9.5 SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Let us consider the system of n homogeneous linear equations with n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0$$

(*i*) If $\det A \neq 0$ where A is the coefficient matrix, then the homogeneous system admits of a unique solution by Cramer's rule and the unique solution is the zero solution. This is called its trivial solution.

(*ii*) If $\det A = 0$, then the system has non-trivial (non zero) solution in which, $x_1, x_2 \dots x_n$ are not all zero simultaneously.

(*iii*) Since $x_1 = x_2 = \dots x_n = 0$ is always a solution of the above system of equations therefore, this system always consistent.

Example 1: Does the system

$$x + y - 3z = 0$$
$$3x - y - z = 0$$
$$2x + y - 4z = 0$$

has non-trivial solutions? If so, find these solutions.

Solution: Here the coefficient matrix $= \begin{pmatrix} 1 & 1 & -3 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} = A$, say

Then $\det A = \begin{vmatrix} 1 & 1 & -3 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{vmatrix}$

$$= 1(4+1) - 1(-12+2) - 3(3+2) = 5 + 10 - 15 = 0$$

$\therefore$ The system has non-trivial solutions. To find these solutions, we solve any two of the given equations, say,

$$x + y - 3z = 0$$
$$3x - y - z = 0$$

Then we get

$$\frac{x}{-1-3} = \frac{y}{-9+1} = \frac{z}{-1-3}$$

or $$\frac{x}{-4} = \frac{y}{-8} = \frac{z}{-4}$$

or $$\frac{x}{1} = \frac{y}{2} = \frac{z}{1} = k \text{ (say)}$$

$\therefore$ $$x = k, \quad y = 2k, \quad z = k.$$

The solution can be written as $\left.\begin{matrix} x = k \\ y = 2k \\ z = k \end{matrix}\right\}$

where, k is an arbitrary real number