CHAPTER

9

Determinants

The notion of determinant is fundamental in algebra and has tremendous applications in many spheres of mathematical activities. We begin with its definition.

Definition: The determinant of a square matrix with real entries, denoted as $\det A$ or $|A|$, is defined to be a real number obtained as follows:

If $A = [a_{11}]$, $\det A = a_{11}$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det A = a_{11}a_{22} - a_{12}a_{21}$.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} - a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

and so on.

Thus, determinant of a matrix can be taken as a function from the set of square matrices to the set of real numbers. Sometimes, the determinant of $A = [a_{ij}]$ will also be denoted by $|a_{ij}|$ *i.e.*,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Note that a determinant can also be defined without reference to a matrix.

Minors and Co-factors: The *minor* of an element in a determinant is the determinant obtained by deleting the row and the column which intersect in that element. Let us consider the determinant of a matrix A as follows:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The minor of a_{11} which lies in first row and first column is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

Similarly, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ and $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

are the minors of the elements a_{12}, a_{22}, a_{33} respectively.

The *minor* of an element a_{ij} is generally denoted by the capital letter M_{ij}. The co-factor of a_{ij} in det A is defined to be $(-1)^{i+j} \times$ (minor of a_{ij})

i.e., $\qquad A_{ij} = (-1)^{i+j} M_{ij}$

The *cofactor* of a_{ij} is generally denoted by A_{ij}. The co-factor of a_{11} is given by

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

Similarly, $A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Let us consider the determinant of a matrix $A = (a_{ij})_{n,n}$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The value of det A can be obtained in terms of the elements of the ith row as det $A = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + A_{in}a_{in}$.

For the values of $i = 1, 2, 3 \ldots n$, there are n different expansions for det A given by

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} \quad \text{(for first row)}$$
$$= a_{21}A_{21} + a_{22}A_{22} + \ldots + a_{2n}A_{2n} \quad \text{(for 2nd row)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= a_{n1}A_{n1} + a_{n2}A_{n2} + \ldots + a_{nn}A_{nn} \quad \text{(for } n\text{th row)}$$

proceeding with similar arguments in respect of columns of A and considering in particular, the elements of the jth column, the expansion of det A can be obtained as

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} \qquad (i = 1, 2, \ldots n)$$

If one row and one column be deleted from $n \times n$ matrix (a_{ij}), the determinant of the remaining $(n-1) \times (n-1)$ matrix is called a *minor* of order $(n-1)$ of A.

For example, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then

Expanding in terms of the first row, we get

$$\Delta = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb)$$

$$= abc - af^2 - h^2c + hgf + hgf + g^2b$$

Similarly, expanding in terms of the 2nd row, we can write

$$\Delta = -h\begin{vmatrix} h & g \\ f & c \end{vmatrix} + b\begin{vmatrix} a & g \\ g & c \end{vmatrix} - f\begin{vmatrix} a & h \\ g & f \end{vmatrix}$$

$$= -h(hc - gf) + b(ac - g^2) - f(af - hg)$$

$$= -h^2c + hgf + abc - bg^2 - af^2 + fgh$$

$$= abc + 2hgf - af^2 - bg^2 - ch^2$$

Expanding in terms of the first column we can similarly get

$$\Delta = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & g \\ f & c \end{vmatrix} + g\begin{vmatrix} h & g \\ b & f \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(bf - bg)$$

$$= abc - af^2 - ch^2 + hfg + hfg - bg^2$$

$$= abc + 2hfg - af^2 - bg^2 - ch^2.$$

9.1 PROPERTIES OF DETERMINANTS

The following properties are true for nth order determinants. We, however, prove them for 3rd order determinants only.

1. The value of det A remains unaltered by changing its rows into columns and columns into rows

i.e., $\quad \det A = \det A^t$

Here $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

$$\therefore \quad \det A^t = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})) \text{ (Expand w.r.t. first column)}$$

$$\therefore \quad \det A = \det A^t.$$

2. The interchange of any two rows (respectively columns) of a matrix A changes the sign of the determinant.

For example,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-1)\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(interchanging } R_1 \text{ and } R_2\text{)}$$

$$= (-1)(-1)\begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} \quad \text{(interchanging } C_1 \text{ and } C_2\text{)}$$

[Note the numerical value remains unchanged.]

[R_i denotes the ith row (i = 1, 2, ... n) and C_j denotes the jth column (j = 1, 2,... n).]

3. If any two rows or columns of a matrix A of order n be identical, then the value of the determinant is zero.

For example,

$$\det A = \begin{vmatrix} a & b & c \\ a & b & c \\ e & f & g \end{vmatrix} = 0. \quad \text{(Since } R_1 \text{ and } R_2 \text{ are identical)}$$

4. If the element of any row or column of a determinant is multiplied by a scalar (number), then the determinant is multiplied by the same scalar (number).

For example,

$$c \det A = \begin{vmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and $$c \det A = \begin{vmatrix} ca_{11} & a_{12} & a_{13} \\ ca_{21} & a_{22} & a_{23} \\ ca_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If a row **(column) of** A **be a scalar multiple of another row (column) then** $\det A = 0$.

For example,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ca_{11} & ca_{12} & ca_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \quad \text{(Since } R_1 \text{ and } R_2 \text{ are identical)}$$

5. If the **elements of any row or column be the sum of the two quantities, then the determinant can be expressed as sum of the two determinants.** For example,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and $$\Delta = \begin{vmatrix} b_{11} & b_{12}+a_{12} & b_{13} \\ b_{21} & b_{22}+a_{22} & b_{23} \\ b_{31} & b_{32}+a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

6. In an $n \times n$ matrix A if a scalar multiple of one row (column) be added to another row (column), then det A remains unchanged.

For example, if $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$ and c be a scalar, let c times the 2nd row of A be added to its 3rd row and $B = (b_{ij})$ be the resulting matrix.

$$\therefore \quad \det B = \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} + ca_{21} & a_{32} + ca_{22} & a_{33} + ca_{23} & a_{34} + ca_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} + c \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$= \det A + c \cdot 0$ $\quad (\because R_1$ and R_2 are identical)

$= \det A.$

7. In an $n \times n$ matrix A, if one row (column) be expressed as a linear combination of the remaining rows (columns) then $\det A = 0$.

Let $\quad A = (a_{ij})$ and $a_{nj} = c_1 a_{1j} + c_2 a_{2j} + \ldots + c_{n-1} a_{n-1j}$

where $c_1, c_2 \ldots c_{n-1}$ are scalars, $j = 1, 2, \ldots n$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ c_1 a_{11} & c_1 a_{12} & \cdots & c_1 a_{1n} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ c_2 a_{21} & c_2 a_{22} & \cdots & c_2 a_{2n} \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ c_{n-1} a_{n-1,1} & c_{n-1} a_{n-1,2} & \cdots & c_{n-1} a_{n-1,n} \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{11} & a_{12} & \cdots & a_{1n} \end{vmatrix} + c_2 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \cdots & a_{2n} \end{vmatrix}$$

$$+ \ldots + c_{n-1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n} \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n} \end{vmatrix}$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_{n-1} \cdot 0 = 0$$

(Since all the determinants have two identical rows.)

Example 1: Prove without expanding

$$\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix} = 0.$$

Solution: Now $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$ Multiplying R_1 by a, R_2 by b, R_3 by c and R_4 by d

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$ Multiplying C_4 by $\frac{1}{abcd}$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

(by three successive interchanges of C_4 to C_1)

$= 0.$

Example 2: Prove without expanding

(*i*) $$\begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix} = (ab + bc + ca)\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

Solution: (*i*) Using the identity $a^3 - a^2\sum a + a\sum ab - abc = 0$,

we get $$\text{L.H.S.} = \begin{vmatrix} a^2\sum a - a\sum ab + abc & a^2 & 1 \\ b^2\sum a - b\sum ab + abc & b^2 & 1 \\ c^2\sum a - c\sum ab + abc & c^2 & 1 \end{vmatrix}$$

$$= \sum a \begin{vmatrix} a^2 & a^2 & 1 \\ b^2 & b^2 & 1 \\ c^2 & c^2 & 1 \end{vmatrix} - \sum ab \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} abc & a^2 & 1 \\ abc & b^2 & 1 \\ abc & c^2 & 1 \end{vmatrix}$$

$$= \sum a.0 + \sum ab \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix}$$

(Since in first determinant has two identical columns and interchange C_1 and C_2 in the 2nd determinant.)

$$= (ab + bc + ca) \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} + 0$$

(Since third part has two identical columns.)

$$= (ab + bc + ca) \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

(ii) $$\begin{vmatrix} 1 & bcd & b+c+d & a^2 \\ 1 & cda & c+d+a & b^2 \\ 1 & dab & d+a+b & c^2 \\ 1 & abc & a+b+c & d^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

Solution: Now $$\begin{vmatrix} 1 & bcd & b+c+d & a^2 \\ 1 & cda & c+d+a & b^2 \\ 1 & dab & d+a+b & c^2 \\ 1 & abc & a+b+c & d^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bcd & a+b+c+d-a & a^2 \\ 1 & cda & b+c+d+a-b & b^2 \\ 1 & dab & c+d+a+b-c & c^2 \\ 1 & abc & d+a+b+c-d & d^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bcd & a+b+c+d & a^2 \\ 1 & cda & a+b+c+d & b^2 \\ 1 & dab & a+b+c+d & c^2 \\ 1 & abc & a+b+c+d & d^2 \end{vmatrix} - \begin{vmatrix} 1 & bcd & a & a^2 \\ 1 & cda & b & b^2 \\ 1 & dab & c & c^2 \\ 1 & abc & d & d^2 \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 1 & bcd & 1 & a^2 \\ 1 & cda & 1 & b^2 \\ 1 & dab & 1 & c^2 \\ 1 & abc & 1 & d^2 \end{vmatrix}$$

$$-\frac{1}{abcd}\begin{vmatrix} a & abcd & a^2 & a^3 \\ b & abcd & b^2 & b^3 \\ c & abcd & c^2 & c^3 \\ d & abcd & d^2 & d^3 \end{vmatrix}$$ Multiplying R_1 by a, R_2 by b, R_3 by c, R_4 by d

$$= (a+b+c+d)\cdot 0 - \begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix}$$ Multiplying C_2 by $\frac{1}{abcd}$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$ (Interchanging C_1 and C_2.)

8. If the elements of a matrix A of order n are real (or complex) polynomials in x (or function of x) and the two rows or columns of A becomes identical when $x = a$, then $(x - a)$ is a factor of det A. Similarly, if r row's (or columns) becomes identical when $x = a$, then $(x - a)^{r-1}$ is a factor of det A.

Example: Prove that $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -(a-b)(b-c)(c-a).$

Solution: Let $A = \begin{pmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{pmatrix}$

Let the elements of the matrix A be polynomials in a. Then two rows of A become identical when $a = b$ $\therefore a - b$ is a factor of det A. Let the elements of the matrix A be polynomials in b. Then two rows of A become identical when $b = c$ $\therefore (b - c)$ is a factor of det A. Similarly, $(c - a)$ is a factor of det A.

Since det A is a polynomial in a, b, c of degree 3.

$\therefore$ $\quad$ det $A = k(a-b)(b-c)(c-a)$ ...(1)

where k is a constant.

Now putting $a = 0$, $b = -1$, $c = 1$ in (1), we get

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = k(0+1)(-1-1)(1-0)$$

$\Rightarrow$ $\quad 2 = -2k \Rightarrow k = -1$

$\therefore$ $\quad$ det $A = -(a-b)(b-c)(c-a)$.