10.8 GAUSS JORDAN METHOD OF INVERSE CALCULATION

The basis of Gauss-Jordan method is the observation that the set of row elementary operations which reduce a non-singular square matrix to the identity matrix and reduce the identity matrix to the inverse of the matrix. The entire calculation is done side by side in order to avoid any mistake in the process and is shown below with an example.

Example 1: Use Gauss-Jordan method to find the inverse of

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution: We write the given matrix and the identity matrix side by side and perform the row operations as shown

$$\left[\begin{array}{ccccccc} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 1 & 3 & -3 & : & 0 & 1 & 0 \\ -2 & -4 & -4 & : & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccccc} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 2 & -6 & : & -1 & 1 & 0 \\ 0 & -2 & 2 & : & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 + 2R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & : & 1 & 0 & \frac{1}{2} \end{bmatrix} \begin{matrix} R_2' = \frac{1}{2}R_2 \\ R_3' = \frac{1}{2}R_3 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 6 & : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & : & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} R_1' = R_1 - R_2 \\ R_3' = R_3 + R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 6 & : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} R_3' = -\frac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & : & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & : & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{matrix} R_1' = R_1 - 6R_3 \\ R_2' = R_2 + 3R_3 \end{matrix}$$

Hence, the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

Observe that:

(1) Every matrix can be reduced to the normal form by finite number of elementary row and column operations.

(2) The rank of matrix is equal to r if the identity matrix in its normal form is of order r.

Example 2: Reduce the following matrix to the (fully reduced) normal form:

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 3 & 6 & 2 & 5 \end{bmatrix}$$

Solution: We observe

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 3 & 6 & 2 & 5 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 2 & 5 \end{bmatrix} \xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3'=R_3-2R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{R_3'=\frac{1}{8}R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_2'=C_2+2C_1} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_4'=C_4-3C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_4'=C_4+6C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_{34}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the fully reduced normal form to the given matrix. Hence, the rank of the given matrix is 3.