CHAPTER

10

Matrix-II

10.1 SUBMATRIX OF A MATRIX

From a matrix, matrices of lower dimensions can be formed by suppressing rows or columns or both. Such matrices are referred to as submatrices of the original matrix. Thus we have:

Definition: A submatrix of a matrix is a matrix obtained from the first by suppressing one or more rows or columns or both.

Thus the matrix $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$ has $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ as a submatrix obtained by suppressing the second column. Similarly [3] is a submatrix obtained by deleting the 1st and 2nd columns and the 2nd row of the original matrix.

Note that every matrix is a submatrix of itself.

10.2 RANK OF A MATRIX

The rank of a matrix has an interesting impact in many results of matrices. This will be evident in some of the following results.

Definition: The **rank** of a matrix is the order of the largest square submatrix whose determinant is non-zero.

The rank of a zero matrix is defined to be zero.

Note that the rank of any identity matrix is the same as its order.

Clearly, the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ is 2 since the submatrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has determinant 1 which is different from 0.

Example: Find the rank of the following matrices:

(*i*) $\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 4 & 3 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 3 & 2 \end{bmatrix}$.

Solution: (*i*) Since $\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 0$ but $\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8 \neq 0$, the rank of the given matrix is 2.

(*ii*) Since the determinant of the given matrix is zero, the rank must be less than 3. Again as the submatrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ has determinant -2, the rank of the given matrix is two.

(*iii*) Clearly the largest square submatrix is of order 3. Let us take the submatrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 3 \end{bmatrix}$. Then its determinant is $1 \neq 0$. Hence, the rank of the given matrix is 3.

For matrices if small dimensions, the above method is quite comfortable, but for matrices of high dimensions, the above method becomes laborious. So we have alternative approaches for determining the rank of a matrix.

10.3 ROW AND COLUMN TRANSFORMATIONS

There are two types of elementary operations *viz.* elementary row operations and elementary column operations.

We enlist them below:

1. Interchanging any two rows of a matrix, denoted by R_{ij} where the i^{th} row and and j^{th} row are interchanged.
2. Multiplying any row by a non-zero constant k, denoted by $k R_i$ where i^{th} row is multiplied by $k \neq 0$.
3. Adding scalar multiple of any row to another row, denoted by $R_i + kR_j$ when k times the j^{th} row is added to the i^{th} row; $k \neq 0$.

The column operations are similarly defined:

C_{ij}: Interchanging the i^{th} and j^{th} columns of a matrix.

kC_i: Multiplying the i^{th} column by $k \neq 0$.

$C_i + kC_j$: Adding k times the j^{th} column to the i^{th} column, $k \neq 0$.

It is a well known result (stated here without proof) that neither row operations nor column operations change the rank of a matrix. Hence, this result may be used to determine the rank of a matrix to our convenience.

Example 1: Determine the rank of

$$\begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 2 & 1 & 5 \\ 7 & 2 & 9 & 17 \end{bmatrix}.$$

Solution: Clearly the given matrix

$$\begin{bmatrix} 2 & 0 & 4 & 6 \\ 3 & 2 & 1 & 5 \\ 7 & 2 & 9 & 17 \end{bmatrix} \xrightarrow{R_1' = \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 1 & 5 \\ 7 & 2 & 9 & 17 \end{bmatrix} \xrightarrow[R_3' = R_3 - 7R_1]{R_2' = R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 2 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, the rank of the reduced matrix is 2. Hence, the rank of the original matrix is also 2.

Remark: One would have got the same result had he applied elementary column operations instead of row operation. One can also apply both row and column operations to reduce the matrix to a simple form or triangular form from which the rank may be determined comfortably.

Definition: If a matrix is of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is the identity matrix of order r and 0 stands for zero matrices, it is said to be in the **fully reduced normal form.**

Example 2: Find the rank of A where $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$.

Solution: Let us apply the elementary operations on the matrix A, we get

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix} \xrightarrow[\substack{R_3' = R_3 - 2R_1 \\ R_4' = R_4 - R_1}]{R_2' = R_2 - 4R_1} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 = \frac{1}{6}R_2} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow[R_4' = R_3 + R_2]{R_1' = R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix} \xrightarrow{R_{34}} \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{C_4' = C_4 - \frac{4}{3}C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_4' = C_4 + \frac{5}{3}C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_4' = C_4 + \frac{8}{3}C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= R$, say $\therefore$ R is the fully reduced normal form. It has 3-non zero rows. Hence, rank of A is 3, *i.e.* $r(A) = 3$.

Example 3: Show that rank of the matrix A is 3 where $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$.

Solution: We apply the elementary operations on the matrix A,

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \xrightarrow{R_1' = \frac{1}{8}R_1} \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{6}{8} \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \xrightarrow{R_3' = R_3 + 8R_1} \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{6}{8} \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\xrightarrow[R_3' = \frac{1}{10}R_3]{R_2' = \frac{1}{3}R_2} \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{6}{8} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1' = R_1 - \frac{1}{8}R_2} \begin{bmatrix} 1 & 0 & \frac{7}{24} & \frac{16}{24} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_{23}} \begin{bmatrix} 1 & 0 & \frac{16}{24} & \frac{7}{24} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[C_4' = C_4 - \frac{7}{24}C_4]{C_3' = C_3 - \frac{16}{24}C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[C_4' = C_4 - \frac{2}{3}C_2]{C_3' = C_3 - \frac{2}{3}C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = R \text{ (say)}$$

$\therefore$ R is fully reduced normal form. It has three-non-zero rows. Hence, rank of A is 3.

Example 4: Show that $r(A) = 2$ and $r(B) = 2$ where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$. Show further that $r(AB) = 1$ while $r(BA) = 2$.

Solution: Now $\det A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{vmatrix} = 1(-9+8) - 1(6-12) - 1(-4+9)$

$$= -1 + 6 - 5 = 0$$

Hence, A is a singular matrix, so rank of $A < 3$. Let us consider the 2nd order minor $\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 \neq 0$. Hence, rank of A is 2.

Now det $B = \begin{vmatrix} 1 & -1 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{vmatrix} = 1(60-60) + 1(30-30) + 3(60-60) = 0 + 0 + 0 = 0$

Hence, B is a singular matrix, so rank of $B < 3$. Let us consider the 2nd order minor $\begin{vmatrix} 1 & -2 \\ 6 & 12 \end{vmatrix} = 12 + 12 = 24 \neq 0$. Hence, rank of B is 2.

Now $$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \\ 6 & 0 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 0 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix}$$

Now det $(AB) = \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \\ 6 & 0 & 12 \end{bmatrix} = 0$. Hence, AB is a singular matrix. So rank of $AB < 3$, but every minor of order 2 is zero $\left(\begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0, \begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 0 \ldots\right)$. So rank of $AB < 2$. Since all the elements of AB are not zero, so rank of AB is 1.

Now $$\det(BA) = \begin{vmatrix} 6 & 1 & 0 \\ 48 & -42 & 360 \\ 40 & -35 & 50 \end{vmatrix} = 6 \times 5 \begin{vmatrix} 6 & 1 & 0 \\ 8 & -7 & 10 \\ 8 & -7 & 10 \end{vmatrix} = 0$$

($\because$ two rows identical)

Hence, BA is a singular matrix, so rank of $BA < 3$. Let us consider the 2nd order minor $\begin{vmatrix} 6 & 1 \\ 48 & -42 \end{vmatrix} = 6\begin{vmatrix} 6 & 1 \\ 8 & -7 \end{vmatrix} = 6(-42-8) = -6 \times 50 \neq 0$. Hence, rank of BA is two.

Example 5: Determine the rank of the matrix A where $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 4 & -6 \\ 0 & 0 & 5 & -2 \\ 3 & 6 & 8 & -1 \end{bmatrix}$.

Solution: We apply the elementary operations on the matrix A,

$$A=\begin{bmatrix}1&2&-1&0\\2&4&4&-6\\0&0&5&-2\\3&6&8&-1\end{bmatrix}\xrightarrow[R_4'=R_4-3R_1]{R_2'=R_2-2R_1}\begin{bmatrix}1&2&-1&0\\0&0&6&-6\\0&0&5&-2\\0&0&9&-1\end{bmatrix}\xrightarrow{R_2'=\frac{1}{6}R_2}\begin{bmatrix}1&2&-1&0\\0&0&1&-1\\0&0&5&-2\\0&0&9&-1\end{bmatrix}$$

$$\xrightarrow{C_{23}}\begin{bmatrix}1&-1&2&0\\0&1&0&-1\\0&5&0&-2\\0&9&0&-1\end{bmatrix}\xrightarrow[\substack{R_3'=R_3-5R_2\\R_4'=R_4-9R_2}]{R_1'=R_1+R_2}\begin{bmatrix}1&0&2&-1\\0&1&0&-1\\0&0&0&3\\0&0&0&8\end{bmatrix}\xrightarrow[R_4'=\frac{1}{8}R_4]{R_3'=\frac{1}{3}R_3}\begin{bmatrix}1&0&2&-1\\0&1&0&-1\\0&0&0&1\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow{C_{34}}\begin{bmatrix}1&0&-1&2\\0&1&-1&0\\0&0&1&0\\0&0&1&0\end{bmatrix}\xrightarrow[\substack{R_2'=R_2+R_3\\R_4'=R_4-R_3}]{R_1'=R_1+R_3}\begin{bmatrix}1&0&0&2\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\xrightarrow{C_4=C_4-2C_1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}=R,\text{(say)}$$

$\therefore$ R is fully reduced normal form. It has 3-non-zero rows. Hence, rank of A is 3.

Example 6: Find the rank of the matrix A where $A=\begin{pmatrix}1&2&1&-3\\2&4&3&1\\3&6&4&-2\end{pmatrix}$.

Solution: We apply the elementary operations on the matrix A,

$$A=\begin{bmatrix}1&2&1&-3\\2&4&3&1\\3&6&4&-2\end{bmatrix}\xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1}\begin{bmatrix}1&2&1&-3\\0&0&1&7\\0&0&1&7\end{bmatrix}\xrightarrow[R_3'=R_3-R_2]{R_1'=R_1-R_2}\begin{bmatrix}1&2&0&-10\\0&0&1&7\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow[C_1'=C_4+10C_1]{C_2'=C_2-2C_1}\begin{bmatrix}1&0&0&0\\0&0&1&7\\0&0&0&0\end{bmatrix}\xrightarrow{C_4'=C_4-7C_3}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\xrightarrow{C_{23}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=$$

R, (say)

$\therefore$ R is fully reduced normal form. It has two-non-zero rows. Hence, rank of A is 2.

Example 7: Reduce the matrix A to the fully reduced normal form where

$$A=\begin{bmatrix}2&4&1&0\\1&2&0&3\\3&6&2&5\end{bmatrix}.$$

Solution: We apply the elementary operations on the matrix A,

$$A=\begin{bmatrix}2&4&1&0\\1&2&0&3\\3&6&2&5\end{bmatrix}\xrightarrow{R_{12}}\begin{bmatrix}1&2&0&3\\2&4&1&0\\3&6&2&5\end{bmatrix}\xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1}\begin{bmatrix}1&2&0&3\\0&0&1&-6\\0&0&2&-4\end{bmatrix}$$

$$\xrightarrow{R_3' = R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{R_3 = \frac{1}{8}R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2' = R_2 + 6R_3]{R_1' = R_1 - 3R_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_2' = C_2 - 2C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_{24}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = R, \text{(say)}$$

$\therefore$ R is fully reduced normal form (and R has 3-non-zero rows, $\therefore$ Rank of A is 3.)