. (Note this expression is not unique.)

(*iv*) Let (x_1, x_2, x_3) be an arbitrary vector of $\mathbb{R}^3$ and let, if possible,

$$(x_1, x_2, x_3) = \alpha(2, 1, 0) + \beta(1, 1, 2) + \gamma(-1, 0, 1) \text{ for some } \alpha, \beta, \gamma \in \mathbb{R}.$$

Then $$2\alpha + \beta - \gamma = x_1,\ \alpha + \beta = x_2,\ 2\beta + \gamma = x_3.$$

Solving, we get

$$\alpha = 3x_2 - x_3 - x_1,\ \beta = x_1 - 2x_2 + x_3,\ \gamma = 4x_2 - x_3 - 2x_1.$$

Thus, every vector of $\mathbb{R}^3$ is expressible as a linear combination of the given vectors. Hence, the given set is a spanning set for $\mathbb{R}^3$.

Example 2: Show that the set of vectors

(*i*) $\{(2, 1, 3), (-1, 1, 0), (1, 2, 3)\}$ is not a spanning set for $\mathbb{R}^3$.

(*ii*) $\{(1, 1, 0), (0, 1, 1)\}$ is not a spanning set for $\mathbb{R}^3$.

(*iii*) $\{(1, 2, -1), (2, 1, 0), (4, 2, 2), (1, 1, 1)\}$ is a spanning set for $\mathbb{R}^3$.

Solution: (*i*) If possible let $(x_1, x_2, x_3) \in \mathbb{R}^3$ be expressible as

$$(x_1, x_2, x_3) = \alpha(2, 1, 3) + \beta(-1, 1, 0) + \gamma(1, 2, 3).$$

or $$2\alpha - \beta + \gamma = x_1 \quad ...(i)$$

$$\alpha + \beta + 2\gamma = x_2 \quad ...(ii)$$

$$3\alpha + 3\gamma = x_3 \quad ...(iii)$$

Adding (*i*) and (*ii*), we get $3\alpha + 3\gamma = x_1 + x_2$.

So if $x_1 + x_2 \neq x_3$, the above system becomes inconsistent. Then the linear expression for the point (x_1, x_2, x_3) will not be possible, (for example, (1, 2, 0) cannot be expressed as a linear combination of the given vectors).

Thus, the given set of vectors is not a spanning set for $\mathbb{R}^3$.

(*ii*) Let, if possible, an arbitrary vector (c_1, c_2, c_3) of $\mathbb{R}^3$ be expressible as a linear combination of the given vectors

i.e., $$(c_1, c_2, c_3) = \alpha(1, 1, 0) + \beta(0, 1, 1)$$

Then $$c_1 = \alpha,\ c_2 = \alpha + \beta,\ c_3 = \beta$$

So, $$c_2 = c_1 + c_3.$$

Evidently, if $c_2 \neq c_1 + c_3$, then such a vector of $\mathbb{R}^3$ cannot be expressed as a linear combination of (1, 1, 0) and (0, 1, 1), *e.g.*, (1, 2, 0). Hence, the given set cannot be a spanning set for $\mathbb{R}^3$.

(*iii*) To show that every vector of $\mathbb{R}^3$ can be expressed as a linear combination of the given vectors, we assume that an arbitrary vector $(c_1, c_2, c_3) \in \mathbb{R}^3$ and write that as

$$(c_1, c_2, c_3) = \alpha(1, 2, -1) + \beta(2, 1, 0) + \gamma(4, 2, 2) + \delta(1, 1, 1).$$

We now prove that for given c_1, c_2, c_3 such $\alpha, \beta, \gamma, \delta$ exist.

From above, we get

$$\alpha + 2\beta + 4\gamma + \delta = c_1$$
$$2\alpha + \beta + 2\gamma + \delta = c_2$$
$$-\alpha + 2\gamma + \delta = c_3$$

This is a system of 3 equations in 4 unknowns having infinitely many solutions.

Hence, for a given vector (c_1, c_2, c_3) there are infinitely many linear combinations of the given vectors to generate (c_1, c_2, c_3). This implies that the given set is a set of generators for $\mathbb{R}^3$.

Remark: From the above two examples, one must have noted that

(1) a vector space may have many sets of generators.

(2) a generator set for $\mathbb{R}^n$ contains at least n generators.

(3) a set of generators need not be linearly independent.

Definition: A basis of a vector space V is a subset of V which is linearly independent and a spanning set as well.

Thus the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{R}^3$ inasmuch as this set of vectors is linearly independent and is also a set of generators.

From the above examples, one should note

1. A vector space can have several bases.
2. The cardinality of each basis is the same.
3. The trivial vector space $\{0\}$ has no basis.

Definition: The dimension of a vector space is defined to be the cardinality of its basis. The dimension of the trivial vector space is defined to be zero. The dimension of the vector space V is denoted by dim (V).

Thus, the dimension of $\mathbb{R}^3$ is 3. Similarly, the dimension of $\mathbb{R}^n$ is n.

The dimension of a vector space may be finite or infinite. The vector space $\mathbb{R}^3$ is finite dimensional but the dimension of $\mathbb{R}[x]$ is infinite. There are many other infinite dimensional vector spaces.

A useful result about dimension is the following.

Theorem: If S and T are two subspaces of a vector space V, then

$$\dim (S + T) = \dim (S) + \dim (T) - \dim (S \cap T).$$

If in particular, $S \cap T = \{\theta\}$, then

$$\dim (S + T) = \dim (S) + \dim (T).$$

The proof is outside the scope of this book.

Example 3: Find the dimension of $S + T$ where $S = \{(x, y, z) \in \mathbb{R}^3; x = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3; y = 0\}$.

Solution: Clearly S is the yz-plane having dimension 2 and T is the zx-plane having dimension 2.

As $S \cap T$ is the z-axis having dimension 1, we get

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(T)$$
$$= 2 + 2 - 1 = 3.$$

Remark: It is easy to observe that every vector of $\mathbb{R}^3$ can be obtained as the sum of a vector of S and a vector of T.

Hence, $S + T$ is nothing but $\mathbb{R}^3$, having dimension 3.

Replacement Theorem: Let $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a basis of a vector space V over $\mathbb{R}$ and β be a non-zero vector of V can be expressed as a linear combination of these vectors as

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n \text{ where } a_i \in \mathbb{R}$$

and if $a_i \neq 0$, then $\{\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots \alpha_n\}$ is a new basis of V.

[*i.e.,* α_i can be replaced by β in the basis.]

The proof is outside the scope of this book.

Theorem: Any two basis of a finite dimensional vector space V have the same number of vectors.

Proof: Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\{\beta_1, \beta_2, \dots, \beta_n\}$ be the two bases of a finite dimnesional vector space V.

Since $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is a basis of V and $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a linearly independent set of vectors in V, then $n \leq m$ ...(1)

Again, since $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a basis of V and $\{\alpha_1, \alpha_2, \dots \alpha_m\}$ is linearly independent set of vectors in V, then $m \leq n$ ...(2)

From (1) and (2), we get

$$m = n$$

This proves the theorem.

Note: (*i*) If $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a basis of a finite dimensional vector space V over $\mathbb{R}$, then any linearly independent set of V contains at most n vectors.

(*ii*) Every finite dimensional vector space has a basis.

(*iii*) Every subset contains n linearly indepedent vector of n dimensional vector space is a basis of that vector space.

(*iv*) Every subset contains more than n vectors of n dimensional vector space is linearly dependent and can not be a basis of that vector space.

Example 1: Prove that the set $S = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$ is a bases of $\mathbb{R}^3$.

Solution: Let us consider the relation

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where $c_i \in \mathbb{R}$ and $\alpha_1 = (1, 0, 1), \alpha_2 = (0, 1, 1), \alpha_3 = (1, 1, 0)$

$\therefore c_1(1, 0, 1) + c_2(0, 1, 1) + c_3(1, 1, 0) = (0, 0, 0)$

or $(c_1 + c_3, c_2 + c_3, c_1 + c_2) = (0, 0, 0)$

This gives $c_1 + c_3 = 0$ ...(1)

$c_2 + c_3 = 0$ ...(2)

$c_1 + c_2 = 0$ ...(3)

Now, we get from (1) + (2) – (3), $2c_3 = 0 \Rightarrow c_3 = 0$ and we also get from (1) and (2), $c_1 = c_2 = 0$.

This proves that the set S is linearly independent.

Let $\xi \in \mathbb{R}^3$ be any arbitrary element where $\xi = (a, b, c)$. Then we shall prove that $\xi \in L(S)$.

If possible, let $\xi = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ where a_1, a_2, a_3 are real.

or $(a, b, c) = a_1(1, 0, 1) + a_2(0, 1, 1) + a_3(1, 1, 0)$

This gives $a_1 + a_3 = a$

$a_2 + a_3 = b$

$a_1 + a_2 = c$

This non-homogeneous system of three equations in a_1, a_2, a_3. The coefficient determinant $= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-1) - 0 + 1(-1) = -1 - 1 = -2 \neq 0.$

Hence, by the Cramer's rule, there exists unique solution for a_1, a_2, a_3.

This proves that $\xi \in L(S)$ and therefore $\mathbb{R}^3 \subset L(S)$ ...(4)

Again $S \subset \mathbb{R}^3$ and $L(S)$ is the smallest subspace containing S, then

$L(S) \subset \mathbb{R}^3$ ...(5)

From (4) and (5), we get $L(S) = R^3$

Since S is linearly independent and $L(S) = R^3$. Hence, S is a basis of $\mathbb{R}^3$.

Example 2: Prove that the set $S = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$ is a basis of R^3.

Solution: Same as Example (1).

Example 3: Show that the set $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$ spans the vector space $\mathbb{R}^3$ but is not a basis set.

Solution: Let $\xi = (a, b, c)$ be any arbitrary element of $\mathbb{R}^3$. Then we shall prove that $\xi \in L(S)$.

If possible, let $(a, b, c) = c_1(1, 0, 0) + c_2(1, 1, 0) + c_3(1, 1, 1) + c_4(0, 1, 0)$ for real c_1, c_2, c_3, c_4

or $(a, b, c) = (c_1 + c_2 + c_3, c_2 + c_3 + c_4, c_3)$

This gives
$$c_1 + c_2 + c_3 = a$$
$$c_2 + c_3 + c_4 = b$$
$$c_3 = c$$

From these equations, we get $c_3 = c$, $c_1 = a - b + c_4$, $c_2 = b - c - c_4$.

If we take $c_4 = 0$, then we get

$$(a, b, c) = (a - b)(1, 0, 0) + (b - c)(1, 1, 0) + c(1, 1, 1) + 0(0, 1, 0)$$

This shows that $\xi = (a, b, c) \in L(S)$

and hence, $\mathbb{R}^3 \subset L(S)$ ...(1)

Again, $S \subset R$ and $L(S)$ is the smallest subspace containing S, then

$$L(S) \subset \mathbb{R}^3 \quad ...(2)$$

From (1) and (2), we get $L(S) = \mathbb{R}^3$ *i.e.*, S spans the vector space $\mathbb{R}^3$.

Again, the relation of linear dependence

$$1(1, 0, 0) + (-1)(1, 1, 0) + 0(1, 1, 1) + 1(0, 1, 0) = (0, 0, 0)$$

Hence, S is not a basis of the vector space $\mathbb{R}^3$.

Example 4: Prove that $S = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ is a basis of $\mathbb{R}^3$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = (2, 1, 1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (1, 1, 2)$.

$\therefore$ $c_1(2, 1, 1) + c_2(1, 2, 1) + c_3(1, 1, 2) = (0, 0, 0)$

or $(2c_1 + c_2 + c_3, c_1 + 2c_2 + c_3, c_1 + c_2 + 2c_3) = (0, 0, 0)$

This gives
$$2c_1 + c_2 + c_3 = 0$$
$$c_1 + 2c_2 + c_3 = 0$$
$$c_1 + c_2 + 2c_3 = 0.$$

This is homogeneous system of three equations with three unknowns c_1, c_2, c_3.

Here the coefficient determinant

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \begin{array}{l} = 2(4-1) - 1(2-1) + 1(1-2) \\ = 6 - 1 - 1 = 4 \neq 0 \end{array}$$

Hence, by the Cramer's rule there exists a unique solution and the solution is $c_1 = c_2 = c_3 = 0$. This proves that the set S is linearly independent.

Since $\mathbb{R}^3$ is a vector space of dimension 3 and S is linearly independent set containing 3 vectors of $\mathbb{R}^3$, so S is a basis of $\mathbb{R}^3$.

Note: Show that the following set of vectors are basis of $\mathbb{R}^3$:

(*i*) $S = \{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$

(*ii*) $S = \{(1, -2, 3), (2, 3, 1), (-1, 3, 2)\}$.

Example 5: Show that the set $S = \{(1, 2, -1, -2), (2, 3, 0, -1), (1, 2, 1, 4), (1, 3, -1, 0)\}$ is basis of $\mathbb{R}^4$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \theta$ where c_1, c_2, c_3, c_4 are real numbers and $\alpha_1 = (1, 2, -1, -2)$, $\alpha_2 = (2, 3, 0, -1)$, $\alpha_3 = (1, 2, 1, 4)$ and $\alpha_4 = (1, 3, -1, 0)$.

$\therefore\ c_1(1, 2, -1, -2) + c_2(2, 3, 0, -1) + c_3(1, 2, 1, 4) + c_4(1, 3, -1, 0) = (0, 0, 0, 0)$

or $(c_1 + 2c_2 + c_3 + c_4, 2c_1 + 3c_2 + 2c_3 + 3c_4, -c_1 + c_3 - c_4, -2c_1 - c_2 + 4c_3) = (0, 0, 0, 0)$

This gives
$$\begin{aligned} c_1 + 2c_2 + c_3 + c_4 &= 0 \\ 2c_1 + 3c_2 + 2c_3 + 3c_4 &= 0 \\ -c_1 + c_3 - c_4 &= 0 \\ -2c_1 - c_2 + 4c_3 &= 0 \end{aligned}$$

This is homogeneous system of four equations with four unknowns c_1, c_2, c_3, c_4.

$$\text{Here the coefficient determinant} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ -1 & 2 & 2 & 0 \\ -2 & 3 & 6 & 2 \end{vmatrix} \begin{matrix} C_2' = C_2 - 2C_1 \\ C_3' = C_3 - C_1 \\ C_4' = C_4 - C_1 \end{matrix} = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 6 & 2 \end{vmatrix}$$

$$= -1(4 - 6 \cdot 0) - 0(4 \cdot 0) + 1(12 - 6)$$

$$= -4 - 0 + 6 = 2 \neq 0.$$

Hence, by the Cramer's rule there exists a unique solution and the solution is $c_1 = c_2 = c_3 = c_4 = 0$. This proves that the set is linearly independent.

Since $\mathbb{R}^4$ is a vector space of dimension 4 and S is linearly independent set containing 4 vectors of $\mathbb{R}^4$, so S is a basis of $\mathbb{R}^4$.

Example 6: Let V be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{2\alpha + 3\beta + \gamma, 3\beta + \gamma, \gamma\}$ is also a basis of V.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = 2\alpha + 3\beta + \gamma$, $\alpha_2 = 3\beta + \gamma$, $\alpha_3 = \gamma$

$\therefore \quad c_1(2\alpha + 3\beta + \gamma) + c_2(3\beta + \gamma) + c_3\gamma = \theta$

or $\quad 2c_1\alpha + 3(c_1 + c_2)\beta + (c_1 + c_2 + c_3)\gamma = \theta$

Since the set $\{\alpha, \beta, \gamma\}$ is linearly independent, then we get

$$2c_1 = 0$$
$$3(c_1 + c_2) = 0$$
$$c_1 + c_2 + c_3 = 0$$

The solution of these equations is $c_1 = 0, c_2 = 0, c_3 = 0$.

This proves that the set of vectors $\alpha_1, \alpha_2, \alpha_3$ is linearly independent.

Since V is a vector space of dimension 3 and $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent set containing 3 vectors of V. Therefore $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of V.

Example 7: Show that the set of vectors $B = \{1, 1 + 3x, 1 + 3x + 2x^2\}$ is a basis of P_3.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = 1, \alpha_2 = 1 + 3x, \alpha_3 = 1 + 3x + 2x^2$.

$\therefore \quad c_1 . 1 + c_2(1 + 3x) + c_3(1 + 3x + 2x^2) = 0$

or $\quad (c_1 + c_2 + c_3) . 1 + 3x(c_2 + c_3) + 2c_3 x^2 = 0 . 1 + 0 . x + 0 . x^2$

This gives $\quad c_1 + c_2 + c_3 = 0$

$$3(c_2 + c_3) = 0$$
$$2c_3 = 0$$

The solution of these equations is $c_1 = c_2 = c_3 = 0$. This proves that the set B is linearly independent.

Since P_3 is a vector space of dimension 3 and B is a linearly independent set containing three vectors in P_3, so B is a basis of P_3.

Example 8: Find the basis of $\mathbb{R}^3$ that contains the vectors (1, 2, 0) and (1, 3, 1).

Solution: Here R_3 is a vector space of dimension 3. The standard basis of R^3 is $\{\xi_1, \xi_2, \xi_3\}$ where $\xi_1 = (1, 0, 0), \xi_2 = (0, 1, 0)$ and $\xi_3 = (0, 0, 1)$.

Let $\quad \alpha = (1, 2, 0)$ and $\beta = (1, 3, 1)$

Now $\quad \alpha = (1, 2, 0) = 1 . \xi_1 + 2 . \xi_2 + 0 . \xi_3$

Then by replacement theorem, α can replace ξ_1 in the basis $\{\xi_1, \xi_2, \xi_3\}$ and $\{\alpha_1, \xi_2, \xi_3\}$ in a new basis of R^3.

Let $\quad \beta = c_1\alpha + c_2\xi_2 + c_3\xi_3$

or $\quad (1, 3, 1) = c_1(1, 2, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (c_1, 2c_1 + c_2, c_3)$

This gives $\quad c_1 = 1, 2c_1 + c_2 = 3, c_3 = 1$

$\therefore \quad c_1 = c_2 = c_3 = 1$ and $\beta = \alpha + \xi_2 + \xi_3$

Then by replacement theorem β can replace ξ_2 in the basis $\{\alpha, \xi_2, \xi_3\}$ and $\{\alpha, \beta, \xi_3\}$ is a new basis of R^3.

Note: Find the basis of $\mathbb{R}^3$ containing the vectors (*i*) (1, 1, 0), (1, 1, 1) and (*ii*) (1, 2, 1), (3, 6, 2).

Solution: Same as Example (8).

Example 9: Show that $S = \{(1, 0, 1, 1), (-1, -1, 0, 0), (0, 1, 1, 0)\}$ is a linearly independent. Subset of R^4 (vector space of dimension 4 over $\mathbb{R}$). Extend the subset to a basis of $\mathbb{R}^4$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = (1, 0, 1, 1)$, $\alpha_2 = (-1, -1, 0, 0)$, $\alpha_3 = (0, 1, 1, 0)$.

$\therefore \quad c_1(1, 0, 1, 1) + c_2(-1, -1, 0, 0) + c_3(0, 1, 1, 0) = (0, 0, 0, 0)$

or $\quad (c_1 - c_2, -c_2 + c_3, c_1 + c_3, c_1) = (0, 0, 0, 0)$

This gives
$$c_1 - c_2 = 0$$
$$-c_2 + c_3 = 0$$
$$c_1 + c_3 = 0$$
$$c_1 = 0$$

The solution of these equations is $c_1 = c_2 = c_3 = 0$. This proves that the set S subset of $\mathbb{R}^4$ is linearly independent.

Let $\quad \beta = (1, 0, 0, 0) \in R^4 - L(S)$

Then we shall prove that the set $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \beta\}$ is linearly independent.

Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_4\alpha_3 + c_3\beta = \theta$ where c_1, c_2, c_3 and c_4 are real numbers.

Now, we assume $c_4 = 0$, if $c_4 \neq 0$, then c_4^{-1} exists in R.

$$\therefore \quad \beta = -c_4^{-1} c_1 \alpha_1 - c_4^{-1} c_2\alpha_2 - c_4^{-1}c_3\alpha_3$$
$$= d_1\alpha_1 + d_2\alpha_2 + d_3\alpha_3 \text{ where } d_i = -c_4^{-1}c_i \in R$$

$\therefore \ \beta \in L(S)$ which gives the contradiction and hence the assumption is true *i.e.*, $c_4 = 0$.

Since $\{\alpha_1, \alpha_2, \alpha_3\}$ is linear independent set. So $c_1 = c_2 = c_3 = 0$.

Therefore, the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\beta = \theta$ implies $c_1 = c_2 = c_3 = c_4 = 0$.

This proves that the set S_1 is linearly independent.

Since $\mathbb{R}^4$ is a vector space of dimension 4 and S_1 is a linearly independent set containing $\mathbb{R}^4$ vectors of $\mathbb{R}^4$, so S_1 is a basis of $\mathbb{R}^4$.

[2nd part: Let $\beta = (1, 0, 0, 0) \in \mathbb{R}^4$]

Let us consider the set $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \beta\}$. Then we shall prove that S_1 is a basis of $\mathbb{R}^4$.

Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\beta = \theta$ where c_1, c_2, c_3, c_4 are real numbers.

$\therefore \quad c_1(1, 0, 1, 1) + c_2(-1, -1, 0, 0) + c_3(0, 1, 1, 0) + c_4(1, 0, 0, 0) = (0, 0, 0, 0)$

This gives
$$c_1 - c_2 + c_4 = 0$$
$$-c_2 + c_3 = 0$$
$$c_1 + c_3 = 0$$
$$c_1 = 0$$

The solution of these equations is $c_1 = c_2 = c_3 = c_4 = 0$. This proves that the set S_1 is linearly independent.

Since $\mathbb{R}^4$ is a vector space of dimension is 4 and S_1 is linearly independent set containing 4 vectors of $\mathbb{R}^4$, so S_1 is a basis of $\mathbb{R}^4$.

Example 10: Find the basis and dimension of the subspace W of $\mathbb{R}^3$ where
$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

Solution: Let $\xi = (a, b, c) \in W$, then $a + b + c = 0$ and $a, b, c, \in \mathbb{R}$

$\therefore \quad \xi = (a, b, -a-b) \quad (\because c = -a - b)$

$\quad = a(1, 0, -1) + b(0, 1, -1)$

Let $\quad \alpha = (1, 0, -1)$ and $\beta = (0, 1, -1)$, then $\xi = a\alpha + b\beta \in L(S)$

where $\quad S = \{\alpha, \beta\}$

$\therefore \quad W \subset L(S)$ ...(1)

Again $\quad \alpha \in W, \beta \in W$. This implies $L(S) \subset W$ ...(2)

From (1) and (2), we get $W = L(S)$.

Then, we shall prove that the set S is linearly independent.

Let us consider the relation $c_1\alpha + c_2\beta = \theta$ where $c_1, c_2 \in \mathbb{R}$

$\therefore \quad c_1(1, 0, -1) + c_2(0, 1, -1) = (0, 0, 0)$

$\Rightarrow \quad (c_1, c_2, -c_1 - c_2) = (0, 0, 0)$

This gives $\quad c_1 = 0, c_2 = 0, -c_1 - c_2 = 0$

$\therefore \ c_1 = c_2 = 0$ and this proves that the set S is a linearly independent set.

Hence, $\quad S = \{\alpha, \beta\}$ is a basis of W and dim $W = 2$.

Example 11: Find the basis and dimension of the subspace W or $\mathbb{R}^3$ where
$$W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0,\ 2x + y + 3z = 0\}.$$

Solution: Let $\xi = (a, b, c)$ be an arbitrary vector of W, then

$$a + 2b + c = 0 \text{ and}$$
$$2a + b + 3c = 0 \text{ where } a, b, c \in \mathbb{R}$$

By cross-multiplication, we get

$$\frac{a}{6-1} = \frac{b}{2-3} = \frac{c}{1-4}$$

i.e., $$\frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} = k \text{ (say)}$$

$\therefore$ $a = 5k, b = -k, c = -3k.$

$\therefore$ $\xi = (5k, -k, -3k) = k(5, -1, -3)$ where k is arbitrary real number.

$\therefore$ $W = L\{\alpha\}$ where $\alpha = (5, -1, -3)$.

Since $\{\alpha\}$ is linearly independent set, $\{\alpha\}$ is a basis of W and dim $W = 1$.

Example 12: S and T are subspace of $\mathbb{R}^4$ given by

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$$
$$T = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y + z + 3w = 0\}$$

Find dim $S \cap T$.

Solution: Now $S \cap T = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0, x + 2y + z + 3w = 0\}$

Let $\xi = (a, b, c, d) \in S \cap T$, then $a, b, c, d \in \mathbb{R}$ and

$$2a + b + 3c + d = 0 \quad ...(1)$$
$$a + 2b + c + 3d = 0 \quad ...(2)$$

Solving (1) and (2), we get

$$a = -5b - 8d, c = 3b + 5b$$

$\therefore$ $\xi = (a, b, c, d) = (-5b - 8d, b, 3b + 5d, d)$

$= b(-5, 1, 3, 0) + d(-8, 0, 5, 1)$

Let $\alpha = (-5, 1, 3, 0)$ and $\beta = (-8, 0, 5, 1)$, then $\xi = b\alpha + d\beta$ where $b, d \in \mathbb{R}$

$\therefore$ $\xi \in L(S)$ where $S = \{\alpha, \beta\}$

$\therefore$ $S \cap T \subset L(S)$ ...(1)

Again, $\alpha \in S, \alpha \in T; \beta \in S, \beta \in T$. This implies $\alpha \in S \cap T, \beta \in S \cap T$

$\therefore$ $L(S) \subset S \cap T$ ...(2)

From (1) and (2) $L(S) = S \cap T$.

Then, we shall prove that the set of vectors α, β is linearly independent.

Let us consider the relation

$$c_1\alpha + c_2\beta = \theta$$

i.e., $c_2(-8, 0, 5, 1) + c_1(-5, 1, 3, 0) = (0, 0, 0, 0)$

or $(-8c_2 - 5c_1, c_1, 5c_2 + 3c_1, c_2) = (0, 0, 0, 0)$

This gives $-8c_2 - 5c_1 = 0$

$$c_1 = 0$$

$$5c_2 + 3c_1 = 0$$

$$c_2 = 0$$

$\therefore$ $c_1 = c_2 = 0$. Hence, S is linearly independent set.

$\therefore$ $\dim S \cap T = 2$.

Example 13: Find the dimension of the subspace S of $\mathbb{R}^4$ where

$$S = \{(x, y, z, w) : x + y + z + w = 0\}.$$

Solution: Let $\xi = (a, b, c, d) \in S$, then $a, b, c, d \in \mathbb{R}$ and $a + b + c + d = 0$

$\therefore$ $d = -a - b - c$

$\therefore$ $\xi = (a, b, c, -a - b - c)$

$= a(1, 0, 0, -1) + b(0, 1, 0, -1) + c(0, 0, 1, -1)$

Let $\alpha = (1, 0, 0, -1), \beta = (0, 1, 0, -1)$ and $\gamma = (0, 0, 1, -1)$,

then $\xi = a\alpha + b\beta + c\gamma$

$\therefore$ $\xi \in L\{S_1\}$ where $S_1 = \{\alpha, \beta, \gamma\}$

$\therefore$ $S \subset L(S_1)$ ...(1)

Again, $\alpha, \beta, \gamma \in S$. This implies $L(S_1) \subset S$ ...(2)

From (1) and (2), we get $L(S_1) = S$.

Then, we shall prove that the set S_1 is linearly independent.

Let us consider the relation

$$c_1\alpha + c_2\beta + c_3\gamma = \theta$$

or $c_1(1, 0, 0, -1) + c_2(0, 1, 0, -1) + c_3(0, 0, 1, -1) = (0, 0, 0, 0)$

or $(c_1, c_2, c_3, -c_1 - c_2 - c_3) = (0, 0, 0, 0)$

This gives $c_1 = 0, c_2 = 0, c_3 = 0, -c_1 - c_2 - c_3 = 0$

$\therefore$ $c_1 = c_2 = c_3 = 0$. This proves that S_1 is linearly independent set.

$\therefore$ S_1 is a basis of S and $\dim S = 3$.

Example 14: Find a basis and determine the dimension of the following subspace of the vector space $\mathbb{R}_{2\times 2}$:

(*i*) $S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : a + d = 0\right\}$.

(*ii*) $S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : a = d = 0\right\}$.

Solution: (*i*) Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in S$, then $x, y, z, w \in \mathbb{R}$ and $x + w = 0$

$\therefore \quad w = -x$

$$\therefore \quad A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = M,\ N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\therefore \quad A = xM + yN + zP$

$\therefore \quad A \in L(S_1)$ where $S_1 = \{M, N, P\}$

$\therefore \quad S \subset L(S_1)$ ...(1)

Again, $M \in S, N \in S, P \in S$. This implies $L(S_1) \subset S$ ...(2)

From (1) and (2), we get $L(S_1) = S$.

Then, we shall prove that S_1 is linearly independent set.

Let us consider the relation $aM + bN + cP = \theta$ where $a, b, c \in \mathbb{R}$

$$\therefore \quad a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This gives $a = 0, b = 0, c = 0$.

This shows that the set S_1 is linearly independent.

$\therefore \quad S_1$ is a basis of S and $\dim S = 3$.

(*ii*) Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in S$, then $x, y, z, w \in \mathbb{R}$ and $x = w = 0$

$$\therefore \ A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = y\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\therefore \quad A = yM + zN$

$\therefore \quad A \in L(S_1)$ where $S_1 = \{M, N\}$

$\therefore \quad S \subset L(S_1)$ ...(1)

Again $M \in S, N \in S$. This implies $L(S_1) \subset S$ ...(2)

From (1) and (2), $L(S_1) = S$.

Then we shall prove that S_1 is linearly independent.

Let us consider the relation $aM + bN = \theta$ where $a, b \in \mathbb{R}$

$$\therefore \quad a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$. This gives $a = b = 0$.

This shows that the set S_1 is linearly independent.

$\therefore$ S_1 is a basis of S and dim $S = 2$.

Example 15: (*a*) Let W be a subspace of $\mathbb{R}^4$ defined by

$W = \{(a, b, c, d) \in \mathbb{R}^4 : a + b = 0, c = 2d\}$. Find dim W.

(*b*) Let $S = \{(x, y, z) \in \mathbb{R}^3 : 3x - y + z = 0\}$. Show that S is a sub-space of $\mathbb{R}^3$. Find a basis of S.

Solution: (*a*) Let $\xi = (x, y, z, w) \in W$, then $x, y, z, w \in \mathbb{R}$ and $x + y = 0, z = 2w$

$$\therefore \quad \xi = (x, y, z, w) = (x, -x, 2w, w)$$
$$= x(1, -1, 0, 0) + w(0, 0, 2, 1)$$

Let $\alpha = (1, -1, 0, 0)$ and $\beta = (0, 0, 2, 1)$

$\therefore$ $\xi = x\alpha + w\beta \in L(S)$ where $S = \{\alpha, \beta\}$

$\therefore$ $W \subset L(S)$ ...(1)

Again $\alpha \in W, \beta \in W$, this implies $L(S) \subset W$ ...(2)

From (1) and (2), we get $L(S) = W$

Then we shall prove that the set S is linearly independent.

Let us consider the relation $c_1\alpha + c_2\beta = \theta$ where c_1, c_2 are real numbers.

$$\therefore \quad c_1(1, -1, 0, 0) + c_2(0, 0, 2, 1) = (0, 0, 0, 0)$$

$$\Rightarrow \quad (c_1, -c_1, 2c_2, c_2) = (0, 0, 0, 0)$$

This gives $c_1 = 0, -c_1 = 0, 2c_2 = 0, c_2 = 0$

$$\therefore \quad c_1 = c_2 = 0$$

This proves that the set S is linearly independent.

Hence, S is the basis of W and it contains two vectors.

$$\therefore \quad \dim W = 2.$$

(*b*) Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$3x_1 - y_1 + z_1 = 0$ and $3x_2 - y_2 + z_2 = 0, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$

Now, $\alpha + \beta = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2 + z_1 + z_2) \in S$ because $3(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2)$

$= (3x_1 - y_1 + z_1) + (3x_2 - y_2 + z_2)$

$= 0 + 0 = 0$

Let $c \in \mathbb{R}$, then $c\alpha = c(x_1, y_1, z_1)$

$= (cx_1, cy_1, cz_1) \in S$ because $3(cx_1) - cy_1 + cz_1$

$= c(3x_1 - y_1 + z_1) = c \,.\, 0 = 0$

Since $\alpha, \beta \in S$ and $c \in \mathbb{R} \Rightarrow \alpha + \beta \in S$ and $c\alpha \in S$

$\therefore$ S is a subspace of $\mathbb{R}^3$

Let $\xi = (x, y, z) \in S$, then $x, y, z \in \mathbb{R}$ and $3x - y + z = 0$

$\therefore \quad \xi = (x, y, z) = (x, y, y - 3x)$

$= x(1, 0, -3) + y(0, 1, 1)$

Let $\alpha = (1, 0, -3)$ and $\beta = (0, 1, 1)$, then $\xi = x\alpha + y\beta \in L(S_1)$ where $S_1 = \{\alpha, \beta\}$

$\therefore \quad S \subset L(S_1)$ ...(1)

Again $\alpha \in S, \beta \in S$. This implies $L(S_1) \subset S$ ...(2)

From (1) amd (2), we get

$S = L(S_1)$

Then, we shall prove that the set S_1 is linearly independent.

Let us consider the relation $c_1\alpha + c_2\beta = \theta$ where $c_1, c_2 \in \mathbb{R}$

$\therefore \quad c_1(1, 0, -3) + c_2(0, 1, 1) = (0, 0, 0)$

$\Rightarrow \quad (c_1, c_2, -3c_1 + c_2) = (0, 0, 0)$

This gives $c_1 = 0, c_2 = 0, -3c_1 + c_2 = 0$

$\therefore \quad c_1 = c_2 = 0$

This proves that the set S_1 is linearly independent.

Hence, S_1 is the basis of S.