CHAPTER

8 Theory of Matrices

Matrices arise in many situations of mathematics. In representing rotation analytically, in solving systems of linear equations, in the study of extrema of functions of two or more variables and in many other ways matrices come up as natural habitats of the domain of mathematics. So the study of matrices is an indispensable component of any modern curriculum. In this chapter we discuss the related concepts of matrices.

8.1 REAL AND COMPLEX MATRIX

Definition: A matrix is a rectangular array (*i.e.* arrangement) of objects. The number of rows and the number of columns are called its dimensions. The objects are called its entries. If the objects are real numbers, it is called a **real matrix**; if the objects are complex numbers, it is called a **complex matrix**. If the objects are polynomials then it is called a **polynomial matrix**. It is to be noted that the entries themselves can be matrices also.

For example, $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -3 \end{bmatrix}$ is a matrix with 2 rows and 3 columns. It is normally referred to as a 2×3-matrix, read as 2 by 3 matrix. By convention the number of rows comes first followed by the number of columns. The above matrix can be called an integral matrix if all the entries are integers.

The matrix $\begin{bmatrix} x^2+1 & 2x-1 & x^3 \\ 3x+2 & 5 & x^4-1 \end{bmatrix}$ is a polynomial matrix as the entries are polynomials.

Of special significance are the matrices like $[1, 2, 0]$, $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}$.

The first one has only one row and three columns. Such a matrix is called a *row matrix*. The second one is likewise a *column matirx* having 2 rows and 1column. The third one is a square matrix having the same number of rows as the number of columns.

8.2 OTHER DEFINITIONS

Definition: A matrix whose number of rows equals the number of columns is called a **square matrix**.

A convention has it that for square matrices, we don't refer to the dimensions but to the order which is, by definition, is the number rows and is therefore also the number of columns.

Thus $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ is a square matrix of order 2.

In general, a matrix will be referred to by capital letters such as A, B, U, V, etc. and their corresponding entries by $a_{ij}, b_{ij}, u_{ij}, v_{ij}$, etc. Thus $A = [a_{ij}]_{m \times n}$ stands for the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Note a_{32} is the element which belongs to the 3rd row and the 2nd column. Similarly, a_{ij} is the element at the crossing of the i^{th} row and the j^{th} column. Thus a square matrix A of order n should look like

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The elements $a_{11}, a_{22}, \dots, a_{ii}, \dots, a_{nn}$ are said to form the *principal diagonal* of the matrix and the elements $a_{1n}, a_{2n-1}, \dots, a_{n1}$ are said to form its *second diagonal* of the square matrix A. The sum of the elements in the principal diagonal is called the *trace* of A and is denoted by tr A. Thus

$$\text{tr } A = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

For the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -1 \\ 3 & 2 & 2 \end{bmatrix}$, tr $A = 2 + 4 + 2 = 8$.

Definition: A matrix is called a **zero matrix**, if every element of it is zero.

Thus $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of dimensions 2×3.

A square matrix is called an *identity matrix* if all its principal diagonal elements are 1 and its non-diagonal elements are zeros.

Thus $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

A square matrix is called a *diagonal matrix* if all elements other than the principal diagonal elements are zero and at least one of the principal diagonal elements is non-zero.

Thus $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix; so also $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition: Two matrices A and B are said to be equal if they have the same dimensions and the corresponding elements are equal.

For example, $\begin{bmatrix} 2 & a & 3 \\ 4 & 0 & b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & 1 \end{bmatrix}$ implies $a = 0, b = 1$.

A square matrix is called a *scalar matrix* if every non-principal diagonal elements is zero and all principal diagonal elements are non-zero and equal.

Thus $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix but $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ is not a scalar matrix.

Note that the identity matrix is also a scalar matrix.

A square matrix is called an *upper triangular* if all the elements below the principal diagonal are zero and at least one above the principal diagonal is non-zero.

Note $A = [a_{ij}]$ is uppertriangular if $a_{ij} = 0$ for all $i > j$

Clearly $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ is an upper triangular matrix.

A square matrix is called a *lower triangular* matrix if all the elements above the principal diagonal are zero and at least one below it is non-zero.

Note $A = [a_{ij}]$ is lower triangular if $a_{ij} = 0$ for all $i < j$.

Thus $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 2 & 3 \end{bmatrix}$ is a lower triangular matrix.

8.3 SYMMETRIC AND SKEW SYMMETRIC MATRIX

A square matrix is called **symmetric** if the elements situated symmetrically with respect to the principal diagonal are equal.

Note a square matrix which is also, a zero matrix is symmetric.

Thus $A = [a_{ij}]$ is symmetrix if $a_{ij} = a_{ji}$ for all i and j.

Clearly $A = \begin{bmatrix} 4 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{bmatrix}$ is symmetric, as the elements $a_{12} = a_{21} = 3$, $a_{31} = a_{13} = 5$, $a_{23} = a_{32} = -1$, the elements $\{a_{12}, a_{21}\}$, $\{a_{31}, a_{13}\}$ and $\{a_{23}, a_{32}\}$ being symmetrically situated with respect to the principal diagonal.

A square matrix is called **skew symmetric** if the elements situated symmetrically with respect to the principal diagonal have opposite signs but the same magnitude and the principal diagonal elements are zero. Thus a matrix $A = [a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

Clearly, $\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & 3 \\ 5 & -3 & 0 \end{bmatrix}$ is skew-symmetric since $a_{12} = -a_{21}$, $a_{13} = -a_{31}$, $a_{31} = -a_{13}$ and $a_{11} = a_{22} = a_{33} = 0$.

Note a square matrix which is also a zero matrix is skew symmetric.

8.4 MATRIX OPERATIONS

With matrices, we can perform a number of operations defined below:

Transposing: If A is an $m \times n$-matrix, then its transpose is an $n \times m$-matrix obtained by converting rows into corresponding columns. The transpose of A is denoted by A^t. Clearly, if $A = [a_{ij}]_{m \times n}$, then $A^t = [a_{ji}]_{n \times m}$.

Thus if $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ 3 & -2 \end{bmatrix}$

Note: (1) $(A^t)^t = A$.

(2) The tranpose of an upper triangular matrix is a lower triangular matrix.

(3) The transpose of a symmetric matrix is the matrix itself, *i.e.*, $S^t = S$ if S is symmetric.

(4) The trace of a square matrix remains invariant order transposition *i.e.*, $\text{tr}(A^t) = \text{tr}(A)$.

(5) The transpose of an indentity matrix is the matrix itself.

Addition: If A and B have the same dimensions, then the sum $A + B$ is defined as a matrix of the same dimensions obtained by adding the corresponding elements. Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 8 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} -3 & 2 & -1 \\ 1 & 3 & 2 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 8+(-3) & 2+2 & 3+(-1) \\ 4+1 & 0+3 & 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 2 \\ 5 & 3 & 7 \end{bmatrix}.$$

Observe that (1) $A + B = B + A$ where A and B are real matrices.

(2) $(A + B)^t = A^t + B^t$.

(3) $A + 0 = A = 0 + A$ where 0 is the zero matrix of the same dimensions of those of A.

Subtraction: If A and B are of the same dimensions, then $A - B$ is a matrix of the same dimensions, obtained by subtracting the elements of B from the corresponding elements of A.

Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, then $A - B = [a_{ij} - b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then $A - B = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$.

Observe that (1) $A - B \neq B - A$.

(2) $(A - B)^t = A^t - B^t$.

(3) $A - 0 = A$ where 0 is the zero matrix.

Scalar Multiplication: If real numbers are taken as scalar and $\lambda \in \mathbb{R}$, then the scalar multiplication λA is a matrix of the same dimensions of that of A and is obtained by multiplying each element of A by λ. Thus, if $A = [a_{ij}]_{m \times n}$, then $\lambda A = [\lambda a_{ij}]_{m \times n}$

For example, if $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$, then $2A = \begin{bmatrix} 4 & 2 & 6 \\ 0 & 8 & 4 \end{bmatrix}$.

Observe that (1) $\lambda(A \pm B) = \lambda A \pm \lambda B$ where A and B are two matrices.

(2) $(\lambda A)^t = \lambda A^t$.

(3) Every scalar matrix is of the form λI.

(4) The transpose of a skew-symmetric matrix is negative of the matrix, *i.e.*, if T is skew-symmetric, $T^t = -T = (-1)T$.

A result of importance is the following:

Theorem: Every square matrix can be expressed as the sum of two matrices, one of which is symmetric and the other skew-symmetric.

Proof: Let A be a square matrix.

Then we can write

$$A = \frac{1}{2}.2A = \frac{1}{2}[A + A] = \frac{1}{2}(A + A^t + A - A^t)$$

i.e.,
$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

$$= S + T, \text{ writing } S = \frac{1}{2}(A + A^t), T = \frac{1}{2}(A - A^t).$$

It now suffices to prove that S is symmetric and T is skew-symmetric. To this end, we observe

$$S^t = \left[\frac{1}{2}(A + A^t)\right]^t = \frac{1}{2}(A + A^t)^t = \frac{1}{2}(A^t + A) = \frac{1}{2}(A + A^t) = S.$$

So S is symmetric

$$T^t = \left[\frac{1}{2}(A - A^t)\right]^t = \frac{1}{2}(A - A^t)^t = \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t) = -T.$$

So T is skew-symmetric.

Hence the proof.

Example 1: If $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, verify

(*i*) $(A + B)^t = A^t + B^t$ (*ii*) $(A - B)^t = A^t - B^t$

(*iii*) $(3A)^t = 3A^t$ (*iv*) $(2A + 3B)^t = 2A^t + 3B^t$

(*v*) $A + B = B + A$.

Solution: (*i*) and (*ii*) Here

$$A + B = \begin{bmatrix} 4 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix}, A - B = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -3 & 2 \end{bmatrix}, A^t = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}, B^t = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

$$\therefore \quad (A + B)^t = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, (A - B)^t = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 3 & 2 \end{bmatrix},$$

$$A^t + B^t = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, A^t - B^t = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 3 & 2 \end{bmatrix}.$$

Clearly, $(A + B)^t = A^t + B^t$ and $(A - B)^t = A^t - B^t$.

(*iii*) Now, $3A = 3\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 6 & -3 & 9 \end{bmatrix}$

$$\therefore \qquad (3A)^t = \begin{bmatrix} 3 & 6 \\ 3 & -3 \\ 6 & 9 \end{bmatrix}, 3A^t = 3\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & -3 \\ 6 & 9 \end{bmatrix}$$

Clearly, $(3A)^t = 3A^t$.

(*iv*) Now, $2A + 3B = 2\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 & 4 \\ 4 & -2 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & -3 \\ 3 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 & 1 \\ 7 & 4 & 9 \end{bmatrix}$$

$$\therefore \qquad (2A + 3B)^t = \begin{bmatrix} 11 & 7 \\ 2 & 4 \\ 1 & 9 \end{bmatrix}$$

Again, $2A^t + 3B^t = 2\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} + 3\begin{bmatrix} 9 & 3 \\ 0 & 6 \\ -3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 11 & 7 \\ 2 & 4 \\ 1 & 9 \end{bmatrix}$$

Clearly, $(2A + 3B)^t = 2A^t + 3B^t$.

(*v*) Now $B + A = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \text{ evidently } A + B = B + A.$$

Example 2: Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(*i*) $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}$.

Solution: (*i*) Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$.

$$\therefore \quad A^t = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}.$$

Hence $\frac{1}{2}(A + A^t) = \frac{1}{2}\begin{bmatrix} 4 & 7 \\ 7 & 4 \end{bmatrix} = S$, say and

$$\frac{1}{2}(A - A^t) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = T, \text{ say}$$

Clearly, $A = S + T$, where S is symmetric and T is skew-symmetric.

(*ii*) Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then $B^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

Hence $\frac{1}{2}(B + B^t) = \frac{1}{2}\cdot\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} = P$, say.

and $\frac{1}{2}(B - B^t) = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = Q$, say.

Clearly $B = P + Q$, where P is symmetric and Q is skew-symmetric.

(*iii*) Let $C = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$. Then $C^t = \begin{bmatrix} 0 & 4 & -3 \\ -4 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$.

Hence $\quad \frac{1}{2}(C + C^t) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$, say.

and $\quad \frac{1}{2}(C - C^t) = \frac{1}{2}\begin{bmatrix} 0 & -8 & 6 \\ 8 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} = V$, say.

Clearly, $C = U + V$, where U is symmetric and V is skew-symmetric.

(*iv*) Let $D = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}$. Then $D' = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix}$.

Hence $\frac{1}{2}(D + D') = \frac{1}{2}\begin{bmatrix} 6 & -2 & 10 \\ -2 & 4 & 8 \\ 10 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ -1 & 2 & 4 \\ 5 & 4 & 1 \end{bmatrix} = S$, say.

$$\frac{1}{2}(D - D') = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T, \text{ say.}$$

Clearly, $D = S + T$, where S is symmetric, and T is skew-symmetric.

Example 3: Given an example of a matrix which is

(*i*) symmetric but not skew-symmetric

(*ii*) skew-symmetric but not symmetric

(*iii*) both symmetric and skew-symmetric

(*iv*) neither symmetric nor skew-symmetric.

Solution:

(*i*) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & 4 \\ -5 & -4 & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 7 & 4 \\ 4 & 1 & 3 \\ 6 & 2 & 3 \end{bmatrix}$

Product of Matrices: If A and B are matrix, the product AB is defined only when the number of columns of A equals the number of rows of B. This condition is called the conformability condition for multiplication. Thus, if A and B are conformable for the product AB, then the product is a matrix obtained by the following rule:

If $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$, then $AB = [c_{ik}]$ where $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$

For example, if

$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, A and B are conformable for the product AB and clearly AB has the dimensions 2×3.

$$\text{Now } AB = \begin{bmatrix} 2\times1+1\times2+3\times1 & 2\times1+1\times(-1)+3\times3 & 2\times0+1\times2+3\times4 \\ 4\times1+0\times2+2\times1 & 4\times1+0\times(-1)+2\times3 & 4\times0+0\times2+2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 14 \\ 6 & 10 & 8 \end{bmatrix}.$$

Observe that to obtain the element c_{23}, the 2^{nd} row of A and 3^{rd} column of B have been multiplied by coordinate-wise and then added (*i.e.* dot product has been taken). Similarly, other entries of C are calculated.

Observe that for any three matrices A, B and C and a scalar λ:

(1) $AB \neq BA.$ [*Non-commutativity*]

(2) $A(B \pm C) = AB \pm AC$ [*Left Distributivity*]

(3) $\lambda(AB) = (\lambda A)B = A(\lambda B)$ [*Homogeneity*]

(4) $(A + B)C = AC + BC$ [*Right Distributivity*]

(5) $AB = BA \Leftrightarrow A^tB^t = B^tA^t$

(6) $AI = IA = A$ [*Identity Property*]

(7) $(AB)C = A(BC)$ [*Associativity*]

(8) $(AB)^t = B^tA^t.$

Example 1: If A and B be two symmetric matrices of the same order, then prove that

(*i*) $A + B$ is symmetric.

(*ii*) AB is symmetric if and only if $AB = BA$.

Solution: Since A and B are symmetric matrices, then $A^t = A$, $B^t = B$

(*i*) Now $(A + B)^t = A^t + B^t = A + B$

Hence $A + B$ is symmetric.

(*ii*) Let AB be symmetric, then $(AB)^t = AB$

$\therefore \quad AB = (AB)^t = B^tA^t = BA$

Conversely, let $AB = BA$, then $(AB)^t = B^tA^t = BA = AB$

Hence, AB is symmetric.

Example 2: If A and B be commuting matrices, then prove that A^t and B^t commute.

Solution: Since A and B commute, then $AB = BA$.

Now $\quad A^tB^t = (BA)^t = (AB)^t \quad (\because AB = BA)$

$\quad = B^tA^t$

Hence, A^t and B^t commute.

Example 3: If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, then prove that $A^2 - 2A + I_2 = 0$. Hence find A^{50}.

Solution: Now $A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

$$\therefore \quad A^2 - 2A + I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Here $A^2 = 2A - I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$$\therefore \quad A^3 = A.A^2 = A(2A - I_2) = 2A^2 - A = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$A^4 = A^2.A^2 = A^2(2A - I_2) = 2A^3 - A^2 = \begin{pmatrix} 2 & 0 \\ -6 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$$

and so on.

Hence, $\quad A^{50} = \begin{pmatrix} 1 & 0 \\ -50 & 1 \end{pmatrix}$.

Example 4: (*i*) If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, then find AB or BA if they exist.

(*ii*) If A and B are square matrices of the same order, does the equality $(A + B) \times (A - B) = A^2 - B^2$ hold good? Give reasons.

Solution: (*i*) Since A is a 3×4 matrix and B is a 3×3 matrix, then the number of columns of $A \neq$ number of rows of B.

Hence AB is not defined.

But the number of columns of B = number of rows of A.

Hence BA is defined.

$$\therefore \quad BA = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+0 & 4+0+0 & 6+1+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 9+2+0 & 12+4+5 \\ 1+0+3 & 2+0+1 & 3+0+0 & 4+0+5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{pmatrix}.$$

(*ii*) Now $(A + B)(A - B) = A(A - B) + B(A - B) = AA - AB + BA - BB$

$$= A^2 - AB + BA - B^2$$

If $AB = BA$, then only the equality $(A + B)(A - B) = A^2 - B^2$ holds good otherwise not.

Example 5: (*i*) If A, B and C are matrices of appropriate order with $AB = AC$, then does it imply that $B = C$? Give an example in support of your conclusion.

(*ii*) If A, B and C are three matrices of order n such that $AB = I_n$ and $BC = I_n$, then prove that $A = C$.

Solution: (*i*) If A, B and C three matrices of appropriate order with $AB = AC$, then it does not necessarily imply that $B = C$.

For example, let $A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

and $\quad C = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix}$

Now $\quad AB = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 11 & 0 & -5 \end{pmatrix}$

and $\quad AC = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 11 & 0 & -5 \end{pmatrix}.$

$\therefore \quad AB = AC \quad$ But $B \neq C$.

(*ii*) Now $AB = I_n$

$\therefore \quad (AB)C = (I_n)C$

$\Rightarrow \quad A(BC) = C$ [$\because$ matrix product is associative and $I_nC = C$]

$\Rightarrow \quad AI_n = C$

$\Rightarrow \quad A = C$ [$\because AI_n = A$]

EXERCISES

1. If $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 \\ 0 & 10 \end{bmatrix}$

Find

(*i*) $3A + 2B$ (*ii*) $2(A + B) - 5C$

(*iii*) $A + B^t$ (*iv*) $(2A - 3B)^t + 2C$

(*v*) $(2A)^t - 2A^t$ (*vi*) $(A + B)^t - A^t - B^t$

2. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 4 & 0 \\ 1 & 2 \end{bmatrix}$

Then verify the following:

(*i*) $(A - B)^t = A^t - B^t$ (*ii*) $(A + B)^t = A^t + B^t$

(*iii*) $(-3A)^t = -3A^t$ (*iv*) $(2A + 3B)^t = 2A^t + 3B^t$

(*v*) $A + B = B + A$ (*vi*) $(A + B) + C = A + (B + C)$.

3. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$

Then verify that

(*i*) $AB \neq BA$ (*ii*) $A(BC) = (AB)C$

(*iii*) $A(B + C) = AB + AC$ (*iv*) $(AB)^t = B^tA^t$.

4. If $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, prove that $A(\theta)\,A(\phi) = A(\phi)\,A(\theta) = A(\theta + \phi)$.

5. Express the following matrices as the sum of two matrices one of which is symmetric and the other skew-symmetric:

(*i*) $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, (*ii*) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, (*iii*) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 4 \\ 5 & 4 & 3 \end{bmatrix}$.

6. (*i*) If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, prove that $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$

(*ii*) If $A = \begin{pmatrix} K & 1 \\ 0 & K \end{pmatrix}$, show that $A^n = \begin{pmatrix} K^n & nK^{n-1} \\ 0 & K^n \end{pmatrix}$.

7. (*i*) Find the matrix A if

$$A^2 = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}.$$

(*ii*) Find all non-null matrices

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$

so that $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(*iii*) If $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$, then prove that $A^2 = I_3$.

(*iv*) If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$, then prove that $A^2 = A$.

(*v*) If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$, then prove that $A^2 = 0$.

8. (*i*) If $$A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

then show that $$A^n = \begin{bmatrix} \cos nt & \sin nt \\ -\sin nt & \cos nt \end{bmatrix}.$$

(*ii*) If $$P = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Then compute A^n if (*i*) n is even, (*ii*) n is odd.

9. (*i*) Find the values of a, b, c, d if the matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ are commutative.}$$

(*ii*) If $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, then prove that $(A - 2I_2)(A - 3I_2) = 0$.

(*iii*) If $A = \begin{bmatrix} \cos\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\psi & \cos\psi\sin\psi \\ \cos\psi\sin\psi & \sin^2\psi \end{bmatrix}$

then prove that $AB = 0$ if ϕ and ψ differ by an odd multiple of $\pi/2$.

(*iv*) If $2A + B = \begin{pmatrix} 2 & 2 & 5 \\ 5 & 4 & 3 \\ 1 & 1 & 4 \end{pmatrix}$ and $A - 2B = \begin{pmatrix} 1 & 6 & 5 \\ 5 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}$, then prove

that $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(*v*) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, then prove that $A^3 - 23A - 40I_3 = 0$

(*vi*) If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ and $f(t) = t^2 - 5t + 6$, then prove that

$$f(A) = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

(*vii*) If $A = \begin{pmatrix} 5 & 4 & -2 \\ 4 & 5 & -2 \\ -2 & -2 & 2 \end{pmatrix}$, then prove that $A^2 - 2A + 10\,I_3 = 0$.

10. If A is symmetric or skew-symmetric, prove that

(*i*) A^2 is symmetric (*ii*) $AA^t = A^tA$.

11. If A and B are both symmetric or skew-symmetric of the same order and $AB = BA$, then show that AB is symmetric.

12. Find all matrices which commute with the matrix

$$\begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}.$$

13. Show that the matrix

$$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$

is a solution of the equation $A^2 - 5A + 7I = 0$.

14. Find all scalar matrices of order 2 which satisfy the following matrix equations:

(*i*) $A^2 - 5A + 7I = 0$ (*ii*) $A^3 - A = 0$.

15. Show that for all values of *a*, *b*, *c*, *d*, the following matrices commute:

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}.$$

16. If $AB = -BA$, the matrices A and B are said to be anti-commutative. Show that each of the matrices

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $i^2 = -1$, anticommutes with the others. These matrices are called *Pauli spin matrices* used in quantum mechanics.

Answers

1. (*i*) $\begin{bmatrix} 8 & 21 \\ 10 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 11 & 8 \\ 6 & -48 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 3 & 2 \\ 10 & 1 \end{bmatrix}$

(*iv*) $\begin{bmatrix} -1 & 15 \\ -12 & 12 \end{bmatrix}$. (*v*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. (*i*) $\begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 4 \\ 5 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. (*i*) $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ or $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix}$; $b, c \in \mathbb{R}$

9. (*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$; $a, b \in \mathbb{R}$

14. (*i*) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a = \frac{1}{2}(5 \pm i\sqrt{3})$ (*ii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$