10.7 GAUSS-ELIMINATION METHOD

It is a direct method for finding the solution of a system of linear equations and is based on the principle of elimination of unknowns in successive steps. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution.

Now we consider a system of 3-equations with 3-unknowns as:

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z + d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{....(1)}$$

where a_i, b_i, c_i, d_i $(i = 1, 2, 3)$ are known constants.

Step I: To eliminate x from 2nd and 3rd equations

Let us assume $a_1 \neq 0$, then multiplying the first equation of (1) successively by $-\frac{a_2}{a_1}$ and $-\frac{a_3}{a_1}$ and adding respectively with 2nd and 3rd equations of (1), then the system (1) become as :

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ b_2y + c_2z + d_2 \\ b_3y + c_3z + d_3 \end{aligned}\right\} \quad \text{....(2)}$$

where $b_2' = b_2 - \frac{b_1a_2}{a_1}, c_2' = c_2 - \frac{c_1a_2}{a_1}, d_2' = d_2 - \frac{a_2d_1}{a_1}$

and $b_3' = b_3 - \frac{b_1a_3}{a_1}, c_3' = c_3 - \frac{c_1a_3}{a_1}, d_3' = d_3 - \frac{d_1a_3}{a_1}$

Here the first equation is called the pivotal equation and a_1 is called the first pivot.

Step II: To eliminate y from 3rd equation in (2)

Let us assume $b_2' \neq 0$. Multiplying the second equation of (2) by $-\frac{b_3'}{b_2'}$ and adding with the 3rd equation of (2), then the system (2) become as:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad ....(3)$$

where $c_3'' = c_3' - \frac{b_3'c_2'}{b_2'}$ and $d_3'' = d_3' - \frac{d_2'b_3'}{b_2'}$

Here the second equation is called the pivotal equation of b_2' is the new pivot.

Step III: To evaluate the unknowns

The values of x, y, z are found from the reduced system (3) by back substitution.

Note : The method will fail if any one of the pivots a_1, b_2', c_3'' becomes zero. In such a case; we rewrite the equations in a different order so that the pivots are non-zero.

Example 1: Solve the following system by Gauss elimination method,

$$\left.\begin{aligned} x + 3y + 2z &= 5 \\ 2x - y + z &= -1 \\ x + 2y + 3z &= 2 \end{aligned}\right\} \quad ....(1)$$

Solution: Step 1: To eliminate x from 2nd and 3rd equations, we multiply successively the first equation by

$$-\frac{2}{1} \text{ and } -\frac{1}{1}$$

and then added to the second and third equations of the systems. Then we get the system as:

$$\left.\begin{aligned} x+3y+2z &= 5 \\ -7y-3z &= -11 \\ -y+z &= -3 \end{aligned}\right\} \quad ....(2)$$

$$\left[\begin{aligned} -2x-6y-4z &= -10 \\ 2x-y+z &= -1 \\ \hline -7y-3z &= -11 \\ -x-3y-2z &= -5 \\ x+2y+3z &= 2 \\ \hline -y+z &= -3 \end{aligned}\right]$$

Step II: To eliminate from third equation of (2), we multiply second equation of (2) by $-\frac{-1}{-7} = -\frac{1}{7}$ and then be added to the third equation of (2). We get the system of equations as follows:

$$\left.\begin{aligned} -3y+2z &= 5 \\ -7y-3z &= -11 \\ \frac{10}{7}z &= -\frac{10}{7} \end{aligned}\right\} \quad ....(3)$$

$$\left[\begin{aligned} y+\frac{3}{7}z &= \frac{11}{7} \\ -y+z &= -3 \\ \hline z\left(\frac{3}{7}+1\right) &= \frac{11}{7}-3 = -\frac{10}{7} \\ z\left(\frac{10}{7}\right) &= -\frac{10}{7} \Rightarrow z = -1 \end{aligned}\right]$$

Thus $z = -1$

$\therefore \quad y = \dfrac{11-3z}{7} = \dfrac{11+3}{7} = 2$

and $x = 5 - 3y - 2z = 5 - 6 + 2 = 1$

$\therefore$ Thus, the solution of the system is: $x = 1, y = 2, z = -1$

Example 2: Solve, by Gauss-elimination method, the system of linear equations

$$\left.\begin{aligned} x+4y-z &= 5 \\ x+y-6z &= -12 \\ 3x-y-z &= 4 \end{aligned}\right\} \quad ....(1)$$

Solution: Step I: To eliminate x from 2^{nd} and 3^{rd} equations; we multiply successively the first equation by $-\frac{1}{1}$ and $-\frac{3}{1}$ and then add to the second and third equations of the system. Then we get the system as

$$\left.\begin{aligned} x+4y-z&=5\\ -3y-5z&=-17\\ -13y+2z&=-11\end{aligned}\right\}\ldots(2)\quad \left[\begin{array}{ll} -x-4y+z=-5 & -3x-12y+3z=-15\\ \underline{x+y-6z=-12} & \underline{3x-y-z=4}\\ -3y-5z=-17 & -13y+2z=-11\end{array}\right]$$

Step II: To eliminate y from third equation of (2), we multiply second equation of (2) by $-\frac{-13}{-3}=-\frac{13}{3}$ and then add to the third equation of (2). We get the system as follows:

$$\begin{aligned} x+4y-z&=5\\ -3y-5z&=-17\\ \frac{71}{3}z&=\frac{188}{3}\end{aligned}\quad \left[\begin{array}{l} 13y+\frac{65}{3}z=\frac{17\times 3}{3}\\ \underline{-13y+2z=-11}\\ \left(\frac{65}{3}+2\right)z=\frac{17\times 13}{3}-11=\frac{188}{3}\end{array}\right]$$

$$\therefore \quad z=\frac{188}{71}$$

$$y=\frac{1}{3}[17-5z]=\frac{1}{3}\left[17-\frac{5\times 188}{71}\right]$$

$$=\frac{1}{3}\left[\frac{1207\times 940}{71}\right]=\frac{267}{3\times 71}=\frac{89}{71}$$

$$x=5-4y+z=5-\frac{4\times 89}{71}+\frac{188}{71}=\frac{543-356}{71}=\frac{187}{71}$$

Thus, the solution of the system is: $x=\frac{187}{71}$, $y=\frac{89}{71}$, $z=\frac{188}{71}$

Example 3: Solve the following system of equations by Gauss elimination method:

$$\begin{aligned} 2x-y+3z&=4\\ x+z&=2\\ 2y+z&=3\end{aligned}$$

Solution: Step I: To eliminate x from 2nd equation, we multiply the first equation by $-\frac{1}{2}$ and then add to the 2nd equation of the system. Then we get the system as:

$$\left.\begin{aligned} 2x - y - 3z &= 4 \\ \frac{1}{2}y - \frac{1}{2}z &= 0 \\ 2y + z &= 3 \end{aligned}\right\} \quad ....(2) \qquad \left[\begin{array}{c} -x + \frac{1}{2}y - \frac{3}{2}z = -2 \\ x + z = 2 \\ \hline \frac{1}{2}y - \frac{1}{2}z = 0 \end{array}\right]$$

Step II: To eliminate y from the 3rd equation, we multiply the 2nd equation of (2) by $-\frac{2}{1/2} = -4$ and then be added to the 3rd equation of the system (2). We get the system of equation as follows:

$$\left.\begin{aligned} 2x - y + 3z &= 4 \\ \frac{1}{2}y - \frac{1}{2}z &= 0 \\ 3z &= 3 \end{aligned}\right\} \quad ....(3) \qquad \left[\begin{array}{c} -2y + 2z = 0 \\ 2y + z = 3 \\ \hline 3z = 3 \end{array}\right]$$

Step III: We find x, y, z from (3) by back substitution

$\therefore z = 1$

$y = z = 1$

$2x = 4 - 3z + y = 4 - 3 + 1 = 2 \Rightarrow x = 1$

Thus, the solution of the system is: $x = 1, y = 1, z = 1$.

Example 4: Solve by Gauss-elimination method, the following system

$$x + 2y + 3z = 10$$
$$x + 3y - 2z = 7$$
$$2x - y + z = 5$$

Solution: Step I: To eliminate x from 2nd and 3rd equations, we multiply the first equation by -1, $-\frac{2}{1}$ respectively and then add to the 2nd and 3rd equations of the system. Then we get the system as:

$$\left.\begin{aligned} x + 2y + 3z &= 10 \\ y - 5z &= -3 \\ -5y - 5z &= -15 \end{aligned}\right\} \quad ....(2) \qquad \left[\begin{array}{cc} -x - 2y - 3z = -10 & -2x - 4y - 6z = -20 \\ x + 3y - 2z = 7 & 2x - y + z = 5 \\ \hline y - 5z = -3 & -5y - 5z = -15 \end{array}\right]$$

Step II: To eliminate y from 3rd equation of (2), we multiply the 2nd equation of (2) by $-\frac{-5}{1} = 5$ and then add to the 3rd equation of (2), we get the system of equations as follows :

$$\left.\begin{aligned} x + 2y + 3z &= 10 \\ y - 5z &= -3 \\ -30z &= -30 \end{aligned}\right\} \quad \ldots(3)$$

$$\left[\begin{aligned} 5y - 25z &= -15 \\ -5y - 5z &= -15 \\ \hline -30z &= -30 \end{aligned}\right]$$

Step III: We find x, y, z from (3) by back substitution

$\therefore z = 1$

$y = -3 + 5z = -3 + 5 = 2$

$x = 10 - 2y - 3z = 10 - 4 - 3 = 3$

$\therefore$ Thus, the solution of the system is $\left.\begin{aligned} x &= 3 \\ y &= 2 \\ z &= 1 \end{aligned}\right\}$