7.30 TORSIONAL LOADING

7.30.1 Torque

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft.

The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft.

A torque causes twist or torsion in the member. Therefore, it is also known as twisting moment or turning moment. It is denoted by 'T'.

It the shaft is subjected to two opposite turning moments it is said to be in pure tension and as a result, it has a tendency to shear off at every cross-section perpendicular to the longitudinal axis. So the effect of torsion is to produce shear stress in the material of the shaft.

7.30.2 Torsion Equation

The torsion equation is based on the following assumptions:

1. The material of the shaft is uniform throughout
2. The shaft is of uniform circular section throughout
3. Cross-section of the shaft, which are plane before twist remain plane after twist.
4. The twist along the length of shaft is uniform throughout.
5. The distance between any two normal cross-section remains the same after the application of torque.
6. Maximum shear stress induced in the shaft due to application of torque does not exceed its elastic limit value.

Consider a shaft fixed at one end AA' and free at the other and BB' as shown in Fig. 45.

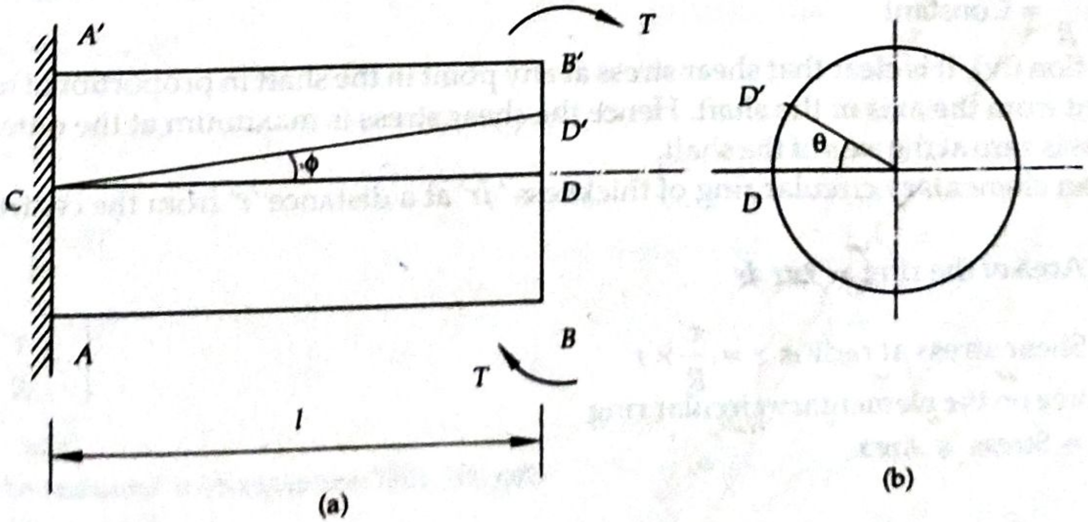

Fig. 45

Let T = Torque or twisting moment
R = Radius of shaft
l = Length of the shaft
τ = Shear stress
G = Modulus of rigidity of the material of the shaft.
θ = Angle of twist.

Let a line CD on the surface of the shaft take a new position CD' after the application of torque. Thus cross-sections will be twisted through angle θ and surface by angle ϕ.

$$\text{Shear strain} = \frac{DD'}{l}$$

$$= \frac{DD'}{CD} = \tan\phi$$

$$= \phi \qquad \text{(If } \phi \text{ is very small than } \tan\phi = \phi)$$

$$\therefore \quad \text{Shear strain, } \phi = \frac{DD'}{l}$$

$$\text{Arc } DD' = OD \times \theta = R\theta \qquad (\because OD = R = \text{Radius of shaft})$$

$$\therefore \quad \text{Shear strain, } \phi = \frac{R\theta}{l} \qquad \text{... (i)}$$

Also $$\text{Shear strain} = \frac{\text{Shear stress}}{\text{Modulus of rigidity}}$$

or $$\phi = \frac{\tau}{G} \qquad \text{... (ii)}$$

from Eq. (i) and (ii), we get

$$\frac{\tau}{G} = \frac{R\theta}{l}$$

or $$\frac{\tau}{R} = \frac{G\theta}{l} \qquad \text{... (iii)}$$

$$\tau = \frac{RG\theta}{l}$$

In a given shaft, under a given torque, G, θ, and l are constant.

$$\therefore \quad \tau \propto R$$

or $$\frac{\tau}{R} = \text{Constant} \qquad \text{... (iv)}$$

from equation (iv), It is clear that shear stress at any point in the shaft in proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

Consider an elementary circular ring of thickness 'dr' at a distance 'r' from the centre as shown in Fig. 46.

$$\text{Area of the ring} = 2\pi r\, dr$$

$$\text{Shear stress at radius } r = \frac{\tau}{R} \times r \qquad \left[\because \frac{\tau}{R} = \text{Constant}\right]$$

Turning force on the elementary circular ring

= Stress $\times$ Area

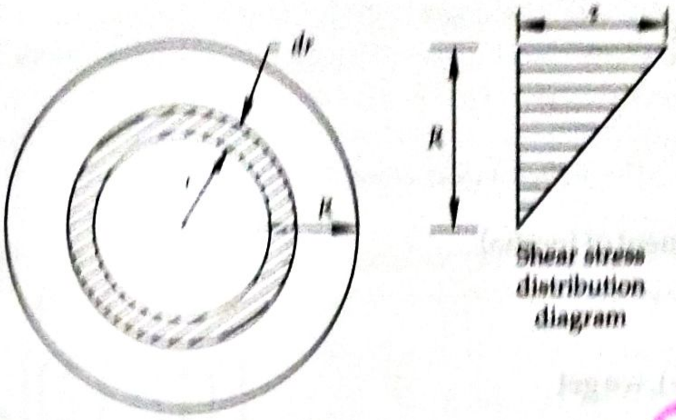

Fig. 46

$= \frac{\tau}{R} \times r \times 2\pi r \cdot dr$

$= \frac{\tau}{R} \times 2\pi r^2 \cdot dr$

Turning moment due to turning force on the elementary ring,

$$dT = \frac{\tau}{R} \times 2\pi r^2 \cdot dr \cdot r$$

$$= \frac{\tau}{R} \times 2\pi r^3 \cdot dr$$

Total turning moment, $T = \int dT$

$$= \int_0^R \frac{\tau}{R} \times 2\pi r^3 \cdot dr$$

$$T = \frac{\tau}{R} \times 2\pi \int_0^R r^3 \cdot dr$$

$$= \frac{\tau}{R} \times 2\pi \left| \frac{r^4}{4} \right|_0^R$$

$$= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4}$$

$$= \tau \times \frac{\pi}{2} \times R^3$$

$$= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} \qquad \left[\because R = \frac{D}{2} \right]$$

$$= \frac{\pi}{16} \tau D^3$$

$$T = \frac{\pi}{16} \tau D^3$$

$$\tau = \frac{16T}{\pi D^3}$$

Substituting the value of τ in equation (iii), we get

$$\frac{16T}{\pi D^3 \cdot R} = \frac{G\theta}{l}$$

$$\frac{16T \cdot 2}{\pi D^3 \cdot D} = \frac{G\theta}{l}$$

$$\frac{32T}{\pi D^4} = \frac{G\theta}{l}$$

But $\frac{\pi D^4}{32} = J$ (polar moment of Inertia)

$$\therefore \quad \frac{T}{J} = \frac{G\theta}{l} \qquad \text{...(v)}$$

from equation (iii) and (v), we get

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau}{R}$$

This equation is known as torsion equation.