CHAPTER

4

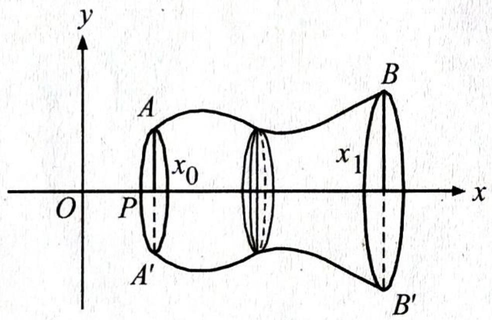
Applications of Definite Integral

4.1 EVALUATION OF VOLUMES AND SURFACE AREAS OF SOLIDS OF REVOLUTION

If a curve not perpendicular to the x-axis is revolved about the x-axis, it generates a surface of revolution. If the curve is finite, the area of the surface of revolution thus generated can be evaluated and also the volume of the region enclosed by this surface can be evaluated. Let $y = f(x)$ describe the curve bounded between $x = x_0$ and $x = x_1$ (see figure).

Then the total volume of the solid of revolution

$$= \pi \int_{x_0}^{x_1} y^2 dx$$



The total surface area of this solid of revolution

$$= 2\pi \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad \text{[Cartesian Form]}$$

If, instead of the x-axis, the curve $x = g(y)$, not parallel to the x-axis, is revolved round the y-axis, the corresponding volume and surface area of the solid of revolution will be given by

$$\text{Volume} = \pi \int_{y_0}^{y_1} x^2 dy$$

$$\text{Surface area} = 2\pi \int_{y_0}^{y_1} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{[Cartesian Form]}$$

where the end points A and B are (x_0, y_0) and (x_1, y_1) respectively.

If the curve is given in polar coordinate system by $r = f(\theta)$ and the ends of the finite curve are $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$, then

$$\text{Volume} = \pi\int_{x_1}^{x_2} y^2 dx = \pi\int_{\theta_1}^{\theta_2} r^2 \sin^2\theta \cdot d(r\cos\theta)$$

$$\text{Surface area} = 2\pi\int_{\theta_1}^{\theta_2} r\sin\theta.\sqrt{dr^2 + r^2\, d\theta^2}$$

If the curve is given in the parametric form

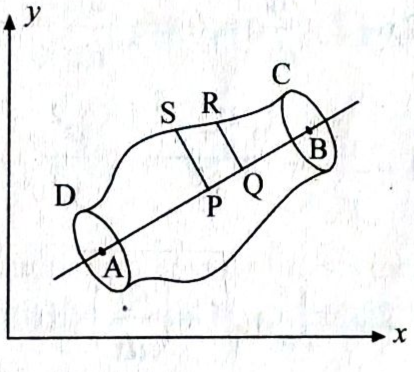
$x = f(t)$ and $y = \phi(t)$, then

$$\text{Volume} = V = \pi\int_{t_1}^{t_2} [\varphi(t)]^2\, f'(t)\,dt$$

$$\text{Surface} = S = 2\pi\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where t_1 and t_2 are values corresponding to $x = a$ and $x = b$.

Rotation about Any Line in the Plane: If the curve *DSRC* be revolved about any line *AB* in its plane, and *PQRS* be the infinitely small strip which is perpendicular to *AB*, then



$$\text{Volume} = V = \int \pi(SP)^2\, PQ$$

$$= \int \pi(SP)^2\, d(AP)$$

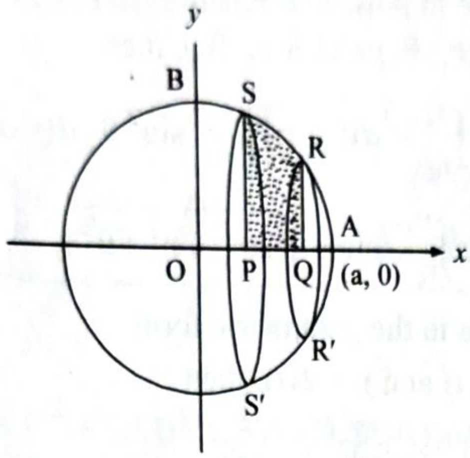
where integration extends from *A* to *B*.

The surface area is given by

$$\text{Surface} = S = 2\pi\int PS\,.\,dS.$$

Example 1: Find the volume and surface area of the sphere generated by the revolution of the circle $x^2 + y^2 = a^2$ about the x-axis.

Solution: Let the quadrant *OAB* be rotated about *OX*. The volume described will be that of a hemisphere, we get



$$V = 2 \times \pi \int_0^a y^2 dx$$

$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{4}{3}\pi a^3$$

Since $x^2 + y^2 = a^2$, $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \frac{a^2}{y^2}$$

$\therefore$ Surface area, $S = 2 . 2\pi \int_0^a y \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$

$$= 4\pi \int_0^a y \frac{a}{y} dx = 4\pi a^2.$$

Example 2: Find the surface area of the surface generated by revolving about the y-axis the part of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, that lies in the first quadrant.

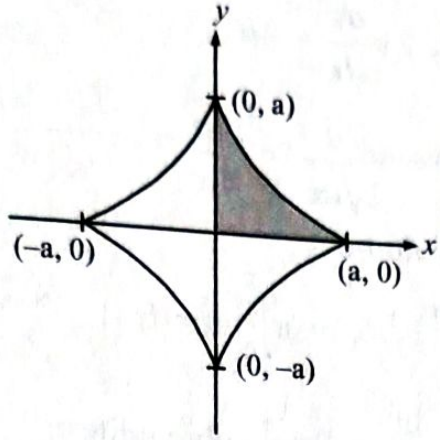
Solution: The parametric equation of the astroid is

$$x = a \cos^3 \theta$$

$$y = a \sin^3 \theta$$

$$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$



Now $\dfrac{dx}{dy} = -\dfrac{3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta$

$\therefore$ $$S = 2\pi\int_0^{\pi/2} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

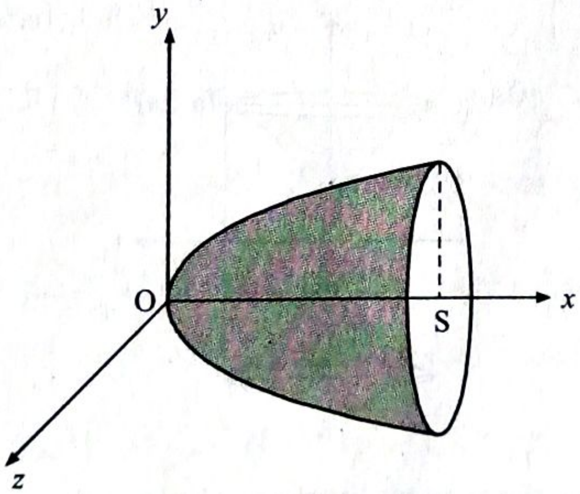
$$= 2\pi\int_0^{\pi/2} a\cos^3\theta\operatorname{cosec}\theta\,.\,(3a\sin^2\theta\cos\theta)\,d\theta$$

$$= 6a^2\pi\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta, \text{ Let } \cos\theta = z, -\sin\theta\,d\theta = dz$$

$$= 6a^2\pi\int_0^1 z^4dz = 6a^2\pi\left(\frac{z^5}{5}\right)_0^1 = \frac{6a^2\pi}{5}.$$

Example 3: Find the volume and area of the surface generated by revolving the parabola $y^2 = 4ax$ about the x-axis bounded by the section $x = a$.

Solution:



Since $\qquad y^2 = 4ax,\ 2y\dfrac{dy}{dx} = 4a$

or $\qquad \dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{2a}{2\sqrt{ax}} = \sqrt{\dfrac{a}{x}}.$

Hence, the required volume

$$= \pi\int_0^a y^2 dx = \pi\int_0^a 4ax\,dx$$

$$= 4a\pi\frac{a^2}{2} = 2\pi a^3 \text{ cubic units}$$

The required surface area

$$= 2\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

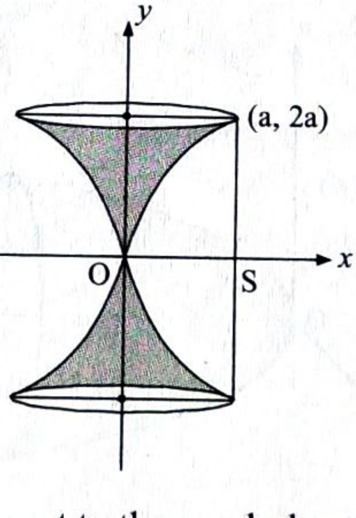
$$= 2\pi\int_0^a \sqrt{4ax}\,\sqrt{1+\frac{a}{x}}\,dx$$

$$= 2\pi\int_0^a \sqrt{a+x}\,dx = \frac{8}{3}\pi\,\{(2a)^{3/2} - a^{3/2}\}$$

$$= \frac{8}{3}(2\sqrt{2}-1)\pi a^{3/2} \text{ sq. units}$$

Example 4: Find the volume and surface area of the solid of revolution generated by revolving the parabola $y^2 = 4ax$ bounded by the lotus rectum about its tangent at the vertex.

Solution:



Since the y-axis is the tangent to the parabola at the vertex, the parabola is rotated about the y-axis.

Hence, the required volume

$$= \pi\int_{-2a}^{2a} x^2 dy$$

$$= \pi\int_{-2a}^{2a} \frac{y^4}{16a^2} dy = \frac{4\pi}{5} a^3 \text{ cubic units}$$

The required area of the surface

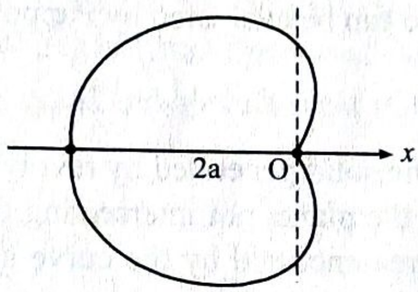
$$= 2\pi\int_{-2a}^{2a} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi\int_{-2a}^{2a} \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= 4\pi a^2\int_{-\pi/4}^{\pi/4} \tan^2\theta \sec^2\theta\, d\theta, \text{ putting } y = 2a\tan\theta$$

$$= \pi a^2\{3\sqrt{2} - \ln(\sqrt{2}+1)\}.$$

Example 5: If the cardioide $r = a(1 - \cos\theta)$ is rotated about the initial line, find the volume and the area of the solid of revolution generated.

Solution:



We first observe that the curve is symmetric with respect to the initial line. So the solid of revolution generated by revolving a cardioide is same as the one generated by the upper half of the cardioide with end points given by (0, 0) and $(2a, \pi)$.

Hence, the required volume

$$= \pi\int y^2 dx = \pi\int r^2 \sin^2\theta\, d(r\cos\theta)$$

$$= \pi a^2\int (1-\cos\theta)^2 \sin^2\theta . d(a(1-\cos\theta)\cos\theta)$$

$$= \pi a^3\int_{\pi}^{0} (1-\cos\theta)^2 \sin^2\theta\,(-\sin\theta + 2\sin\theta\cos\theta)\, d\theta$$

[$\because$ x increases as θ diminishes from π to 0]

$$= \pi a^3\int_{-1}^{1} (1-t)^2\,(1-t^2)\,(1-2t)dt, \text{ putting } t = \cos\theta.$$

$$= \frac{8}{3}\pi a^3 \text{ cubic units.}$$

Also the required surface area

$$= 2\pi\int y\,ds = 2\pi\int r\sin\theta\sqrt{dr^2 + r^2 d\theta^2}$$

$$= 2\pi\int_0^{\pi} a(1-\cos\theta)\sin\theta\sqrt{(a\sin\theta\, d\theta)^2 + a^2(1-\cos\theta)^2 d\theta^2}$$

$$= 2\pi a^2\int_0^{\pi}(1-\cos\theta)\sin\theta\sqrt{2(1-\cos\theta)}\,d\theta$$

$$= 2\sqrt{2}\,\pi a^2\int_0^{2} t^{3/2}\,dt, \text{ putting } t = 1-\cos\theta$$

$$= \frac{32}{5}\pi a^2 \text{ sq. units.}$$

4.2 PAPPUS' THEOREM

If a plane closed curve is rotated through an angle about a straight line lying in its plane and not intersecting the curve, then the area of the surface thus generated and its volume can be evaluated by Pappus' theorem. The theorem is stated as follows:

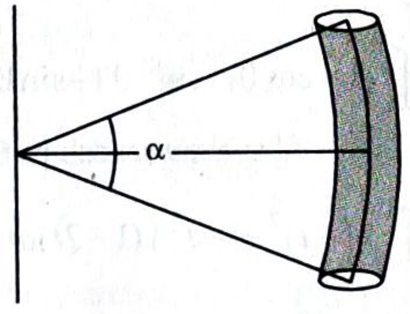
Theorem:

(*i*) The volume of the solid generated by revolving a plane closed curve about an axis in the plane, not intersecting the curve, is equal to the product of the area enclosed by the curve and the length of the arc described by the centroid of the area.

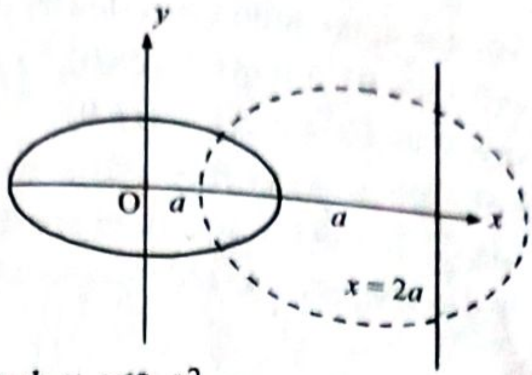
(*ii*) The surface area of the solid is equal to the product of the perimeter of the curve and the length of the arc described by the centroid of the perimeter.

Example 6: Find the volume of the solid formed by revolving the ellipse $x = a\cos\theta$, $y = b\sin\theta$ about the line $x = 2a$.

Solution:



Since the centre of the ellipse is also the centroid of the area enclosed by the ellipse, by Pappus theorem.



Volume = $\pi ab \times \pi(2a)^2$
$= 4\pi^2 a^3 b$ cubic units.

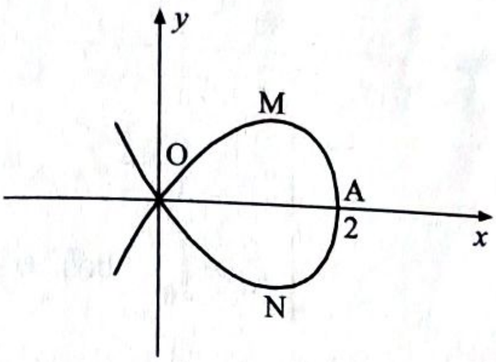
Example 7: Find the volume generated by the revolution about x-axis the area bounded by the loop of the curve $y^2 = x^2 (2 - x)$.

Solution: We see that $y = 0$ for $x = 0, 2$. The curve cuts the x-axis at $O(0, 0)$ and $A(2, 0)$.

When $x > 2$, y is imaginary. That is there is no part of the loop to the right of A.

When $0 < x < 2$, y has two equal and opposite finite values. Thus the loop formed within interval. Which is symmetric about x-axis.

When $x < 0$, y, has two equal and opposite values, these values of y increase in magnitude as $x \to -\infty$.



The required volume is

$$V = \pi\int_0^2 y^2 dx$$

$$= \pi\int_0^2 x^2(2-x)\,dx$$

$$= \pi\int_0^2 [2x^2 - x^3]\,dx$$

$$= \pi\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \pi\left(\frac{2}{3}\cdot 8 - \frac{1}{4}.16\right)$$

$$= 16\,\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{16\pi}{12} = \frac{4}{3}\pi.$$

Example 8: Find the volume of the solid generated by revolving the cycloid.

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta).$$

Solution: Now $x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$

The extreme values of x are given by $\theta = \pm\pi$,

i.e. $x = \pm a\pi$ [$\because 0 = a(1 + \cos\theta) \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi$]

$$dx = a(1 + \cos\theta)\,d\theta$$

The required volume

$$V = \pi\int_{-a\pi}^{a\pi} y^2 dx = \pi\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2\, a(1+\cos\theta)\,d\theta$$

$$= \pi a^3\int_{-\pi}^{\pi}(1+\cos\theta)^3 d\theta = \pi a^3\int_{-\pi}^{\pi}(2\cos^2\theta/2)^3 d\theta$$

$$= 8\pi a^3\int_{-\pi}^{\pi}\cos^6\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^3\int_0^{\pi}\cos^6\frac{\theta}{2}\,d\theta \quad \left[\int_{-a}^{a} f(x)dx = 2\int_0^{\pi} f(x)dx \text{ when } f(x) \text{ is even function}\right]$$

$$= 16\pi a^3 \times 2\int_0^{\pi/2}\cos^6\phi\, d\phi \text{ when } \frac{\theta}{2} = \phi$$

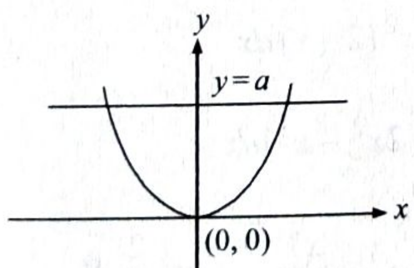
$$= 32\pi a^3\,\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\left[J_n = \frac{n-1}{2}J_{n-2} \text{ where } J_n = \int_0^{\pi/2}\cos^n\theta\, d\theta,\ J_0 = \int_0^{\pi/2} d\theta = \frac{\pi}{2}\right]$$

$$= 5\pi^2 a^3.$$

Example 9: Find the volume of the solid generated by revolving the part of the parabola $x^2 = 4ay$, $a > 0$, between the ordinates $y = 0$ and $y = a$ about its axis.

Solution: The required volume bounded by $y = 0, y = a$ is given by



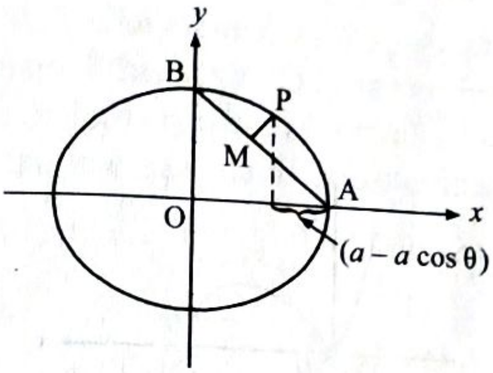
$$V = \pi\int_0^a x^2 dy$$

$$= \pi\int_0^a 4ay\,dy = 4\pi a\left[\frac{y^2}{2}\right]_0^a$$

$$= 2\pi a^3 \text{ cubic units.}$$

Example 10: The smaller segment of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, cut off by the chord $\frac{x}{a}+\frac{y}{b}=1$ revolves completely about this chord. Find the volume generated.

Solution: AB represents the chord $\frac{x}{a}+\frac{y}{b}=1$. Let P be any point on the ellipse, whose coordinates are $(a\cos\theta, b\sin\theta)$. The length of the perpendicular PM is $\frac{ab}{\sqrt{a^2+b^2}}\{\sin\theta+\cos\theta-1\}$. Let A be the fixed point on the axis of revolution AB, then



$$AM^2 = AP^2 - PM^2$$

$$= \{(-a\cos\theta+a)^2+(b\sin\theta)^2\} - \frac{a^2b^2}{a^2+b^2}(\sin\theta+\cos\theta-1)^2$$

$$= a^2(1-\cos\theta)^2+b^2\sin^2\theta$$

$$-\frac{a^2b^2}{a^2+b^2}\quad[(1-\cos\theta)^2+\sin^2\theta-2\sin\theta\,(1-\cos\theta)]$$

$$= \frac{a^4(1-\cos\theta)^2+b^4\sin^2\theta+2a^2b^2\sin\theta\,(1-\cos\theta)}{a^2+b^2}$$

$$= \frac{[a^2(1-\cos\theta)+b^2\sin\theta]^2}{a^2+b^2}$$

$$\therefore \qquad AM = \frac{a^2(1-\cos\theta) + b^2 \sin\theta}{\sqrt{a^2+b^2}}$$

$$d(AM) = \frac{a^2 \sin\theta + b^2 \cos\theta}{\sqrt{a^2+b^2}} d\theta$$

Thus, the required volume

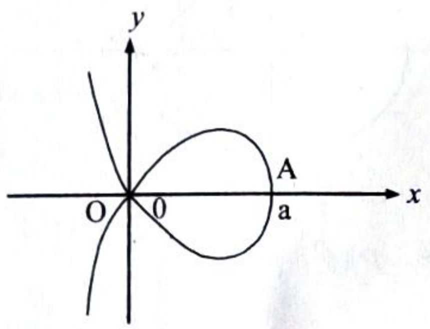
$$= \pi\int (PM)^2\, d(AM) \text{ [range being throughout } AB]$$

$$= \pi\int_0^{\pi/2} \frac{a^2b^2}{a^2+b^2}(\sin\theta+\cos\theta-1)^2 \times \frac{a^2\sin\theta + b^2\cos\theta}{\sqrt{a^2+b^2}} d\theta$$

$$= \frac{\pi a^2 b^2}{\sqrt{a^2+b^2}}\left(\frac{5}{3}-\frac{\pi}{2}\right).$$

Example 11: Show that the volume of the solid produced by the revolution of the loop of the curve $y^2(a+x) = x^2(a-x)$ about the x-axis is $2\pi a^3$ (log 2 – 2/3).

Solution: Now $y^2 = \dfrac{x^2(a-x)}{a+x}$



$\therefore$ $y = 0$ for $x = 0, a$. The curve cuts the x-axis at $A(a, 0)$ and $O(0, 0)$.

When $x > a$, y is imaginary.

When $0 < x < a$, then y has two equal and opposite values. Thus a loop is formed which is symmetric about the x-axis.

$\therefore$ The required volume is

$$V = \pi\int_0^a y^2\, dx = \pi\int_0^a \frac{x^2(a-x)}{a+x} dx$$

$$= \pi\int_0^a \left\{-x^2 + 2ax - 2a^2 + \frac{2a^3}{a+x}\right\} dx$$

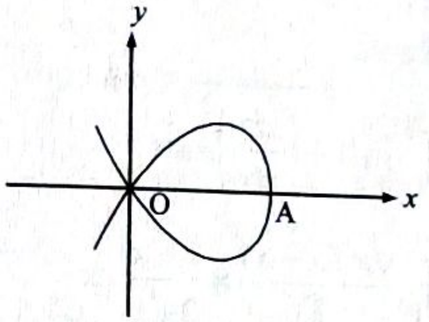
$$= \pi\left[-\frac{x^3}{3} + ax^2 - 2a^2x + 2a^3 \log(a+x)\right]_0^a$$
$$= \pi a^3\,[-1/3 + 1 - 2 + 2\log 2a - \log a]$$
$$= \pi a^3\,(2\log 2 - 4/3)$$
$$= 2\pi a^3\,(\log 2 - 1/3).$$

Example 12: Show that the surface area and the volume of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2, y = t - \frac{1}{3}t^2$ are respectively 3π and $\frac{3}{4}\pi$.

Solution: Since $x = t^2$ and $y = t - \frac{1}{3}t^3$

$$\therefore \qquad y^2 = t^2\left(1 - \frac{1}{3}t^2\right)^2 = \left(1 - \frac{1}{3}x\right)^2$$

Now $y = 0$ for $x = 0, 3$. The curve cuts the x-axis at $A(3, 0)$ and $O(0, 0)$. When $x > 3$, y is imaginary. When $0 < x < 3$, y has two equal and opposite finite values. Thus, the loop is formed within interval which is symmetric about x-axis and for $x < 0$, y has two equal and opposite finite values. This value of y increases as $x \to \infty$.



The required volume is

$$V = \pi\int_0^3 y^2\,dx = \pi\int_0^3 x\left(1 - \frac{1}{3}x\right)^2 dx$$

$$= \frac{\pi}{9}\int_0^3 x[9 - 6x + x^2]\,dx$$

$$= \frac{\pi}{9}\left[\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right]_0^3$$

$$= \frac{\pi}{9}\left[\frac{9.9}{2} - \frac{6 \times 3.9}{3} + \frac{9 \times 9}{4}\right]$$

$$= \frac{\pi}{9}\left[\frac{3}{4} \times 9 \times 9 - 6 \times 9\right] = \frac{\pi}{9} \times 3 \times 9\left[\frac{9}{4} - 2\right] = \frac{3}{4}\pi$$

$$\therefore \quad \frac{dy}{dx} 2y = \frac{1}{9}[(3-x)^2 - 2x(3-x)]$$

$$= \frac{1}{9}(3-x)[3 - x - 2x]$$

$$= \frac{1}{9} 3(3-x)(1-x) = \frac{(3-x)(1-x)}{3}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(3-x)^2 (1-x)^2}{94y^2}$$

$$= 1 + \frac{(3-x)^2 (1-x)^2}{36x(3-x)^2} 9$$

$$= 1 + \frac{(1-x)^2}{4x} = \frac{4x + 1 - 2x + x^2}{4x}$$

$$= \frac{(1+x)^2}{4x}$$

The required surface area

$$= 2\pi \int_0^3 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^3 \frac{\sqrt{x}\,(3-x)}{3} \times \frac{1+x}{2\sqrt{x}}\, dx$$

$$= \frac{\pi}{3}\int_0^3 (1+x)(3-x)\, dx$$

$$= \frac{\pi}{3}\int_0^3 [3 - x + 3x - x^2]\, dx$$

$$= \frac{\pi}{3}\left[3x + \frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{\pi}{3}[9 + 9 - 9]$$

$$= 3\pi.$$

Example 13: Find the volume of the solid generated by revolving, about the x-axis, the areas bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$; $x = 0, y = 0$.

Solution: Since $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a parabola lying in the first quadrant and touching the axes at $(a, 0), (0, a)$. Then required volume is

$$V = \pi\int_0^a y^2\,dx = \pi\int_0^a (\sqrt{a} - \sqrt{x})^4\,dx$$

$$= \pi\int_0^a \left(a^2 - 4a^{3/2}\,x^{1/2} + 6ax - 4a^{1/2}\,x^{3/2} + x^2\right)dx$$

$$= \left[a^2 x - \frac{8a^{3/2}}{3}\,x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}\,x^{5/2} + \frac{x^3}{3}\right]_0^a$$

$$= \pi\left[a^3 - \frac{8}{3}a^3 + 3a^3 - \frac{8}{5}a^3 + \frac{a^3}{3}\right]$$

$$= \pi a^3\,(4 - 8/3 - 8/5 + 1/3) = \pi a^3\left(\frac{60 - 40 - 24 + 5}{15}\right)$$

$$= \pi a^3 \times \frac{1}{15}.$$

Example 14: Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis and bounded by the section $x = x_1$.

Solution: The required volume is given by

$$V = \pi\int_0^{x_1} y^2\,dx = \pi\int_0^{x_1} 4ax\,dx = 4\pi a\,\frac{x^2}{2}\Big|_0^{x_1}$$
$$= 2\pi a\,x_1^2$$

Now $$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y} = \frac{2a}{\sqrt{4ax}} = \sqrt{\frac{a}{x}}$$

The required surface area is given by

$$S = 2\pi\int_0^{x_1} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{x_1}\sqrt{4ax}\,\sqrt{1 + \frac{a}{x}}\,dx$$

$$= 2\pi\,2\sqrt{a}\int_0^{x_1}\sqrt{x + a}\,dx = 4\pi\sqrt{a} \times \frac{2}{3}(x + a)^{3/2}\Big|_0^{x_1}$$

$$= \frac{8}{3}\pi\sqrt{a}\left[(a + x_1)^{3/2} - a^{3/2}\right]$$

Example 15: Find the volume V of a solid bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution: In this case, z varies from 0 to $1-x-y$, then y varies from 0 to $1-x$ and then x varies from 0 to 1.

$$\therefore \quad V = \iiint dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}[z]_0^{1-x-y}\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x}(1-x-y)\,dy\,dx$$

$$= \int_{x=0}^{1}\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 [1 - 2x + x^2]\,dx = \frac{1}{2}\left[x^2 - x^2 + \frac{x^3}{3}\right]_0^1.$$

$$= \frac{1}{2}\left[1 - 1 + \frac{1}{3}\right] = \frac{1}{6}$$

EXERCISES

1. Find the surface of revolution generated by the curve $y = x^3, 0 \le x \le 1/2$, about x-axis.
2. Find the volume and the area of surface of revolution of the solid obtained by revolving the enclosed by the curve $y = \sqrt{x}$ and the lines $x = 1, x = 4$ about x-axis.
3. Find the volume of the solid generated by the revolution of the cardioide $r = a\,(1 + \cos\theta)$ about the initial line.
4. Show that the curved surface and volume of the catenoid formed by the revolution, about x-axis, of the area bounded by the catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, the y-axis, the x-axis and an ordinate are respectively $\pi(sy + ax)$ and $\frac{\pi a}{2}(sy + ax)$, s being the length of the arc between $(0, a)$ and (x, y).
5. Find the volumes of the solids generated by revolving, about the x-axis, the area bounded by the curve $y = 5x - x^2$ and lines $x = 0$ and $x = 5$.

6. Find the volume of solid generated by revolving $r^2 = a^2 \cos 2\theta$ about the line $\theta = \pi/2$.
7. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about x-axis.
8. Find the volume of when the loop of the curve $y^2 = x(2x-1)^2$ revolves about x-axis.
9. Find the volume of the solid by the revolution of the plane area bounded by $y^2 = 9y$ and $y = 3x$ about x-axis.
10. Find the area of surface of revolution generated by the following region about y-axis:
    (*i*) the region enclosed by $2y = x + 1, \quad 1 \le x \le 3$
    (*ii*) the region enclosed by $y^3 = 3x$, $y = 0$ and $y = 1$.
11. Find the volume of the solid generated by the curve $y^2 = x^2(1 - x^2)$ rotated about the y-axis.
12. Find the volume formed by the revolution of loop $y^2(a + x) = x^2(3a - x)$ about the x-axis.

Answers

1. $\dfrac{61\pi}{1728}$ 2. $\dfrac{15\pi}{2}, \dfrac{\pi}{4}\{17^{3/2} - 5^{3/2}\}$ 3. $\dfrac{8}{3}\pi a^3$

5. $\dfrac{625}{6}\pi$ 6. $\dfrac{\pi a^3}{4\sqrt{2}}$ 7. $\left(\dfrac{4}{3}\right)\pi ab^2$

8. $\dfrac{\pi}{48}$ 9. $\dfrac{3\pi}{2}$

10. (*i*) $\dfrac{\pi}{9}(\sqrt{8} - 1)$ (*ii*) $\dfrac{35\pi\sqrt{5}}{3}$ 11. $\dfrac{\pi^2}{4}$

12. $\pi a^3 (8 \log 2 - 3)$