CHAPTER

11

# Vector Space Theory

The notion of vector space is of prime importance in mathematics, not only because it has a beautiful structure but also because it leads itself to the development of very many methods of solving basic problems of mathematics. Though vector spaces have innumerable properties, we restrict our discussions to only a fundamental few here but before we come to these we begin with the definition of a vector space.

## 11.1 BASIC NOTIONS

**Definition:** A *vector space* (also called a *linear space*) over $\mathbb{R}$ is a non-empty set $V$ endowed with two operations *viz.* addition and scalar multiplication (also called exterior product) which satisfy the following conditions:

(1) $x + y \in V$ for all $x, y \in V$ [*Closure property of addition*]

(2) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$ [*Associative property of addition*]

(3) $\exists\ \theta \in V$ such that $\theta + x = x + \theta = x$ for every $x \in V$ [*Identity property of addition*]

(4) For every $x \in V$, there exists $-x \in V$ such that
$$x + (-x) = \theta = (-x) + x$$
[*Inverse property of addition*]

(5) $x + y = y + x$ for all $x, y \in V$ [*Commutative property of addition*]

(6) $\alpha x \in V$ for all $\alpha \in \mathbb{R}$, all $x \in V$ [*Closure property of scalar multiplication*]

(7) $\alpha(x + y) = \alpha x + \alpha y$ for every $\alpha \in \mathbb{R}$, every $x, y \in V$. [*Scalar distributive property*]

(8) $(\alpha + \beta)x = \alpha x + \beta x$ for every $\alpha, \beta \in \mathbb{R}$, every $x \in V$ [*Vector distributive property*]

(9) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in \mathbb{R}, x \in V$. [*Scalar vector associative property*]

(10) $1\,x = x$ for all $x \in V$ [*Scalar identity property*]

The set $\mathbb{R}$ of all real numbers is usually referred to as the scalar field of $V$ and the elements of $V$ are called *vectors*.

Note that a vector space $V$ may consist of only one element $\{\theta\}$. Such a vector space is called a *trivial vector space*.

**Example 1:** The set $\mathbb{R}$ is a vector space over $\mathbb{R}$:

This is clear as all the ten conditions follow easily for $\mathbb{R}$.

**Remark:** The set $\mathbb{R}$ is also a scalar field. Thus a real number can be treated as a vector as well.

**Example 2:** The set $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ where addition and scalar multiplication are defined on $\mathbb{R}^2$ as follows

$$\left.\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ \alpha(x_1, y_1) &= (\alpha x_1, \alpha y_1)\end{aligned}\right\} \text{ where } (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2,\ \alpha \in \mathbb{R}$$

Clearly, the closure properties of addition and scalar multiplication follow directly from the definition.

For associativity of addition, we observe for $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$.

$$\begin{aligned}\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) &= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)\\ &= (\{x_1 + x_2\} + x_3, \{y_1 + y_2\} + y_3)\\ &= (x_1 + \{x_2 + x_3\}, y_1 + \{y_2 + y_3\})\\ &\qquad \text{since } \mathbb{R} \text{ is associative w.r.t. } +\\ &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)\\ &= (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}.\end{aligned}$$

Since $(0, 0) \in \mathbb{R}^2$ and $(x_1, y_1) + (0, 0) = (x_1, y_1) = (0, 0) + (x_1, y_1)$, the identity property of addition follows. Note $\theta = (0, 0)$ here.

For $(x_1, y_1) \in \mathbb{R}^2$, we have $(-x_1, -y_1) \in \mathbb{R}^2$ satisfying

$$(x_1, y_1) + (-x_1, -y_1) = (0, 0).$$

Hence, the inverse property of addition follows.

For the commutative property of addition, we note

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ &= (x_2 + x_1, y_2 + y_1) \text{ since } \mathbb{R} \text{ is commutative w.r.t. } +\\ &= (x_2, y_2) + (x_1, y_1).\end{aligned}$$

Next, let $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$, then

$$\alpha\{(x_1, y_1) + (x_2, y_2)\} = \alpha(x_1 + x_2, y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$$
$$= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$$
$$= \alpha(x_1, y_1) + \alpha(x_2, y_2).$$

Hence, the scalar distributive property follows.

Again, $(\alpha + \beta)(x_1, y_1) = (\{\alpha + \beta\}x_1, \{\alpha + \beta\}y_1)$
$$= (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1)$$
$$= (\alpha x_1, \alpha y_1) + (\beta x_1, \beta y_1) = \alpha(x_1, y_1) + \beta(x_1, y_1)$$

Thus, the vector distributivity follows.

Finally, $\alpha\{\beta(x_1, y_1)\} = \alpha(\beta x_1, \beta y_1) = (\alpha\{\beta x_1\}, \alpha\{\beta y_1\})$
$$= (\{\alpha\beta\}x_1, \{\alpha\beta\}y_1) \text{ since } \alpha, \beta, x_1, y_1 \in \mathbb{R}$$
$$= (\alpha\beta)(x_1, y_1)$$

and $1(x_1, y_1) = (x_1, y_1)$.

Hence, $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

**Example 3:** The set $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ when $n \geq 1$.

That $\mathbb{R}$ and $\mathbb{R}^2$ are vector spaces has been proved in examples 1 and 2. That $\mathbb{R}^3$ is a vector space can be proved similarly. We prove here the general case.

Let $x, y, z \in \mathbb{R}^n$ and $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n)$, and let $\alpha, \beta \in \mathbb{R}$.

Then by definition

$$x + y = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \in \mathbb{R}^n.$$
$$\alpha x = \alpha(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n) \in \mathbb{R}^n.$$

So the closure properties follow.

The associative and commutative properties follow from the respective properties of $\mathbb{R}$.

The element $\theta = (0, 0, ..., 0) \in \mathbb{R}^n$ is the additive identity of $\mathbb{R}$.

Then element $-x = (-x_1, -x_2, ..., -x_n) \in \mathbb{R}^n$ is the additive inverse of $\mathbb{R}$.

Further, $\alpha(x + y) = \alpha(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
$$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, ..., \alpha x_n + \alpha y_n)$$
$$= (\alpha x_1, \alpha x_2, ..., \alpha x_n) + (\alpha y_1, \alpha y_2, ..., \alpha y_n)$$
$$= \alpha(x_1, x_2, ..., x_n) + \alpha(y_1, y_2, ..., y_n)$$
$$= \alpha x + \alpha y.$$
$$(\alpha + \beta)x = (\alpha + \beta)(x_1, x_2, ..., x_n)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, ..., \alpha x_n + \beta x_n)$$
$$= (\alpha x_1, \alpha x_2, ..., \alpha x_n) + (\beta x_1, \beta x_2, ..., \beta x_n) = \alpha x + \beta y.$$

Thus scalar distributivity and vector distributivity follow.

Finally, $\alpha(\beta x) = \alpha(\beta x_1, \beta x_2, ..., \beta x_n) = (\alpha\beta x_1, \alpha\beta x_2, ..., \alpha\beta x_n)$
$= (\alpha\beta)(x_1, x_2, ..., x_n) = \alpha\beta x.$

and $1x = 1(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n).$

Hence, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

**Example 4:** The set $\mathbb{R}[x]$ of all polynomials in a real variable $x$ with real coefficients is a vector space over $\mathbb{R}$.

To see the above one should note first that the sum of any two such polynomials is also a polynomial with real coefficients and the product of such a polynomial and a real number is also a polynomial of the same time. These observations had to the closure properties of addition and scalar multiplication. The associativity and commutativity of addition of polynomials are obvious. The constant 0 is a polynomial of degre zero, which plays the role of additive identity. The additive inverse of a polynomial is a polynomial with coefficients negative of the coefficients of the polynomial. The scalar and vector distributive properties follow easily since

$(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x)$ where $f(x)$ is a polynomial, $\alpha, \beta \in \mathbb{R}$ and $\alpha(f(x) + g(x)) = \alpha f(x) + \alpha g(x)$ where $f(x)$ and $g(x)$ are two polynomials in $x$ and $\alpha \in \mathbb{R}$. Clearly $1 f(x) = f(x)$.

Hence, $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$.

**Example 5:** Let $M_{m \times n}(\mathbb{R})$ denote all $m \times n$-matrices with real entries. Then $M_{m \times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Since the sum of two $m \times n$-matrices is again and $m \times n$-matrix, the closure property of addition follows. If an $m \times n$-matrix, is multiplied by a real number, then the product is also an $m \times n$-matrix. This proves the closure property of scalar multiplication. The associativity and commutativity of addition follow from the corresponding properties of real numbers. The $m \times n$-matrices with all entries zero is the additive identity and the $m \times n$ matrix with all entries negative of the entries of an $m \times n$ matrix plays the role of its inverse. The scalar vector associativity, the scalar distributivity and the vector distributivity follow easily.

The multiplication of an $m \times n$ matrix by the real numbers gives the same matrix.

Thus $M_{m \times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

The following result is an immediate consequence of the definition.

**Theorem:** The following are true for a vector space $V$.

(*i*) $ox = \theta \quad \forall\, x \in V$

(*ii*) $\alpha\theta = \theta \quad \forall\, \alpha \in \mathbb{R}$

(*iii*) $\alpha(-x) = -\alpha x \quad \forall\, \alpha \in \mathbb{R}, x \in V$

(*iv*) $\alpha(x - y) = \alpha x - \alpha y \quad \forall\, \alpha \in \mathbb{R}, x, y \in V$

(*v*) $(\alpha - \beta)x = \alpha x - \beta x \quad \forall\, \alpha, \beta \in \mathbb{R}, x \in V.$

**Proof:** (*i*) We know

$$ox = (o + o)x = ox + ox.$$

$\therefore$ $ox = \theta$, by cancellation property

(*ii*) Again,

$$\alpha(\theta + \theta) = \alpha\theta$$

or $\alpha\theta + \alpha\theta = \alpha\theta$, by scalar distributivity

$\therefore$ $\alpha\theta = \theta$, by cancellation property.

(*iii*) Now,

$$\alpha\{x + (-x)\} = \alpha\theta = \theta$$

or $\alpha x + \alpha(-x) = \theta$ by scalar distributivity.

or $-\alpha x + \{\alpha x + \alpha(-x)\} = -\alpha x + \theta$

or $\{-\alpha x + \alpha x\} + \alpha(-x) = -\alpha x$

or $\theta + \alpha(-x) = -\alpha x$

or $\alpha(-x) = -\alpha x$

(*iv*) Here we see

$$\begin{aligned}\alpha(x - y) &= \alpha\{x + (-y)\}\\ &= \alpha x + \alpha(-y)\\ &= \alpha x - \alpha y, \text{ by } (iii)\end{aligned}$$

(*v*) Finally, $(\alpha - \beta)x + \beta x = \{(\alpha - \beta) + \beta\}x = \alpha x.$

Hence, $(\alpha - \beta)x = \alpha x - \beta x.$

## 11.2 SUBSPACE

Subspaces of vector spaces play important role in the development of vector space theory. We begin with its definition.

**Definition:** A subset $S$ of a vector space $V$ over $\mathbb{R}$ is called a (vector) *subspace* of $V$ if $S$ is a vector space with respect to the same operations of addition and scalar multiplication.

Since many of the properties of a vector space are transmitted automatically to its subsets, to verify whether a subset of a vector space is a subspace, we need not verify all the ten conditions. Indeed we need to verify only two closure properties which can be merged into one single condition. Thus we get

**Theorem:** A subset $S$ of a vector space $V$ over $\mathbb{R}$ is a subspace if and only if $\alpha x + \beta y \in S$ for all $\alpha, \beta \in \mathbb{R}, x, y \in \mathbb{S}$.

**Proof:** *'If' part:*

Taking $\alpha = \beta = 1, x, y \in \mathbb{S}$, we see $x + y \in S$.

So $S$ has the closure property of addition.

If $x, y, z \in \mathbb{S}$, then $x, y, z \in V$ also and hence $(x + y) + z = x + (y + z)$ and $x + y = y + x$.

Thus, the associative and commutative properties of addition follow in $S$.

Taking $x \in S$, we see $1x - 1\,x \in S$ (here $\alpha = 1, \beta = -1$).

*i.e.*, $\theta \in S$. Thus, the identity property of addition is proved.

Again, if $x \in S$, then $ox - 1\,x \in S$ (here $\alpha = 0, \beta = -1$).

*i.e.*, $-x \in S$. The inverse property thus follows.

Next, $\alpha, \beta \in \mathbb{R}, x \in S$, then $x \in V$ also as $S \subset V$.

Hence, $\alpha(\beta x) = (\alpha\beta)x$. This proves scalar vector associativity.

Further, if $\alpha, \beta \in \mathbb{R}, x \in S$, then $x \in V$ also.

Therefore, $(\alpha + \beta)x = \alpha x + \beta x$. This proves vector distributivity.

If $\quad \alpha \in \mathbb{R}, x, y \in S$ then $x, y \in V$ and so

$$\alpha(x + y) = \alpha x + \alpha y.$$

This proves scalar distributivity.

Finally, $x \in S$ implies $x \in V$ and hence

$1x = x$, which proves the scalar identity property.

Thus $S$ is a vector space over $\mathbb{R}$.

*'Only if' part:*

Let $S$ be a subspace of $V$ and let $\alpha, \beta \in \mathbb{R}, x, y \in S$.

Then by the closure property of scalar multiplication $\alpha x \in S, \beta y \in S$. By the closure property of addition we then get $\alpha x + \beta y \in S$.

**Corollary 1:** If $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ and $x_1, x_2, ..., x_n \in S$ then $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \in S$.

**Corollary 2:** If $x \neq \theta, x \in V$, then $S = \{\alpha x; \alpha \in \mathbb{R}\}$ is a subspace of $V$.

**Corollary 3:** If $x_1 \neq \theta, x_2 \neq \theta, ..., x_n \neq \theta$ all in $V$, then $S = \{\alpha, x_1 + \alpha_2 x_2 + ... + \alpha_n x_n; \alpha, \alpha_2, ..., \alpha_n \in \mathbb{R}\}$ is a subspace of $V$.

**Corollary 4:** The necessary and sufficient conditions for a non-empty subset $S$ of a vector space $V$ over $\mathbb{R}$ to be a subspace of $V$ are

(*i*) $x \in S, y \in S \Rightarrow x + y \in S$

(*ii*) $a \in \mathbb{R}, x \in S \Rightarrow ax \in S$

A result that follows directly from the above is the following:

**Theorem:** If $S_1$ and $S_2$ are two subspaces of a vector space $V$, then

(*i*) $S_1 \cap S_2$ is a subspace of $V$.

(*ii*) $S_1 + S_2$ is a subspace of $V$ where $S_1 + S_2 = \{x + y; x \in S_1; y \in S_2\}$.

(*iii*) $S_1 \cup S_2$ need not be a subspace of $V$.

**Proof:** (*i*) Let $\alpha, \beta \in \mathbb{R}, x, y \in S_1 \cap S_2$.

Then $\quad x, y \in S_1$ and $x, y \in S_2$

So $\quad \alpha x + \beta y \in S_1$ and $\alpha x + \beta y \in S_2$

$\therefore \quad \alpha x + \beta y \in S_1 \cap S_2$

This implies $S_1 \cap S_2$ is a subspace of $V$.

(*ii*) Let $\alpha, \beta \in \mathbb{R}, x_1 + y_1 \in S_1 + S_2, x_2 + y_2 \in S_1 + S_2$

Then
$$\alpha(x_1 + y_1) + \beta(x_2 + y_2)$$
$$= \{\alpha x_1 + \alpha y_1\} + \{\beta x_2 + \beta y_2\}, \text{ by scalar distributivity}$$
$$= \{\alpha x_1 + \beta x_2\} + \{\alpha y_1 + \beta y_2\}, \text{ by commutativity of addition}$$
$$= S_1 + S_2$$

Since $\alpha x_1 + \beta x_2 \in S_1$ and $\alpha y_1 + \beta y_2 \in S_2$.

Hence, $S_1 + S_2$ is a subspace of $V$.

(*iii*) Consider $S_1 = \{(x, 0); x \in \mathbb{R}\}$ and $S_2 = \{\alpha(1, 1); \alpha \in \mathbb{R}\}$

Clearly $S_1$ and $S_2$ are two subspaces of $\mathbb{R}^2$.

Now, $(1, 0) \in S_1, (1, 1) \in S_2$ but $(1, 0) + (1, 1) = (2, 1) \neq S_1 \cup S_2$

Since $S_1 \cup S_2$ consists of elements either of elements whose 2nd coordinate is zero or of elements whose both coordinates are equal.

Thus, the closure property of addition having failed, $S_1 \cup S_2$ cannot be a (vector) subspace of $\mathbb{R}^2$.

**Theorem:** Let $V$ be a vector space over $\mathbb{R}$.

(*i*) If $x$ be a non-zero vector of $V$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha x = \beta x \Rightarrow \alpha = \beta$.

(*ii*) If $x, y \in V$ and a be *a* non-zero value of $\mathbb{R}$, then $ax = ay \Rightarrow x = y$.

**Proof:** (*i*) Here $\quad \alpha x = \beta x$

$\therefore \quad \alpha x - \beta x = \theta$

or $\quad (\alpha - \beta)x = \theta$

Since $\quad x \neq \theta$, then $\alpha - \beta = 0 \Rightarrow \alpha = \beta$.

(*ii*) Here $\quad ax = ay$

$\therefore \quad ax - ay = \theta$

or $\quad a(x - y) = \theta$

Since $x \neq 0$, then $x - y = \theta \Rightarrow x = y$.

**Note:** *Field*: Field is a algebraic structure with respect to the two arithmetic operation addition (+) and multiplication $(\cdot)$.

**Example 1:** Let $S$ be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$, then show that $S$ is a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (0, 0, z_1)$ and $\beta = (0, 0, z_2)$ where $z_1, z_2 \in \mathbb{R}$, then $\alpha, \beta \in S$.

Let $a, b \in \mathbb{R}$, then $c\alpha + d\beta = a(0, 0, z_1) + b(0, 0, z_2) = (0, 0, a z_1) + (0, 0, b z_2)$

$$= (0, 0, az_1 + bz_2) \in S \qquad (\because az_1 + bz_2 \in \mathbb{R})$$

Hence, $S$ is a subspace of $\mathbb{R}^3$.

**Example 2:** Let $S$ be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = y^2\}$, then show that $S$ is not a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$$x_1^2 + z_1^2 = y_1^2 \text{ and } x_2^2 + z_2^2 = y_2^2$$

Now $\alpha + \beta = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

since $(x_1 + x_2)^2 + (z_1 + z_2)^2$ may not be equal to $(y_1 + y_2)^2$, hence

$(\alpha + \beta)$ may not belong to $S$.

For example, let $\alpha = (4, 5, -3)$ and $\beta = (-4, 5, 3)$, then

$$\alpha + \beta = (0, 10, 0) \notin S \quad (\because 0^2 + 0^2 \neq 10^2)$$

Hence, $S$ is not a subspace of $\mathbb{R}^3$.

**Example 3:** Let $S$ be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 2\}$, then show that $S$ is not a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$$x_1 + y_1 + z_1 = 2 \text{ and } x_2 + y_2 + z_2 = 2$$

$\therefore \alpha + \beta = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin S$

because $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$

$$= 2 + 2 = 4 \neq 2$$

Hence, $S$ is not a subspace of $\mathbb{R}^3$.

**Example 4:** Let $S$ be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, 0) \in \mathbb{R}^3 : x, y, \in \mathbb{R}\}$, then show that $S$ is a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x_1, y_1, 0) \in S$ and $\beta = (x_2, y_2, 0) \in S$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Let $a, b \in \mathbb{R}$, then $a\alpha + b\beta = a(x_1, y_1, 0) + b(x_2, y_2, 0)$

$$= (ax_1, ay_1, 0) + (bx_2, by_2, 0)$$

$$= (ax_1 + bx_2, ay_1 + by_2, 0) \in S$$

$$(\because ax_1 + by_1 \in \mathbb{R} \text{ and } ax_2 + by_2 \in \mathbb{R})$$

Hence, $S$ is a subspace of $\mathbb{R}^3$.

**Example 5:** Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x - 3y + 4z = 0\}$ is a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$$x_1 - 3y_1 + 4z_1 = 0 \text{ and } x_2 - 3y_2 + 4z_2 = 0 \quad ...(1)$$

Let $a, b \in \mathbb{R}$, then $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

Now, $(ax_1 + bx_2) - 3(ay_1 + by_2) + 4(az_1 + bz_2)$

$$= (ax_1 - 3ay_1 + 4az_1) + (bx_2 - 3by_2 + 4bz_2)$$

$$= a(x_1 - 3y_1 + 4z_1) + b(x_2 - 3y_2 + 4z_2)$$

$$= a \cdot 0 + b \cdot 0, \text{ by (1)}$$

$$= 0$$

$\therefore \quad a\alpha + b\beta \in S$

Hence, $S$ is a subspace of $\mathbb{R}^3$.

**Example 6:** Show that $S = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0 \text{ and } x - 2y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x, y, z) \in S$ and $\beta = (a, b, c) \in S$, then

$$x + y - z = 0, x - 2y + z = 0 \text{ and } a + b - c = 0, a - 2b + c = 0 \quad ...(1)$$

Let $p, q \in \mathbb{R}$, then $p\alpha + q\beta = p(x, y, z) + q(a, b, c) = (px + qa, py + qb, pz + qc)$

Now $\quad (px + qa) + (py + qb) - (pz + qc)$

$$= (px + py - pz) + (qa + qb - qc) = p(x + y - z) + q(a + b - c)$$

$$= p \cdot 0 + q \cdot 0 \text{ by (1)}$$

$$= 0$$

and $\quad (px + qa) - 2(py + qb) + (pz + qc)$

$$= (px - 2py + pz) + (qa - 2qb + qc)$$

$$= p(x - 2y + z) + q(a - 2b + c) = p \cdot 0 + q \cdot 0, \quad \text{(by (1))}$$

$$= 0$$

$\therefore \quad p\alpha + q\beta \in S$

Hence, $S$ is a subspace of $\mathbb{R}^3$.

**Example 7:** Show that $S = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$ is a subspace of $\mathbb{R}^3$ where $a, b, c \in \mathbb{R}$.

**Solution:** Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$$ax_1 + by_1 + cz_1 = 0 \text{ and } ax_2 + by_2 + cz_2 = 0 \quad ...(1)$$

Let $p, q \in \mathbb{R}$, then $p\alpha + q\beta = p(x_1, y_1, z_1) + q(x_2, y_2, z_2)$

$$= (px_1 + qx_2, py_1 + qy_2, pz_1 + qz_2)$$

Now $a(px_1 + qx_2) + b(py_1 + qy_2) + c(pz_1 + qz_2)$

$$= (apx_1 + bpy_1 + cpz_1) + (aqx_2 + bqy_2 + cqz_2)$$

$$= p(ax_1 + by_1 + cz_1) + q(ax_2 + by_2 + cz_2) = p.0 + q.0 \text{ (by (1))}$$

$$= 0$$

$\therefore$ $p\alpha + q\beta \in S$. Hence, $S$ is a subspace of $\mathbb{R}^3$.

**Example 8:** Show that $S = \{(x, y, z) \in \mathbb{R}^3 : xy = z\}$ is not a subspace of $\mathbb{R}^3$.

**Solution:** Let $\alpha = (x_1, y_1, z_1) \in S$ and $\beta = (x_2, y_2, z_2) \in S$, then

$$x_1\, y_1 = z_1, \text{ and } x_2\, y_2 = z_2$$

Let $a, b \in \mathbb{R}$, then $a\alpha + b\beta = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Since $(ax_1 + bx_2)(ay_1 + by_2)$ may not equal to $az_1 + bz_2$, hence, $a\alpha + b\beta$ may not belong to $S$.

For example, $\alpha = (1, 2, 2)$, $\beta = (3, 1, 3)$ and $a = b = 1$, then

$$a\alpha + b\beta = 1(1, 2, 2) + 1(3, 1, 3) = (4, 3, 5) \notin S \text{ because } 4 \,.\, 3 \neq 5.$$

Hence, $S$ is not a subspace of $\mathbb{R}^3$.

**Example 9:** Prove that the union of two subspaces of $V$ is not, in general, a subspace of $V$.

**Solution:** Let us consider the two subspaces $S$ and $T$ of the vector space $\mathbb{R}^3$ where $S = \{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3 : x = z = 0\}$.

Let $\alpha = (1, 0, 0) \in S$ and $\beta = (0, 1, 0) \in T$, then $\alpha + \beta = (1, 1, 0)$

$\therefore$ $\alpha + \beta \notin S \cup T$ $(\because \alpha + \beta \notin S \text{ and } \alpha + \beta \notin T)$

Here $\alpha \in S \cup T$ and $\beta \in S \cup T$ but $\alpha + \beta \notin S \cup T$

Hence, $S \cup T$ is not a subspace of $\mathbb{R}^3$.

**Example 10:** Let $V$ be a vector space over $\mathbb{R}$ and $\alpha \in V$, then prove that

$$S = \{a\alpha : a \in \mathbb{R}\} \text{ is a subspace of } V.$$

**Solution:** Let $\alpha = \theta$, then $S = \{\theta\}$ and $S$ is a subspace of $V$.

Let $\alpha \neq \theta$, then $S$ is a non-empty subset of $V$ $(\because \alpha \in V)$

Let $\beta \in S$ and $\gamma \in S$, then $\beta = a_1\, \alpha$ and $\gamma = a_2\, \alpha$ where $a_1$ and $a_2$ are some scalar.

$\therefore \beta + \gamma = (a_1 + a_2)\, \alpha \in S$ $(\because a_1 + a_2 \in \mathbb{R})$

Let $c$ be a scalar in $\mathbb{R}$, then $c\beta = c(a_1\, \alpha) = (ca_1)\, \alpha \in S$ $(\because ca_1 \in \mathbb{R})$

$\therefore \beta, \gamma \in S \Rightarrow \beta + \gamma \in S$ and $c \in \mathbb{R}, \beta \in S \Rightarrow c\beta \in S$

Hence, $S$ is a subspace of $V$.

**Note:** This subspace is generated by the vector $\alpha$, and $\alpha$ is called the generator of this subspace.

**Example 11:** Show that $S = \left\{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{R}_{2\times 2} : x + y = 0\right\}$ is a subspace of $\mathbb{R}_{2\times 2}$.

**Solution:** Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{R}_{2\times 2}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2}$, then $x + y = 0$, $a + b = 0$

Let $p, q \in \mathbb{R}$, then

$$pA + qB = p\begin{pmatrix} x & y \\ z & w \end{pmatrix} + q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} px + qa & py + qb \\ pz + qc & pw + qd \end{pmatrix} \in \mathbb{R}_{2\times 2}$$

$(\because (px + qa) + (py + qb) = p(x + y) + q(a + b) = p \cdot 0 + q \cdot 0 = 0)$

Hence, $S$ is a subspace of $\mathbb{R}_{2\times 2}$.

**Linear Combination:** Let $V$ be a vector space over $\mathbb{R}$ and let $\alpha_1, \alpha_2, \ldots \alpha_r \in V$, then a vector $\beta$ in $V$ is said to be a *linear combination* of the vectors $\alpha_1, \alpha_2, \ldots \alpha_r$ if $\beta$ can be expressed as

$$\beta = c_1 \alpha_1 + c_2 \alpha_2 + \ldots + c_r \alpha_r \text{ for some scalars } c_1, c_2, \ldots, c_r \in \mathbb{R}$$

**Example:** Let $V$ be a real vector space and $\alpha, \beta, \gamma \in V$, then $2\alpha + \beta + \gamma$, $\alpha + 4\beta + 3\gamma$, $0 . \alpha + 2\beta + 3\gamma$, $0\alpha + 2\beta + 0 . \gamma$, $0\alpha + 0\beta + 0\gamma$, ..., are linear combinations of $\alpha, \beta, \gamma$.

**Theorem:** Let $V$ be vector space over $\mathbb{R}$ and let $S$ be a non-empty finite subset of $V$. Then the set $W$ of all linear combinations of the vectors in $S$ forms a subspace of $V$ and it is smallest subspace containing the set $S$.

**Proof:** Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, then $W = \{c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n : c_1, c_2, \ldots, c_n \in \mathbb{R}\}$.

$\therefore$ $W$ is a non-empty subset of $V$, since $\alpha_1 \in W$.

Let $\alpha = r_1 \alpha_1 + r_2 \alpha_2 + \ldots + r_n \alpha_n \in W$ and $\beta = s_1\alpha_1 + s_2\alpha_2 + \ldots + s_n\alpha_n \in W$

where $\quad r_1, r_2, \ldots, r_n \in \mathbb{R}$ and $s_1, s_2, \ldots, s_n \in \mathbb{R}$

$$\therefore \quad \alpha + \beta = (r_1\alpha_1 + r_2\alpha_2 + \ldots + r_n \alpha_n) + (s_1\alpha_1 + s_2\alpha_2 + \ldots + s_n\alpha_n)$$
$$= (r_1 + s_1)\alpha_1 + (r_2 + s_2)\alpha_2 + \ldots + (r_n + s_n)\alpha_n \in W$$
$$(\because r_i + s_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n)$$

Let $a \in \mathbb{R}$, then $a\alpha = a(r_1\alpha_1 + r_2\alpha_2 + \ldots + r_n\alpha_n)$

$$= (ar_1)\alpha_1 + (ar_2)\alpha_2 + \ldots + (ar_n)\alpha_n \in W$$
$$(\because ar_i \in \mathbb{R}, i = 1, 2, \ldots, n)$$

$\therefore$ $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$ and $p \in \mathbb{R}, \alpha \in W \Rightarrow p\alpha \in W$

Hence, $W$ is a subspace of $V$.

Let $U$ be any subspace of $V$ containing the set $S$.

Let $\xi \in W$, then $\xi = x_1\alpha_1 + x_2\alpha_2 + ... + x_n\alpha_n$ for some $x_i \in \mathbb{R}$

Since $U$ is a subspace and $x_1\alpha_1, x_2\alpha_2, ... x_n\alpha_n \in U$, then

$$x_1\alpha_1 + x_2\alpha_2 + ... + x_n\alpha_n \in U$$

$\because$ $\xi \in W \Rightarrow \xi \in U$

$\therefore$ $W \subset U$

This proves that $W$ is the smallest subspace containing $S$.

**Linear Span:** The smallest subspace containing a finite set $S$ of a vector space $V$ is said to be the linear span of $S$ and it is denoted by $L(S)$.

$L(S)$ is generated (or spanned) by the set $S$ and $S$ is said to be the set of generators of $L(S)$.

**Note:** Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, then $L(S)$ is the set of all linear combinations of the vectors $\alpha_1, \alpha_2, ..., \alpha_n$.

**Theorem:** Let $S$ and $T$ be two non-empty finite subset of a vector space $V$ over $\mathbb{R}$ and $S \subset T$, then $L(S) \subset L(T)$.

**Proof:** Let $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\xi \in L(S)$, then

$$\xi = r_1\alpha_1 + r_2\alpha_2 + ... + r_n\alpha_n \text{ for some scalars } r_i \in \mathbb{R}, i = 1, 2, ..., n$$

$\therefore$ $\alpha_1 \in S \Rightarrow \alpha_1 \in T \quad (\because S \subset T)$

$\therefore$ $\alpha_1 \in L(T)$

$\because$ $L(T)$ is a subspace of $V$, $r_1 \in \mathbb{R}$ and $\alpha_1 \in L(T) \Rightarrow r_1\alpha_1 \in L(T)$

Similarly, $r_2\alpha_2 \in L(T), ..., r_n\alpha_n \in L(T)$

Since $L(T)$ is a subspace of $V$, then $r_1\alpha_1 + r_2\alpha_2 + ... + r_n\alpha_n \in L(T)$

$\therefore$ $\xi \in L(T)$

$\therefore$ $\xi \in L(S) \Rightarrow \xi \in L(T)$

Hence, $L(S) \subset L(T)$.

**Result:** Let $S$ and $T$ be two non-empty finite subsets of a vector space $V$ over $\mathbb{R}$ and each element of $T$ is a linear combination of the elements of $S$, then $L(T) \subset L(S)$.

**Example 1:** If $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$, $\gamma = (2, 1, 1)$, $\delta = (4, 2, 2)$ in $\mathbb{R}^3$, then prove that

(*i*) $\alpha$ is a linear combination of $\beta$ and $\gamma$.

(*ii*) $\beta$ is not a linear combination of $\gamma$ and $\delta$.

**Solution:** (*i*) Let $\alpha = c\beta + d\gamma$ where $c, d \in \mathbb{R}$

$\therefore \quad (4,3,5) = c(0,1,3) + d(2,1,1) = (0, c, 3c) + (2d, d, d)$

$= (2d, c+d, 3c+d)$

$\therefore \quad 4 = 2d,\ 3 = c + d$ and $3c + d = 5$

$\therefore \quad d = 2, \quad c = 3 - d = 3 - 2 = 1,$

$\therefore \quad \alpha = \beta + 2\gamma$, hence $\alpha$ is the linear combination of $\beta$ and $\gamma$.

(*ii*) Let $\beta = a\gamma + b\delta$ where $a, b \in \mathbb{R}$

$\therefore \quad (0,1,3) = a(2,1,1) + b(4,2,2) = (2a, a, a) + (4b, 2b, 2b)$

$= (2a + 4b, a + 2b, a + 2b)$

$\therefore \quad 2a + 4b = 0, \quad a + 2b = 1$ and $a + 2b = 3$.

The equations are inconsistent. Hence, $\beta$ cannot be expressed as $a\gamma + b\delta$ for real $a, b$.

$\therefore$ $\beta$ is not a linear combination of $\gamma$ and $\delta$.

**Example 2:** Find the subspace of $R^3$ spanned by the vectors $\alpha = (1, 2, 3)$, $\beta = (3, 1, 0)$. Examine if $\gamma = (2, 1, 3)$, $\delta = (-1, 3, 6)$ are in the subspace.

**Solution:** Now $L\{\alpha, \beta\}$ is the set of vectors $\{c\alpha + c\beta : c, d \in \mathbb{R}\}$

If $\gamma \in L\{\alpha, \beta\}$, then there exists real numbers $c, d$ such that

$\gamma = (2, 1, 3) = c\alpha + d\beta = c(1, 2, 3) + d(3, 1, 0) = (c + 3d, 2c + d, 3c + 0)$

or $\quad (2, 1, 3) = (c + 3d, 2c + d, 3c + 0)$

$\therefore \quad c + 3d = 2, 2c + d = 1, 3c = 3$

$[\because\ 3c = 3 \Rightarrow c = 1, 2c + d = 1$

$\Rightarrow d = 1 - 2c = 1 - 2 = -1$ and $c = 2 - 3d = 2 + 3 = 5 \neq 1]$

These equations are inconsistent and so $\gamma$ is not in $L\{\alpha, \beta\}$.

If $\delta \in L\{\alpha, \beta\}$, then there exists real numbers $c, d$ such that

$\delta = c\alpha + d\beta$ or $(-1, 3, 6) = (c + 3d, 2c + d, 3c)$

$\therefore \quad -1 = c + 3d, 3 = 2c + d, 3c = 6$

Solving these equations, we get $c = 2, d = -1$

The equations are consistent.

$\therefore\ \delta = 2(1, 2, 3) - 1(3, 1, 0) \quad \therefore\ \delta \in L\{\alpha, \beta\}$.

**Example 3:** Let $S = \{\alpha, \beta, \gamma\}$, $T = \{\alpha, \beta, \alpha + \beta, \beta + \gamma\}$ be subsets of a real vector space $V$. Show that $L(S) = L(T)$.

**Solution:** Since $S$ and $T$ are finite subsets of $V$ and each element of $T$ is a linear combination of the vectors of $S$, then $L(T) \subset L(S)$.

Again $\quad \alpha = \alpha + 0\,.\,\beta + 0\,.\,(\alpha + \beta) + 0(\beta + \gamma)$

$$\beta = 0\,.\,\alpha + \beta + 0(\alpha + \beta) + 0(\beta + \gamma)$$

$$\gamma = 0\,.\,\alpha - \beta + 0(\alpha + \beta) + (\beta + \gamma)$$

This shows that each element of $S$ is a linear combination of the elements of $T$ and therefore $L(S) \subset L(T)$.

$\because \quad L(T) \subset L(S)$ and $L(S) \subset L(T)$

$\therefore \quad L(S) = L(T)$.

**Example 4:** Express $(-1, 2, 4)$ as a linear combination of $\alpha = (-1, 2, 0)$, $\beta = (0, -1, 1)$ and $\gamma = (3, -4, 2)$ in the vector $V_3$ of real numbers.

**Solution:** Let $(-1, 2, 4) = a(-1, 2, 0) + b(0, -1, 1) + c(3, -4, 2)$ for $a, b, c \in \mathbb{R}$.

$$= (-a, 2a, 0) + (0, -b, b) + (3c, -4c, 2c)$$

$$= (-a + 3c, 2a - b - 4c, b + 2c)$$

$$\therefore \quad -a + 3c = -1$$

$$2a - b - 4c = 2$$

$$b + 2c = 4$$

Solving these equations by Cramer's rule, we get $a = 4, b = 2$ and $c = 1$.

$$\therefore \quad (-1, 2, 4) = 4(-1, 2, 0) + 2(0, -1, 1) + 1(3, -4, 2).$$

**Example 5:** Examine if the set $S$ is a subspace of the vector space $\mathbb{R}_{2\times 2}$ where

$$S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} : \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\right\}.$$

**Solution:** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in S$, then $\det A = 0$ and $\det B = 0$

Now, $A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$

$A + B$ may not belong to $S$ because $\det(A + B) = \det\begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$ may not be equal to zero.

For example, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 12 \\ 1 & 4 \end{pmatrix} \in S$ where $\det A = 0$, $\det B = 0$.

Now $\quad A + B = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 12 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 14 \\ 4 & 10 \end{pmatrix}$

But $\quad \det(A + B) = 40 - 56 = -16 \neq 0$

Hence, $S$ is not a subspace of $\mathbb{R}_{2 \times 2}$.