9.4 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY CRAMER'S RULE

Let

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

be a system of n linear equations with n unknowns $x_1, x_2, \dots x_n$ where $\det A \neq 0$ and A is the coeffficient matrix. Then there exists a unique solution of the system is given by

$$x_1 = \frac{\det A_1}{\det A},\ x_2 = \frac{\det A_2}{\det A},\ \dots\ x_n = \frac{\det A_n}{\det A}$$

where A_i is the $n \times n$ matrix obtained from A by replacing its ith column by the column $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, $i = 1, 2, \dots n$.

Proof: Here

$$x_1 \det A = \begin{vmatrix} x_1a_{11} & a_{12} & \cdots & a_{1n} \\ x_1a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ x_1a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

[Multiplying c_1 by x_1]

$$= \begin{vmatrix} x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} & a_{12} \ldots a_{1n} \\ x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} & a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ x_1a_{n1} + x_2a_{n2} + \ldots + x_na_{nn} & a_{n2} \ldots a_{nn} \end{vmatrix}$$

[where $C_1' = C_1 + x_2C_2 + \ldots + x_nC_n$]

$$= \begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \ldots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix} = \det A_1$$

....

....

Now

$$x_n \det A = \begin{vmatrix} a_{11} & a_{12} & \ldots & x_na_{1n} \\ a_{21} & a_{22} & \ldots & x_na_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \ldots & x_na_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n-1} & x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n-1} & x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} \\ \vdots & \vdots & & \vdots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn-1} & x_1a_{n1} + x_2a_{n2} + \ldots + x_na_{nn} \end{vmatrix}$$

[where $C_n' = x_1C_1 + x_2C_2 + \ldots + x_{n-1}C_{n-1} + C_n$.]

$$= \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n-1} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n-1} & b_2 \\ \vdots & \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn-1} & b_n \end{vmatrix} = \det A_n$$

Since $\det A \neq 0$, therefore

$$x_1 = \frac{\det A_1}{\det A},\ x_2 = \frac{\det A_2}{\det A} \ldots x_n = \frac{\det A_n}{\det A}.$$

Note:

1. **Cramer's rule is applicable only to a system of *n* linear equations in *n* unknowns.**

2. The system of equations has a unique solution if $\det A \neq 0$.
3. If $\det A = 0$ and at least one of $\det A_1$, $\det A_2$ ... $\det A_n$ is zero then the system of equations has no solution.
4. If $\det A = 0$ and also $\det A_1 = \det A_2 = ... = \det A_n = 0$, then the system of equations has an infinite number of solutions.
5. A system of equations is said to be consistent if it has at least one solution. Therefore, a system of equations is inconsistent if it has no solution.

Example: Solve the system of equations by Cramer's rule

$$x + 2y - 3z = 1$$
$$2x - y + z = 4$$
$$x + 3y = 5.$$

Solution: Here the co-efficient matrix $= \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix} = A$, say.

Then $\det A = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$

$$= 1(-3) - 2(0-1) - 3(6+1) = -3 + 2 - 21 = -22 \neq 0.$$

Hence, there exists a unique solution for x, y, z.

By the Cramer's rule we get,

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 5 & 3 & 0 \end{vmatrix}}{\det A}$$

$$= \frac{1\cdot(0-3) - 2(0-5) - 3(12+5)}{-22}$$

$$= \frac{-3+10-51}{-22} = \frac{-44}{-22} = 2$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & 1 \\ 1 & 5 & 0 \end{vmatrix}}{\det A}$$

$$= \frac{1\cdot(0-5)-1\cdot(0-1)-3(10-4)}{-22}$$

$$= \frac{-5+1-18}{-22} = \frac{-22}{-22} = 1$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 3 & 5 \end{vmatrix}}{\det A}$$

$$= \frac{1\cdot(-5-12)-2(10-4)+1(6+1)}{-22}$$

$$= \frac{-17-12+7}{-22} = \frac{-22}{-22} = 1$$

∴ The required solution is $x = 2,\ y = 1,\ z = 1$.