11.3 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

The notions of linear dependence and independence are of vital importance in vector space theory.

Definition: A set of vectors $x_1, x_2, ..., x_n$ is said to be *linearly independent* if $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = \theta$ implies $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

A set of vectors $x_1, x_2, ..., x_n$ is said to be *linearly dependent* if they are not linearly independent. (*i.e.*, at least one $\lambda_i \neq 0$).

It is to be noted that

(1) a set of vectors containing the null vector cannot be linearly independent.

(2) if $x_1, x_2, ..., x_n$ are linearly dependent, then one of the vectors can be expressed as a linear combination of the others.

(3) a subset of a set of linearly independent vectors is always linearly independent.

(4) a superset of a set of linearly dependent vectors is linearly dependent.

Definition: An infinite set of vectors is said to be linearly independent if every finite subset of the set is linearly independent.

An infinite set of vectors is said to be linearly dependent if they are not linearly independent.

Example: Show that the set of vectors

(*i*) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent in $\mathbb{R}^3$.

(*ii*) $\{(1, 2, 0), (2, 0, 3)\}$ is linearly independent in $\mathbb{R}^3$.

(*iii*) $\{(1, 1, 2), (-1, 1, 0), (1, 0, 1)\}$ is linearly dependent in $\mathbb{R}^3$.

(*iv*) $\{(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1)\}$ is linearly dependent in $\mathbb{R}^3$.

(*v*) $\{(1, 2, -1), (0, 1, 1), (1, 4, -3)\}$ is linearly independent in $\mathbb{R}^3$.

(*vi*) $\{(4, -4, 2), (-6, 6, 3)\}$ is linearly dependent in $\mathbb{R}^3$.

(*vii*) $\{(1, 2, 3), (2, 1, 3), (0, 0, 0)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: (*i*) Let $\lambda_1(1, 0, 0) + \lambda_2(0, 1, 0) + \lambda_3(0, 0, 1) = (0, 0, 0)$

Then $$(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$

$\therefore$ $$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Hence, the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent.

(*ii*) Let $\alpha(1, 2, 0) + \beta(2, 0, 3) = (0, 0, 0)$

or $(\alpha + 2\beta, 2\alpha, 3\beta) = (0, 0, 0)$

$\therefore$ $\alpha + 2\beta = 0, 2\alpha = 0, 3\beta = 0.$

Hence, $\alpha = 0 = \beta.$

This implies that $\{(1, 2, 0), (2, 0, 3)\}$ is linearly independent.

(*iii*) Let $\lambda_1(1, 1, 2) + \lambda_2(-1, 1, 0) + \lambda_3(1, 0, 1) = (0, 0, 0)$

or $(\lambda_1 - \lambda_2 + \lambda_3, \lambda_1 + \lambda_2, 2\lambda_1 + \lambda_3) = (0, 0, 0)$

$\therefore$

$$\lambda_1 - \lambda_2 + \lambda_3 = 0$$
$$\lambda_1 + \lambda_2 = 0$$
$$2\lambda_1 + \lambda_3 = 0.$$

Clearly, apart from the trivial solution (0, 0, 0), the above system has a non-trivial solution $(1, -1, -2)$, *i.e.*, $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2.$

Thus, $\lambda_1(1, 1, 2) + \lambda_2(-1, 1, 0) + \lambda_3(1, 0, 1) = (0, 0, 0)$ does not imply $\lambda_1 = \lambda_2 = \lambda_3 = 0$, *i.e.*, the vectors $(1, 1, 2), (-1, 1, 0), (1, 0, 1)$ are not linearly independent.

Therefore, the given set of vectors is linearly dependent.

Remark: Note that the determinant formed by the given vectors as three rows has value zero. It is easy to prove that the vanishing of the determinant implies dependence of the vectors.

(*iv*) Let $\lambda_1(1, 1, 0) + \lambda_2(1, 0, 1) + \lambda_3(0, 1, 1) + \lambda_4(1, 1, 1) = (0, 0, 0)$ where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$.

Then

$$\lambda_1 + \lambda_2 + \lambda_4 = 0$$
$$\lambda_1 + \lambda_3 + \lambda_4 = 0$$
$$\lambda_2 + \lambda_3 + \lambda_4 = 0.$$

Solving we get

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}, \lambda_4 = -1 \text{ as a non-trivial solution.}$$

Thus, $\lambda_1(1, 1, 0) + \lambda_2(1, 0, 1) + \lambda_3(0, 1, 1) + \lambda_4(1, 1, 1) = (0, 0, 0)$ does not imply $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_3 = 0$.

Hence, the given set of vectors are linearly dependent in $\mathbb{R}^3$.

(*v*) Let $\alpha(1, 2, -1) + \beta(0, 1, 1) + \gamma(1, 4, -3) = (0, 0, 0)$ where $\alpha, \beta, \gamma \in \mathbb{R}$.

Then

$$\alpha + \gamma = 0$$
$$2\alpha + \beta + 4\gamma = 0$$
$$-\alpha + \beta - 3\gamma = 0.$$

Since the coefficient determinant has value zero, the only solution of the above system is $\alpha = \beta = \gamma = 0$ by Cramer's rule.

Thus, the above relation of the vectors implies $\alpha = \beta = \gamma = 0$, which means that the vectors are linearly independent.

(*vi*) Let $\alpha(4, -4, 2) + \beta(-6, 6, 3) = (0, 0, 0)$ where $\alpha, \beta \in \mathbb{R}$.

Then
$$4\alpha - 6\beta = 0$$
$$-4\alpha + 6\beta = 0$$
$$2\alpha + 3\beta = 0.$$

Apart from the trivial solution, we get $\alpha = 3, \beta = -2$.

Thus, the above relation of vectors does not imply $\alpha = \beta = 0$. Therefore, the vectors are linearly dependent.

(*vii*) Since $0(1, 2, 3) + 0(2, 1, 3) + 1(0, 0, 0) = (0, 0, 0)$, $\lambda_1(1, 2, 3) + \lambda_2 (2, 1, 3) + \lambda_3(0, 0, 0) = (0, 0, 0)$ does not imply that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Hence, the given set of vectors is linearly dependent.

Deletion Theorem: If a vector space V over $\mathbb{R}$ be spanned by a linearly dependent set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, then V can also be generated by a suitable proper subset of $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ [*i.e.*, some vectors can be deleted from linearly dependent spanning set of V].

Proof: Since $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a linearly dependent set, one of the vectors of the set, say α_j, can be expressed as a linear combination of the remaining others.

Let $\alpha_j = a_1 \alpha_1 + a_2 \alpha_2 + ... + a_{j-1} \alpha_{j-1} + a_{j+1} \alpha_{j+1} + ... + a_n\alpha_n$ for some scalars $a_1, a_2, ..., a_{j-1}, a_{j+1} ... a_n \in \mathbb{R}$.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $T = \{\alpha_1\alpha_2, ..., \alpha_{j-1}, \alpha_{j+1}, ..., \alpha_n\}$

$\therefore$ $L(S) = V$

Since $T \subset S$, then $L(T) \subset L(S)$ ...(1)

Again, since each element of S is a linear combination of the vectors of T

$\therefore$ $L(S) \subset L(T)$ ...(2)

Hence, by (1) and (2), $L(S) = L(T)$

But $L(S) = V$

$\therefore$ $V = L(T)$ and V is spanned by a proper subset T of S.

Example 1: Let $\alpha_1 = (1, 2, 0), \alpha_2 = (3, -1, 1), \alpha_3 = (4, 1, 1)$. Show that the set $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent. Apply Deletion theorem to find a proper subset of S that can generate $L(S)$.

Solution: Let $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2 and c_3 are scalars.

Then $c_1(1, 2, 0) + c_2(3, -1, 1) + c_3(4, 1, 1) = (0, 0, 0)$

or $(c_1 + 3c_2 + 4c_3, 2c_1 - c_2 + c_3, c_2 + c_3) = (0, 0, 0)$

$\therefore$
$$c_1 + 3c_2 + 4c_3 = 0$$
$$2c_1 - c_2 + c_3 = 0$$
$$c_2 + c_3 = 0$$

$\Rightarrow$ $c_1 = c_2 = -c_3$.

Let $c_1 = 1$, then $c_2 = 1$ and $c_3 = -1$

$\therefore$ $\alpha_1 + \alpha_2 - \alpha_3 = \theta$

This shows that the set S is linearly dependent.

Now $\alpha_3 = \alpha_1 + \alpha_2$

$\therefore$ By the deletion theorem, α_3 can be deleted from the generating set of $L(S)$.

i.e., $L\{\alpha_1, \alpha_2\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

Note: α_1 can also be deleted by Deletion theorem and $L\{\alpha_1, \alpha_2, \alpha_3\} = L\{\alpha_2, \alpha_3\}$.

α_2 can also be deleted by Deletion theorem and $L\{\alpha_1, \alpha_3\} = L\{\alpha_1, \alpha_2, \alpha_3\}$.

Example 2: Show that the set of vectors $\{(2, 1, 1), (1, 2, 2), (1, 1, 1)\}$ is linearly dependent is $\mathbb{R}^3$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = (2, 1, 1), \alpha_2 = (1, 2, 2), \alpha_3 = (1, 1, 1)$.

$\therefore$ $c_1(2, 1, 1) + c_2(1, 2, 2) + c_3(1, 1, 1) = (0, 0, 0)$

or $(2c_1 + c_2 + c_3, c_1 + 2c_2 + c_3, c_1 + 2c_2 + c_3) = (0, 0, 0)$

$\therefore$
$$2c_1 + c_2 + c_3 = 0$$
$$c_1 + 2c_2 + c_3 = 0$$
$$c_1 + 2c_2 + c_3 = 0$$

From the first two equations (by cross multiplication)

$$\frac{c_1}{-1} = \frac{c_2}{-1} = \frac{c_3}{3} = k \text{ (say)}$$

$\therefore$ $c_1 = -k, c_2 = -k, c_3 = 3k$

The last equation is also satisfied by these.

Since k is arbitrary, there exists c_1, c_2, c_3, not all zero, such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$.

Hence, the set of vectors $\alpha_1, \alpha_2, \alpha_3$ is linearly dependent.

Example 3: Prove that the set of vectors $\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$ is linearly independent in $\mathbb{R}^3$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta$ where c_1, c_2, c_3 are real numbers and $\alpha_1 = (1, 2, 2), \alpha_2 = (2, 1, 2), \alpha_3 = (2, 2, 1)$.

$$\therefore \quad c_1(1, 2, 2) + c_2(2, 1, 2) + c_3(2, 2, 1) = (0, 0, 0)$$

or $$(c_1 + 2c_2 + 2c_3, 2c_1 + c_2 + 2c_3, 2c_1 + 2c_2 + c_3) = (0, 0, 0)$$

$$\therefore \quad \left.\begin{aligned} c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 2c_2 + c_3 &= 0 \end{aligned}\right\} \quad \text{...(1)}$$

Now $\det A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$ where A is the coefficient matrix of the system (1).

$$= 1(1-4) - 2(2-4) + 2(4-2) = -3 + 4 + 4$$
$$= 5 \neq 0.$$

The system (1) is a homogeneous system of three equations in three unknowns c_1, c_2, c_3 and $\det A = 5 \neq 0$, then by Cramer's rule, there exists a unique solution. This solution is

$$c_1 = c_2 = c_3 = 0.$$

This proves the set vectors is linearly independent.

Example 4: If the vectors $(0, 1, a), (1, a, 1), (a, 1, 0)$ of the vector space $\mathbb{R}^3$ over $\mathbb{R}$ be linearly dependent, then find the value of a.

Solution: Let us consider the relation $x(0, 1, a) + y(1, a, 1) + z(a, 1, 0) = \theta$ where $x, y, z \in \mathbb{R}$

or $$(y + za, x + ay + z, xa + y) = (0, 0, 0)$$

$$\therefore \quad y + za = 0 \quad \text{...(1)}$$
$$x + ay + z = 0 \quad \text{...(2)}$$
$$xa + y = 0 \quad \text{...(3)}$$

From (1) $\dfrac{y}{a} = \dfrac{z}{-1} = k$ (say), then $y = ak, z = -k$

Putting these values in (2), we get $x = -(ak)a - (-k) = k(1 - a^2)$

Putting x and y in (3), we get

$$ak(1 - a^2) + ak = 0$$

or $$k(a - a^3 + a) = 0 \text{ or } ka(a^2 - 2) = 0$$

$$\therefore \quad k = 0, a = 0, \pm\sqrt{2}$$

If $k = 0$, then $x = y = z = 0$ so that the given vectors are linearly independent. Hence, $k \neq 0$.

Then, for the vectors to be linearly dependent, the values of a are $0, \pm\sqrt{2}$.

Alternative: Since the vectors $(0, 1, a), (1, a, 1), (a, 1, 0)$ are linearly dependent, then value of the determinant $\begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix}$ must be zero.

i.e.,
$$\begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = 0$$

or $\quad 0(a.0 - 1) - 1(1.0 - a.1) + a(1 - a^2) = 0$

or $\quad a + a - a^3 = 0$

or $\quad a(2 - a^2) = 0$ or $a(a^2 - 2) = 0$

$\therefore \quad a = 0, \pm\sqrt{2}$

Hence, the values of a are $0, \pm\sqrt{2}$.

Example 5: (*a*) Show that the subset $S\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of V_3 over $\mathbb{R}$ generates the entire vector space V_3 over $\mathbb{R}$, that is $L(S) = V$.

(*b*) Show that the vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 2, 3) generate the some space as generated by the vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Solution: (*a*) Let (a, b, c) be any element of V. Then

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

$\therefore (a, b, c) \in L(S)$ where $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Hence, $\quad V \subset L(S)$

But $\quad L(S) \subset V$

Hence, $\quad L(S) = V$.

(*b*) We see that, for real a, b, c, d

$$a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) + d(1, 2, 3)$$
$$= (a + d, b + 2d, c + 3d)$$
$$= (a + d)(1, 0, 0) + (b + 2d)(0, 1, 0) + (c + 3d)(0, 0, 1)$$

$\therefore \quad L\{(1,0,0), (0,1,0), (0,0,1), (1,2,3)\} = L\{(1,0,0), (0,1,0), (0,0,1)\}$

Hence, the two sets of vectors generate the same space.

Example 6: Show that the vectors (1, 1, 2, 4), (2, – 1, – 5, 2), (1, – 1, – 4, 0) and (2, 1, 1, 6) are linearly dependent in $\mathbb{R}^4$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \theta$ where c_1, c_2, c_3, c_4 are real numbers and $\alpha_1 = (1, 1, 2, 4)$, $\alpha_2 = (2, -1, -5, 2)$, $\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$.

$\therefore c_1(1,1,2,4)+c_2(2,-1,-5,2)+c_3(1,-1,-4,0)+c_4(2,1,1,6)=(0,0,0,0)$

or $(c_1+2c_2+c_3+2c_4,\ c_1-c_2-c_3+c_4,\ 2c_1-5c_2-4c_3+c_4,\ 4c_1+2c_2+6c_4)$
$=(0,0,0,0)$

$$\therefore \left.\begin{aligned} c_1+2c_2+c_3+2c_4&=0\\ c_1-c_2-c_3+c_4&=0\\ 2c_1-5c_2-4c_3+c_4&=0\\ 4c_1+2c_2+6c_4&=0 \end{aligned}\right\} \quad ...(1)$$

Now $\det A = \begin{vmatrix} 1 & 2 & 1 & 2\\ 1 & -1 & -1 & 1\\ 2 & -5 & -4 & 1\\ 4 & 2 & 0 & 6 \end{vmatrix}$

where A is the coefficient matrix of system (1)

$$= \begin{vmatrix} 1 & 0 & 0 & 0\\ 1 & -3 & -2 & -1\\ 2 & -9 & -6 & -3\\ 4 & -6 & -4 & -2 \end{vmatrix} \begin{matrix} C_2' = C_2 - 2C_1\\ C_3' = C_3 - C_1\\ C_4' = C_4 - 2C_1 \end{matrix}$$

$$= \begin{vmatrix} -3 & -2 & -1\\ -9 & -6 & -3\\ -6 & -4 & -2 \end{vmatrix} = (-1)^3 \begin{vmatrix} 3 & 2 & 1\\ 9 & 6 & 3\\ 6 & 4 & 2 \end{vmatrix}$$

$$= (-1)^3[3(12-12)-2(18-18)+3(36-36)] = 0.$$

The system (1) is a homogeneous system of four equations in the four unknowns c_1, c_2, c_3, c_4 and $\det A = 0$, then by the Cramer's rule, there exists non-zero solutions. Hence, c_1, c_2, c_3, c_4 are not zero.

This proves that the set of vectors are linearly dependent.

Example 7: Show that the vectors $(2, 6, -1, 8), (0, 10, 4, 3), (0, 0, -1, 4), (0, 0, 0, 8)$ are linearly independent in the four dimensional vector space $\mathbb{R}^4$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \theta$, where c_1, c_2, c_3, c_4 are real numbers and $\alpha_1 = (2, 6, -1, 8)$, $\alpha_2 = (0, 10, 4, 3)$, $\alpha_3 = (0, 0, -1, 4)$, $\alpha_4 = (0, 0, 0, 8)$.

$\therefore c_1(2,6,-1,8)+c_2(0,10,4,3)+c_3(0,0,-1,4)+c_4(0,0,0,8)=(0,0,0,0)$

or $(2c_1,\ 6c_1+10c_2,\ -c_1+4c_2-c_3,\ 8c_1+3c_2+4c_3+8c_4)=(0,0,0,0)$

$\therefore$

$$2c_1 = 0 \quad ...(1)$$

$$6c_1 + 10c_2 = 0 \quad ...(2)$$

$$-c_1 + 4c_2 - c_3 = 0 \quad ...(3)$$

$$8c_1 + 3c_2 + 4c_3 + 8c_4 = 0 \quad ...(4)$$

From (1), $c_1 = 0$

Using $c_1 = 0$ and from (2), $c_2 = 0$

Using $c_1 = c_2 = 0$ and from (3), we get $c_3 = 0$

Using $c_1 = c_2 = c_3 = 0$ and from (4), we get $c_4 = 0$

$\therefore \quad c_1 = c_2 = c_3 = c_4 = 0$

This proves the set of vectors is linearly independent.

Example 8: Show that the four vectors $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, $\gamma = (0, 0, 1)$ and $\delta = (1, 1, 1)$ in V_3 from a linear dependent set in $\mathbb{R}^3$.

Solution: Let us consider the relation $c_1\alpha + c_2\beta + c_3\gamma + c_4\delta = \theta$ where c_1, c_2, c_3, c_4 are real numbers.

$$\therefore \quad c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) + c_4(1, 1, 1) = (0, 0, 0)$$

or $$(c_1 + c_4, c_2 + c_4, c_3 + c_4) = (0, 0, 0)$$

$$\therefore \quad c_1 + c_4 = 0$$
$$c_2 + c_4 = 0$$
$$c_3 + c_4 = 0$$

This system has a solution

$$c_1 = -c_4, c_2 = -c_4, c_3 = -c_4, \text{ for each choice of } c_4$$

Let us take $c_4 = -1$. This gives

$$(1, 0, 0) + (0, 1, 0) + (0, 1, 0) - (1, 1, 1) = (0, 0, 0)$$

Hence, the given set is linearly dependent.

Example 9: Find whether the following set of vectors is linearly dependent or independent, $\{(1, 0, 1), (1, 1, 0), (1, -1, 1), (1, 2, -3)\}$.

Solution: Let us consider the relation $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \theta$ where c_1, c_2, c_3, c_4 are real numbers and $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (1, 1, 0)$, $\alpha_3 = (1, -1, 1)$, $\alpha_4 = (1, 2, -3)$.

$$\therefore \quad c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(1, -1, 1) + c_4(1, 2, -3) = (0, 0, 0)$$

or $$(c_1 + c_2 + c_3 + c_4, c_2 - c_3 + 2c_4, c_1 + c_3 - 3c_4) = (0, 0, 0)$$

$$\therefore \quad \begin{aligned} c_1 + c_2 + c_3 + c_4 &= 0 \\ c_2 - c_3 + 2c_4 &= 0 \\ c_1 + c_3 - 3c_4 &= 0 \end{aligned} \quad \begin{bmatrix} \Rightarrow 3c_4 - c_3 - 2c_4 + c_3 + c_3 + c_4 = 0 \Rightarrow \\ \Rightarrow c_2 = -2c_4 + c_3 \qquad c_3 = -2c_4 \\ \Rightarrow c_1 = 3c_4 - c_3 \quad \therefore c_2 = -4c_4, c_1 = 5c_4 \end{bmatrix}$$

This system has solution $c_1 = 5c_4, c_2 = -4c_4, c_3 = -2c_4$, for each choice of c_4.

Let us take $c_4 = 1$. This gives

$$5(1, 0, 1) - 4(1, 1, 0) - 2(1, -1, 1) + 1(1, 2, -3) = (0, 0, 0)$$

Hence, the given system is linearly dependent.