CHAPTER

3

Improper Integrals

3.1 INTRODUCTION

Definite integration has many limitations. First, for definite integration the integrand has to be a bounded function over the domain of integration. Second, the limits of integration have to be real numbers, *i.e.*, the domain of integration has to be a bounded interval of $\mathbb{R}$. Attempts to define integration over unbounded intervals and also to unbounded integrands have been successful. Such integrals are known as improper integrals.

3.2 IMPROPER INTEGRAL OF TYPE ONE

All improper integrals are divided into two types, the first of which has unbounded intervals as its range of integration. We have three cases here:

1. $\int_a^{\infty} f(x)\,dx$

   This integral is defined as

   $\lim_{X\to\infty} \int_a^X f(x)\,dx$, provided the integral $\int_a^X f(x)\,dx$ is defined for every $X > a$.

Example 1: Evaluate $\int_0^{\infty} e^{-x}\,dx$.

Solution: By definition,

$$\int_0^{\infty} e^{-x}\,dx = \lim_{X\to\infty} \int_0^X e^{-x}\,dx$$

$$= \lim_{X\to\infty} \left[-e^{-x}\right]_0^X = \lim_{X\to\infty} \left[1 - \frac{1}{e^X}\right] = 1$$

2. $\int_{-\infty}^{b} f(x)\,dx.$

This integral is defined as $\lim_{Y\to\infty} \int_{Y}^{b} f(x)\,dx$, provided $\int_{Y}^{b} f(x)\,dx$ is defined for every $Y < b$.

Example 2: Evaluate $\int_{-\infty}^{0} 2^x\,dx$

Solution: By definition $\int_{-\infty}^{0} 2^x\,dx = \lim_{Y\to-\infty} \int_{Y}^{0} 2^x\,dx = \lim_{Y\to-\infty} \left[\frac{2^x}{\ln 2}\right]_Y^0$,

$$= \lim_{Y\to-\infty} \left(\frac{1}{\ln 2} - \frac{2^Y}{\ln 2}\right) = \frac{1}{\ln 2}$$

3. $\int_{-\infty}^{\infty} f(x)\,dx.$

This integral is defined as $\int_{-\infty}^{m} f(x)\,dx + \int_{m}^{\infty} f(x)\,dx$ when $m \in \mathbb{R}$. So if the integrals $\int_{-\infty}^{m} f(x)\,dx$ and $\int_{m}^{\infty} f(x)\,dx$ are defined as per above definition, then the above integral is defined. Thus by definition

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{Y\to-\infty} \int_{Y}^{m} f(x)\,dx + \lim_{X\to\infty} \int_{m}^{X} f(x)\,dx \text{ where } Y < m < X$$

provided $\int_{Y}^{m} f(x)\,dx$ and $\int_{m}^{X} f(x)\,dx$ exist.

Remarks: For definite integral, if $\int_{a}^{b} f(x)\,dx$ is defined, we say $\int_{a}^{b} f(x)\,dx$ exists but for improper integral if any of the improper integrals is defined, we say that it converges.

Example 3: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution: By definition.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{Y \to -\infty} \int_{Y}^{0} \frac{dx}{1+x^2} + \lim_{X \to \infty} \int_{0}^{X} \frac{dx}{1+x^2}$$

$$= \lim_{Y \to -\infty} \left[\tan^{-1} x\right]_{Y}^{0} + \lim_{X \to \infty} \left[\tan^{-1} X\right]_{0}^{X}$$

$$= \lim_{Y \to -\infty} \left[0 - \tan^{-1} Y\right] + \lim_{X \to \infty} \left[\tan^{-1} Y - 0\right]$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

3.3 IMPROPER INTEGRAL OF TYPE TWO

Here also there are three possible cases:

1. $\int_{a}^{b} f(x)\, dx$**, $f(x)$ unbounded only at a.**

   This improper integral is defined as

$$\int_{a}^{b} f(x)\, dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\, dx,$$

provided the integrals. $\int_{a+\varepsilon}^{b} f(x)\, dx$ is defined for $\varepsilon > 0$.

Example 1: Evaluate $\int_{0}^{1} \frac{dx}{x^2}$

Solution: By definition

$$\int_{0}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0+0} \int_{\varepsilon}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \left[-\frac{1}{x}\right]_{0}^{1} = \lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon} - 1\right) = \infty.$$

Hence, $\int_{0}^{1} \frac{dx}{x^2}$ does not converge.

2. $\int_a^b f(x)\,dx$, $f(x)$ unbounded only at b

This integral is defined as

$$\lim_{\varepsilon'\to 0+0} \int_a^{b-\varepsilon'} f(x)\,dx, \text{ provided } \int_a^{b-\varepsilon'} f(x)\,dx \text{ exists.}$$

Example 2: Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

Solution: Clearly the integrand $\frac{1}{\sqrt{1-x^2}}$ is unbounded only at $x = 1$. So by definition,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon'\to 0+} \int_0^{1-\varepsilon'} \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{\varepsilon'\to 0+} [\sin^{-1} x]_0^{1-\varepsilon'}$$

$$= \lim_{\varepsilon'\to 0+} \sin^{-1}(1-\varepsilon') = \frac{\pi}{2}$$

3. $\int_a^b f(x)\,dx$, $f(x)$ unbounded only at c where $a < c < b$

This integral is defined as the sum $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are defined as per above definition. Thus $\int_a^b f(x)\,dx$

$$= \lim_{\varepsilon'\to 0+} \int_a^{c-\varepsilon'} f(x)\,dx + \lim_{\varepsilon\to 0+} \int_{c+\varepsilon}^{b} f(x)\,dx.$$

Example 3: Evaluate $\int_{-1}^1 \frac{dx}{x^2}$.

Solution: The integrand $\frac{1}{x^2}$ is unbounded at 0 where $-1 < 0 < 1$.

By definition, we get

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon' \to 0+} \int_{-1}^{0-\varepsilon'} \frac{1}{x^2}\, dx + \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} \frac{1}{x^2}\, dx$$

$$= \lim_{\varepsilon' \to 0+} \left[-\frac{1}{x} \right]_{-1}^{-\varepsilon'} + \lim_{\varepsilon \to 0+} \left[-\frac{1}{x} \right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon' \to 0+} \left(\frac{1}{\varepsilon'} - 1 \right) + \lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon} - 1 \right) = \infty.$$

Hence $\int_{-1}^{1} \frac{dx}{x^2}$ does not converge, *i.e.*, is not defined.

Definition: The improper integral $\int_a^b f(x)\, dx$ having unboundedness of $f(x)$ at c where $a < c < b$ is said to have the Cauchy's principal value if

$$\lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\, dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^{b} f(x)\, dx \text{ exists finitely.}$$

Example 4: Find the Cauchy's principal value of $\int_{-1}^{1} \frac{dx}{x^3}$ if it exists.

Solution: Clearly $\frac{1}{x^3}$ is unbounded at 0 where $-1 < 0 < 1$.

Now Cauchy's principal value $= \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\varepsilon \to 0+} \int_{-1}^{-\varepsilon} \frac{1}{x^3}\, dx + \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{1}{x^3}\, dx$

$$= \lim_{\varepsilon \to 0+} \left[\frac{-1}{2x^2} \right]_{-1}^{-\varepsilon} + \lim_{\varepsilon \to 0+} \left[\frac{-1}{2x^2} \right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0+} \left(\frac{1}{2} - \frac{1}{2\varepsilon^2} \right) + \lim_{\varepsilon \to 0+} \left(-\frac{1}{2} + \frac{1}{2\varepsilon^2} \right)$$

$$= \lim_{\varepsilon \to 0+} \left(\frac{1}{2} - \frac{1}{2\varepsilon^2} - \frac{1}{2} + \frac{1}{2\varepsilon^2} \right) = 0.$$

3.4 BETA AND GAMMA FUNCTIONS

We begin with the gamma function.

3.4.1 The Gamma Function $\Gamma(n)$

The gamma function is defined as follows:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0$$

Now we write,

$$\Gamma(n) = \int_0^1 e^{-x} x^{n-1}\, dx + \int_1^\infty e^{-x} x^{n-1}\, dx$$

We note that if $n < 1$, then 0 is a point of infinite discontinuity and hence $\int_0^1 e^{-x}x^{n-1}\, dx$ is an improper integral having the integrand unbounded at zero. We, therefore, test convergence at zero when $n < 1$.

Define $f(x) = e^{-x}x^{n-1}$ and $g(x) = \frac{1}{x^{1-n}}$

Clearly, $f \geq 0$, $g \geq 0$ for all $x \in (0, 1)$ and $\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} e^{-x} = 1$

Hence, $\int_0^1 e^{-x}x^{n-1}\, dx$ converges, since $\int_0^1 \frac{1}{x^{1-n}}$ converges for $n > 0$.

Next, we test convergence of $\int_1^\infty e^{-x} x^{n-1}\, dx$.

Let $\phi(x) = \frac{1}{x^2}$, then $\phi(x) > 0$ for all $x \geq 1$.

$$\therefore \quad \lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim_{x \to \infty} \frac{x^{n+1}}{e^x} = 0 \text{ for all } n.$$

Therefore, $\int_1^\infty e^{-x}x^{n-1}\, dx$ converges as $\int_1^\infty \frac{1}{x^2}\, dx$ converges.

Combining, we get $\int_0^\infty e^{-x}x^{n-1}\, dx$ is convergent for $n > 0$.

Proposition 1: $\Gamma(1) = 1$

From the definition of gamma function, we get

$$\Gamma(1) = \int_0^\infty e^{-x}x^0\,dx = \int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx$$

$$= \lim_{A\to\infty}\left[-e^{-x}\right]_0^A = \lim_{A\to\infty}\left[1-e^{-A}\right] = 1$$

Proposition 2: $\Gamma(n+1) = n\,\Gamma(n)$

From the definition of gamma function, we get

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^n\,dx = \lim_{A\to\infty}\int_0^A e^{-x}x^n\,dx$$

$$= \lim_{A\to\infty}\left[-e^{-x}x^n\Big|_0^A + n\int_0^A e^{-x}x^{n-1}dx\right] \text{ [Integrating by parts]}$$

$$= \lim_{A\to\infty}[-e^{-A}A^n] + n\lim_{A\to\infty}\int_0^A e^{-x}x^{n-1}\,dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}\,dx = n\,\Gamma(n)$$

$$\Gamma(n) = \Gamma(n-1+1) = (n-1)\,\Gamma(n-1)$$

$$= (n-1)\,\Gamma(n-2+1)$$

$$= (n-1)\,(n-2)\,\Gamma(n-2)$$

$$= (n-1)\,(n-2)\,\Gamma(n-3+1)$$

$$= (n-1)\,(n-2)\,(n-3)\,\Gamma(n-3)$$

....

....

$$= (n-1)\,(n-1)\,(n-3)\,...\,3.2.1\,\Gamma(1)$$

$$= (n-1)\,(n-2)\,(n-3)\,...\,3.2.1$$

$$= \underline{|n-1}$$

$$\therefore\ \Gamma(n+1) = n\underline{|n-1} = \underline{|n}$$

Proposition 3: For any $a > 0$, $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$.

Let $y = ax$, then $dy = a\,dx$ and $\begin{array}{c|c|c} x & 0 & \infty \\ \hline y & 0 & \infty \end{array}$.

$$\therefore \quad \int_0^\infty e^{-ax} x^{n-1} dx = \int_0^\infty e^{-y} \frac{y^{n-1}}{a^{n-1}} \frac{dy}{a} = \frac{1}{a^n} \int_0^\infty e^{-y} y^{n-1}\, dy = \frac{\Gamma(n)}{a^n}$$

Proposition 4: For $0 < m < 1$, $\Gamma(m)\,\Gamma(1 - m) = \pi \operatorname{cosec} m\pi$.

3.4.2 The Beta Function $B(m, n)$

The Beta function is defined as follows:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx,\ m, n > 0.$$

We note that if $m \geq 1$, $n \geq 1$, the beta function is a proper integral but if $m < 1$, zero is a point of infinite discontinuity and if $n < 1$, one is a point of infinite discontinuity.

Let $m < 1, n < 1$, then we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\, dx$$

To test the convergence of $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\, dx$ at $x = 0$, we take

$f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = \frac{1}{x^{1-m}}$, we observe that $f > 0$, $g > 0$ for all

$x \in \left(0, \frac{1}{2}\right)$ and $\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} (1-x)^{n-1} = 1$

Hence $\int_0^{1/2} x^{m-1}(1-x)^{n-1} dx$ converges as $\int_0^{1/2} \frac{1}{x^{1-m}}\, dx$ converges for $m > 0$

Next, we test the convergence of

$$\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\, dx \text{ at } x = 1.$$

Here we take $f(x) = x^{m-1}(1-x)^{n-1}$, $g(x) = \dfrac{1}{(1-x)^{1-n}}$

Clearly $f > 0, g > 0$ for all $x \in [1/2, 1)$ and $\lim\limits_{x \to 1-} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 1-} x^{m-1} = 1$

Hence $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ is convergent, since $\int_{1/2}^{1} \dfrac{dx}{(1-x)^{1-n}}$ is convergent for $n > 0$.

Combining, we get $\int_{0}^{1} x^{m-1}(1-x)^{n-1}dx$ is convergent for $m > 0, n > 0$.

Proposition 1: $B(m,n) = B(n,m)$

Now $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Let $1-x=z$, then $-dx = dz$ and

x	0	1
z	1	0

$$\therefore \quad B(m,n) = \int_1^0 (1-z)^{m-1} z^{n-1}(-dz)$$

$$= \int_1^0 z^{n-1}(1-z)^{m-1}dz = B(n,m)$$

Proposition 2: $B(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \dfrac{x^{n-1}}{(1+x)^{m+n}}dx$

Now $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Let $x = \dfrac{1}{1+y}$, then $dx = -\left(\dfrac{1}{1+y}\right)^2 dy$ and

x	0	1
y	∞	0

$$\therefore \quad B(m,n) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$\left[1-x = 1-\frac{1}{1+y} = \frac{y}{1+y}\right]$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

Since $B(m, n) = B(n, m)$, then

$$B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Proposition 3: $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

From the definition of Beta function, we get

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Let $x = \sin^2 \theta$, then $dx = 2 \sin \theta \cos \theta\, d\theta$ and

x	0	1
θ	0	$\pi/2$

$$\therefore \quad B(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta\, (1-\sin^2\theta)^{n-1}\, 2\sin\theta \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Note: $B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = [2\theta]_0^{\pi/2} = \pi$

Proposition 4: The relation between Beta and Gamma functions is

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

and $$\Gamma\left(\frac{1}{2}\right) = \sqrt{2}$$

Now $$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

or $$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \pi$$

or $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi\,\Gamma(1) = \pi \qquad [\because \Gamma(1) = 1]$$

or $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proposition 5: $B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p, q > -1$$

From the proposition (3), we get

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Let $2m - 1 = p$ and $2n - 1 = q$, then $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

$$\therefore \quad B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

Note: $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

Proposition 6: For $m > 0$, $2^{2m-1}\Gamma(m)\Gamma\left(m + \frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m)$

From the definition of Beta function, we get

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad ...(1)$$

For $n = m$, $B(n, m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (2\sin\theta \cos\theta)^{2m-1} \, d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} \phi \frac{d\phi}{2} \text{ where } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}} 2 \int_0^{\pi/2} \sin^{2m-1} \phi \, d\phi \quad [\text{As } f(a-x) = f(x)] \quad ...(2)$$

Again putting $n = \frac{1}{2}$ in equation (1), we get

$$B\left(m, \frac{1}{2}\right) = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

$$= \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} 2^{2m-1} \quad [\text{using equation (2)}]$$

or
$$\frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} = \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} 2^{2m-1}$$

or
$$\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) 2^{2m-1} = \Gamma\left(\frac{1}{2}\right) \Gamma(2m)$$

or
$$2^{2m-1}\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \sqrt{\pi}\;\Gamma(2m) \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 1: Prove that $\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt = n^x B(x, n+1)$

Solution: Let $\frac{t}{n} = z$, then $dt = n dz$ and

t	0	n
z	0	1

$$\therefore \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt = \int_0^1 (1-z)^n z^{x-1} n^{x-1}\, n dz$$

$$= \int_0^1 z^{x-1}(1-z)^n n^x\, dz$$

$$= n^x \int_0^1 z^{x-1}(1-z)^{(n+1)-1}$$

$$= n^x B(x, n+1)$$

Example 2: Prove that $B(m+1, n) = \frac{m}{m+n} B(m, n)$

Solution: We know that $B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \quad B(m+1, n) = \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+1+n)} = \frac{m\,\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} = \frac{m}{m+n}\,\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{m}{m+n} B(m, n)$$

Example 3: Prove that $\int_0^\infty e^{-9x} x^{3/2} dx = \frac{9}{4}\sqrt{\pi}$

Solution: Let $9x = z$, then $dx = \frac{dz}{9}$

$$\therefore \quad \int_0^\infty e^{-9x} x^{3/2}\, dx = \int_0^\infty e^{-z}\left(\frac{z}{9}\right)^{3/2} \frac{dz}{9} = \frac{27}{9}\int_0^\infty e^{-z} z^{3/2}\, dz$$

$$= 3\int_0^\infty e^{-z} z^{5/2-1}\, dz$$

$$= 3\,\Gamma\left(\frac{5}{2}\right) = 3\,\Gamma\left(\frac{3}{2}+1\right) = 3.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= 3.\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$= 3.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{9}{4}\sqrt{\pi}$$

Example 4: Evaluate $\int_0^{\pi/2} \sin^5 x \cos^5 x\, dx$

Solution: We know that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

$$\therefore \int_0^{\pi/2} \sin^5 x \cos^5 x \, dx = \frac{1}{2}\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{5+5+2}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma(3)\,\Gamma(3)}{\Gamma(6)} = \frac{\Gamma(2+1)\,\Gamma(2+1)}{2\Gamma(5+1)}$$

$$= \frac{\lfloor 2 \lfloor 2}{2\lfloor 5} = \frac{\not{2}}{5\times4\times3\times\not{2}\times1} = \frac{1}{60}$$

Example 5: Evaluate $\int_0^1 x^2(1-x^2)^{7/2}\,dx$

Solution: Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$ and

x	0	1
θ	0	$\pi/2$

$$\therefore \int_0^1 x^2(1-x^2)^{7/2}\,dx = \int_0^{\pi/2} \sin^2\theta\,(\cos^2\theta)^{7/2}\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\,\cos^8\theta\,d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{\Gamma\left(\frac{2+8+2}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma\left(\frac{7}{2}+1\right)}{\Gamma(6)} = \frac{1}{2}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}{\lfloor 5}$$

$$= \frac{7}{8\lfloor 5}\sqrt{\pi}\,\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$= \frac{7\times5\times3}{64\times5\times4\times3\times2}\pi = \frac{7\pi}{512}$$

Example 6: Evaluate $\int_0^{\pi/2} \sin^9 x\,dx$

Solution: We know that $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$

For $n = 0$, $\int_0^{\pi/2} \sin^m x\, dx = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$

Here $m = 9$, then

$$\int_0^{\pi/2} \sin^9 x\, dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{9+1}{2}\right)}{\Gamma\left(\frac{9+2}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(5)}{\Gamma\left(\frac{11}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\lfloor 4}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{128}{315}$$

Example 7: Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Solution: Since e^{-x^2} is an even function, then $\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx$

Let $x^2 = z$, then $2x\, dx = dz$

$$\therefore \quad 2\int_0^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-z} z^{-\frac{1}{2}} \frac{dz}{2} = \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Example 8: Prove that $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$

Solution: Now $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\pi \operatorname{cosec}\frac{\pi}{4} \qquad (\because \Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi)$$

$$= \frac{\pi}{2}\sqrt{2} = \frac{\pi}{\sqrt{2}}$$

Example 9: Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

Solution: Let $x^2 = \sin\theta$, then $2x\,dx = \cos\theta\, d\theta$ and

x	0	1
θ	0	$\pi/2$

$$\therefore \quad \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\,\frac{\cos\theta\, d\theta}{2\sqrt{\sin\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\,\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

Example 10: Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m} B(m, n)$

Solution: Let $bx = a\tan^2\theta$, then $dx = \frac{a}{b}2\tan\theta\sec^2\theta\, d\theta$ and

x	0	∞
θ	0	$\pi/2$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \int_0^{\pi/2} \frac{\left(\frac{a}{b}\right)^{m-1}\tan^{2m-2}\theta\,\left(\frac{a}{b}\right)2\tan\theta\sec^2\theta\, d\theta}{a^{m+n}\sec^{2(m+n)}\theta}\,dx$$

$$= \frac{1}{a^n b^m}\,2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= \frac{1}{a^n b^m} B(m, n)$$

Example 11: Prove that $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Solution: Let $A=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$

$$=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(1-\frac{3}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \quad \ldots(1)$$

We write (1) in reverse order, we get

$$A=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{3}{n}\right)\ldots\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \quad \ldots(2)$$

We know that $\Gamma(m)\,\Gamma\left(1-\frac{1}{m}\right)=\frac{\pi}{\sin m\pi}$, then by multiplication of equations (1) and (2), we get

$$A^2=\left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right]$$

$$\ldots\left[\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(\frac{2}{n}\right)\right]\left[\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right]$$

$$=\frac{\pi}{\sin\frac{\pi}{n}}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}$$

$$\ldots\left[\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(1-\frac{(n-2)}{n}\right)\right]\left[\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{(n-1)}{n}\right)\right]$$

$$=\frac{\pi}{\sin\frac{\pi}{n}}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}\cdots\frac{\pi}{\sin\frac{(n-2)\pi}{n}}\cdot\frac{\pi}{\sin\frac{(n-1)\pi}{n}}$$

$$=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-2)\pi}{n}\sin\frac{(n-1)\pi}{n}} \quad \ldots(3)$$

We know that $\frac{\sin n\theta}{\sin\theta}=\left\{2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{(n-1)\pi}{n}\right)\right\}$

$$\therefore \lim_{\theta\to0}\frac{\sin n\theta}{\sin\theta}=\lim_{\theta\to0}\left\{2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{(n-1)\pi}{n}\right)\right\}$$

or $$\lim_{\theta\to 0} \frac{\sin n\theta}{n\theta} \cdot \frac{n}{\frac{\sin\theta}{\theta}} = 2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{n}$$

or $$n = 2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{n}$$

$\therefore$ $$\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}} \qquad \ldots(4)$$

From equations (3) and (4), we get

$$A^2 = \frac{\pi^{n-1}}{\frac{n}{2^{n-1}}} = \frac{(2\pi)^{n-1}}{n}$$

$\therefore$ $$A = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

Hence $$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

EXERCISES

1. Define the improper integrals $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\,dx$.

Evaluate

(*i*) $\int_0^{\infty} \frac{dx}{(x+1)(x+2)}$ (*ii*) $\int_{-\infty}^{1} e^x\,dx$.

2. Define the improper integral $\int_a^b f(x)\,dx$ when $f(x)$ is unbounded only at c where $a < c < b$. What is meant by the Cauchy's principal value of this integral?

Prove that $\int_{-1}^{1} \frac{1}{x}\,dx$ does not converge but has the Cauchy's principal value 0.

3. Evaluate

(*i*) $\int_1^{\infty} \frac{dx}{x^{3/2}}$ (*ii*) $\int_{-\infty}^{0} e^{2x}\,dx$.

4. Show that

(i) $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ (ii) $\int_{-\infty}^\infty xe^{-x^2}\, dx = 0$.

5. Prove that

(i) $\int_0^\infty x^3 e^{-x^2}\, dx = \frac{1}{2}$ (ii) $\int_0^\infty 5^{-x^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{\ln 5}}$

(iii) $\int_0^\infty e^{-5x^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{5}}$

6. Show that

(i) $\int_0^\infty e^{-x^4}\, dx \int_0^\infty e^{-x^4} x^2\, dx = \frac{\pi}{8\sqrt{2}}$

(ii) $\int_0^\infty e^{-x^2} \sqrt{x}\, dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$

7. Prove that

(i) $\int_2^\infty \frac{dx}{x^2-1} = \frac{1}{2}\log 3$ (ii) $\int_0^\infty \frac{dx}{(1+x^2)^4} = \frac{5}{32}\pi$

(iii) $\int_1^\infty \frac{x\, dx}{(1+x^2)^2}$ (iv) $\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$

(v) $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2} = \pi$

(vi) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab\,(a+b)}\, (a, b > 0)$

(vii) $\int_0^1 \log x\, dx = -1$

8. Prove that

(i) $\int_0^\infty e^{-2x} x^5 dx = \frac{3}{8}$ (ii) $\int_0^\infty e^{-x} x^{3/2} dx = \frac{3}{4}\sqrt{\pi}$

(iii) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{2}}{4}$ (iv) $\int_0^\infty e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{2a}$

(v) $\int_0^\infty \frac{e^{-pt}}{\sqrt{t}}\,dt=\sqrt{\frac{2\pi}{p}}$

9. Prove that

(i) $B=(m, 1)=\frac{1}{m}$ (ii) $B(m, m)=2^{1-2m}\,B\left(m, \frac{1}{2}\right)$

(iii) $B\left(\frac{1}{2}, \frac{1}{2}\right)=\pi=\int_0^\infty \frac{dt}{\sqrt{t\,(1+t)}}$

(iv) $B(m, 1-m)=\Gamma(m)\,\Gamma(1-m)=\int_0^\infty \frac{x^{n-1}}{1+x}\,dx$

(v) $\int_0^{\pi/2} \cos^n x\,dx=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$

(vi) $\int_0^{\pi/2} \sin^4 x\cos^4 x\,dx=\frac{3\pi}{256}$

(vii) $\int_0^{\pi/2} \cos^5 x\sin^4 x\,dx=\frac{8}{315}$ (viii) $\int_0^1 x^2(1-x^2)^{5/2}\,dx=\frac{2}{63}$

(ix) $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx=\frac{3\pi}{128}$

(x) $\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx=(b-a)^{m+n-1}B(m,n)$

(xi) $\int_0^1 x^{m-1}\,(1-x^2)^{n-1}\,dx=\frac{1}{2}\,B\left(\frac{1}{2}m, n\right), m, n>0$

(xii) $\int_0^1 \frac{dx}{(1-x^6)^{1/6}}=\frac{\pi}{3}$ (xiii) $\int_0^1 \frac{dx}{1+x^4}=\frac{\pi\sqrt{2}}{4}$

10. Prove that $\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{1}{2}\frac{1}{a^m b^n}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m, n > 0$

11. Prove that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} B(m+1, n+1)$

12. Prove that $\int_0^\infty e^{-x^2} x^{2n} dx = \frac{1.3.5 \ldots (2n-1)}{2^{n+1}}\sqrt{\pi}$

13. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{1}{a^n (1+a)^m} B(m, n)$

14. Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \ldots \Gamma\left(\frac{8}{9}\right) = \frac{16\pi^4}{3}$

Answers

1. (*i*) log 2 (*ii*) e
3. (*i*) 2 (*ii*) 1/2