CHAPTER

2

Evaluation of Definite Integral

One way to look at integration is to consider it as a process opposite to differentiation. To see how integration can be performed we, therefore, need to have the following notions.

2.1 DEFINITION

If corresponding to a function $f(x)$, there exists a function such that $\phi'(x) = f(x)$, then $\phi(x)$ is called the primitive of $f(x)$ or the indefinite integral of $f(x)$, a fact expressed symbolically by

$$\int f(x)\,dx = \phi(x).$$

Since the derivative of a constant is zero, it is appropriate to add a constant to the function always and write

$$\int f(x)\,dx = \phi(x) + c.$$

where c is called the constant of integration.

The following table of integration will be of immense help in integration theory.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$ provided $n \neq -1$.

2. $\int x^{-1} dx = \ln x + c$

3. $\int e^x dx = e^x + c$

4. $\int a^x dx = \dfrac{a^x}{\ln a} + c,\ a > 0$

5. $\int \sin mx\, dx = -\dfrac{\cos mx}{m} + c$

6. $\int \cos mx\, dx = \dfrac{\sin mx}{m} + c$

7. $\int \sec^2 mx\, dx = \frac{\tan mx}{m} + c$

8. $\int \text{cosec}^2 mx\, dx = -\frac{\cot mx}{m} + c$

9. $\int \sec mx \tan mx\, dx = \frac{\sec mx}{m} + c$

10. $\int \text{cosec}\, mx \cot mx\, dx = \frac{\text{cosec}\, mx}{m} + c$

11. $\int \sinh mx\, dx = \frac{\cosh mx}{m} + c$

12. $\int \cosh mx\, dx = \frac{\sinh mx}{m} + c$

13. $\int \text{sech}^2 mx\, dx = +\frac{\tanh mx}{m} + c$

14. $\int \text{cosech}^2\, mx\, dx = -\frac{\coth mx}{m} + c$

15. $\int \text{sech}\, mx \tanh mx\, dx = -\frac{\text{sech}\, mx}{m} + c$

16. $\int \text{cosech}\, mx \coth mx\, dx = -\frac{\text{cosech}\, mx}{m} + c$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

18. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x-a}{x+a} + c$

19. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + c$

20. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + c$

21. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + c$

22. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c, |x| < a$

23. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$

24. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$

25. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

The integration by parts can be committed to memory by any one of the following two ways:

1. $\int u\,dv = uv - \int v\,du$ where u and v are functions of x.
2. $\int f(x)\, g(x)\, dx = f(x) \int g(x) dx - \int f'(x) \left(\int g(x) dx \right) dx$

The **fundamental theorem of integral calculus** which plays a key role everywhere is stated below.

If $\phi(x)$ is the primitive of a function $f(x)$ integrable over $[a, b]$, then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a).$$

The above theorem enables us to note the following properties of definite integrals:

D1. $\int_a^b f(x)\, dx = \int_a^b f(z)\, dz$

D2. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

D3. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,\ a < c < b$

D4. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

D5. $\int_0^a f(x)\, dx = \int_0^{a/2} f(x)\, dx + \int_0^{a/2} f(a - x)\, dx$

D6. $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(a + x) = f(x)$.

Corollary $\int_0^{na} f(x)\, dx = n\int_0^a f(x)\, dx$ if $f(a + x) = f(x)$

D7. $\int_{-a}^a f(x)\, dx = \int_0^a \{f(x) + f(-x)\}\, dx$

$$= \begin{cases} 2\int_0^a f(x)\, dx & \text{if } f(x) \text{ is even,} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

We now show a few examples to illustrate the above properties.

Example 1: Evaluate

(*i*) $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$ (*ii*) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$.

Solution: (*i*) Putting $x = \tan\theta$, we get

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} (1+\tan\theta)\,d\theta = \int_0^{\pi/4} \ln\left[1+\tan\left(\frac{\pi}{4}-\theta\right)d\theta\right]$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\}d\theta = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \{\ln 2 - \ln(1+\tan\theta\}\,d\theta = \int_0^{\pi/4} \ln 2\,d\theta - \mathrm{I}$$

$$2I = \ln 2\int_0^{\pi/4} d\theta = \frac{\pi}{4}\ln 2.$$

So $$I = \frac{\pi}{8}\ln 2$$

(*ii*) Here

$$I = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx - I$$

$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx.$$

So $$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Putting $z = \cos x$, we get

$$\mathrm{I} = \frac{\pi}{2}\int_1^{-1} \frac{-dz}{1+z^2} = \frac{\pi}{2}\left[\tan^{-1} z\right]_1^{-1} = \frac{\pi^2}{4}.$$

2.2 MISCELLANEOUS EXAMPLES

Example 1. Evaluate $\int_0^a \sin nx\,dx$ using limit of a sum.

Solution: Now $\int_0^a \sin nx\, dx = \lim_{h\to 0} h \sum_{r=0}^{p-1} \sin(rnh)$ where $ph = a - 0 = a$

$$= \lim_{h\to 0} h[\sin 0 + \sin nh + \sin 2nh + \ldots + \sin (p-1)\, nh]$$

$$= \lim_{h\to 0} h\left[\frac{\cos\frac{hn}{2} - \cos\frac{3}{2}hn}{2\sin\frac{hn}{2}} + \frac{\cos\frac{3}{2}hn - \cos\frac{5}{2}hn}{2\sin\frac{hn}{2}} + \ldots\ldots \right.$$

$$\left. + \frac{\cos\left[(p-1) - \frac{1}{2}\right]nh - \cos\left[(p-1) + \frac{1}{2}\right]nh}{2\sin\frac{nh}{2}}\right]$$

$$\left[\because\ 2\sin nh\ \sin\frac{nh}{2} = \cos\frac{hn}{2} - \cos\frac{3}{2}hn \text{ and } 2\sin (p-1)\, nh\ \sin\frac{hn}{2}\right.$$

$$\left. = \cos\left[(p-1) - \frac{1}{2}\right]nh - \cos\left[(p-1) + \frac{1}{2}\right]nh\right]$$

$$= \lim_{h\to 0} h\left[\frac{\cos\frac{nh}{2} - \cos nh\left[(p-1) + \frac{1}{2}\right]}{2\sin\frac{nh}{2}}\right]$$

$$= \lim_{h\to 0} h\left[\frac{2.\sin\left(\frac{nh(p-1)}{2}\right)\cdot \sin\frac{nph}{2}}{2\sin\frac{nh}{2}}\right]$$

$$= \lim_{h\to 0}\left[\frac{\frac{nh}{2}}{\sin\frac{nh}{2}} \times \frac{2}{n}\sin\frac{nph}{2}\sin (p-1)\frac{nh}{2}\right]$$

$$= \frac{2}{n}\lim_{h\to 0}\left[\sin\frac{an}{2}\sin\left(\frac{an - nh}{2}\right)\right] \quad \because\ ph = a$$

$$= \frac{2}{n}\sin^2\frac{an}{2} = \frac{1}{n}(1 - \cos na).$$

Example 2: Prove that $\int_{-a}^{a} x\,|x|\,dx = 0.$

Solution: Here $\int_{-a}^{a} x\,|x|\,dx = \int_{-a}^{0} x\,|x|\,dx + \int_{0}^{a} x\,|x|\,dx$

$$= \int_{-a}^{0} x(-x)\,dx + \int_{0}^{a} x\,.\,x\,dx$$

$$= \int_{-a}^{0} (-x^2)\,dx + \int_{0}^{a} x^2\,dx$$

$$= \left[-\frac{x^3}{3}\right]_{-a}^{0} + \left[\frac{x^3}{3}\right]_{0}^{a}$$

$$= 0 - \frac{a^3}{3} + \frac{a^3}{3} - 0 = 0.$$

Example 3: Evaluate $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}}.$

Solution: Let $A = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}}$

Taking log on both sides, we get

$$\log A = \log\left[\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}}\right]$$

$$= \lim_{n\to\infty}\left[\log\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots+\left(1+\frac{n^2}{n^2}\right)^n\right\}^{\frac{2}{n^2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n^2}\left[\log\left(1+\frac{1}{n^2}\right) + 2\log\left(1+\frac{2^2}{n^2}\right)\right.\right.$$

$$\left.\left. + 3\log\left(1+\frac{3^2}{n^2}\right) + \dots + n\log\left(1+\frac{n^2}{n^2}\right)\right]\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{n}\left\{\frac{1}{n}\log\left(1+\frac{1}{n^2}\right)+\frac{2}{n}\log\left(1+\frac{2^2}{n^2}\right)\right.\right.$$

$$\left.\left.+\frac{3}{n}\log\left(1+\frac{3^2}{n^2}\right)+\ldots+\frac{n}{n}\log\left(1+\frac{n^2}{n^2}\right)\right\}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2}{n}\cdot\frac{r}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}\log\left(1+\frac{r^2}{n^2}\right) = 2\int_0^1 x\ \log(1+x^2)\,dx$$

Let $1+x^2 = z,\ 2x\,dx = dz$

$$\therefore \qquad I = \int_1^2 \log z\,dx = [z\log z - z]_1^2 = 2\log 2 - 2 - (\log 1 - 1)$$

$$= 2\log 2 - 1 = \log 4 - \log e = \log\frac{4}{e}$$

$$\therefore \quad \log A = \log\frac{4}{e} \quad \therefore\ A = \frac{4}{e}.$$

Example 4: Evaluate $\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$.

Solution: Let $A = \lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

Taking log on both sides, we get

$$\log A = \log\left[\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}\right]$$

$$= \lim_{n\to\infty}\log\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} \log\left(1 + r^2 h^2\right) \text{ where } nh = 1$$

$$= \int_0^1 \log(1 + x^2)\, dx$$

$$= [x \log(1 + x^2)]_0^1 - \int_0^1 \frac{2x}{1 + x^2} \cdot x\, dx \quad \text{[Integrating by parts]}$$

$$= \log 2 - 2 \int_0^1 \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \log 2 - 2\,[x - \tan^{-1} x]_0^1 = \log 2 - 2\,[1 - \tan^{-1} 1]$$

$$= \log 2 - 2 + 2 \cdot \frac{\pi}{4} = \log 2 - 2 + \frac{\pi}{2}$$

$$= \log 2 - \log e^2 + \log e^{\pi/2}$$

$$= \log(2 \cdot e^{\pi/2 - 2})$$

$\therefore \quad \log A = \log(2 \cdot e^{\pi/2 - 2})$

$\therefore \quad A = 2\, e^{\pi/2 - 2} = 2\, e^{\pi - 4/2}.$

Example 5: Evaluate $\lim_{n \to \infty} \left[\frac{\lfloor n}{n^n}\right]^{1/n}$

Solution: Let $A = \lim_{n \to \infty} \left[\frac{\lfloor n}{n^n}\right]^{1/n} = \lim_{n \to \infty} \left[\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n}{n}\right]^{1/n}$

Taking log on both sides, we get

$$\log A = \log\left[\lim_{n \to \infty} \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}\right)^{1/n}\right]$$

$$= \lim_{n \to \infty} \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \log\frac{3}{n} + \ldots + \log\frac{n}{n}\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right) = \lim_{h \to 0} h \sum_{r=1}^{n} \log rh. \text{ where } nh = 1$$

$$= \int_0^1 \log x\, dx = [x \log x - x]_0^1 = [1 \log 1 - 1 - 0] = -1$$

$\therefore \quad \log A = -1 = \log e^{-1} \Rightarrow A = e^{-1} = \frac{1}{e}$

Example 6: Evaluate

(*i*) $\int_0^{\pi} |\sin x + \cos x|\, dx$ (*ii*) $\int_{-1}^{3/2} |x \sin \pi x|\, dx.$

Solution: Let $I = \int_0^{\pi} |\sin x + \cos x|\, dx$,

Now $\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$

$$= \sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

$\therefore$ Let $x + \frac{\pi}{4} = z$, then $dx = dz$,

x	0	π
z	$\pi/4$	$5\pi/4$

$$\therefore \quad I = \int_{\pi/4}^{5\pi/4} \sqrt{2}\, |\sin z|\, dz$$

$$= \int_{\pi/4}^{\pi} \sqrt{2}\, |\sin z|\, dz + \int_{\pi}^{5\pi/4} \sqrt{2}\, |\sin z|\, dz$$

$$= \int_{\pi/4}^{\pi} \sqrt{2}\sin z\, dz - \int_{\pi}^{5\pi/4} \sqrt{2}\sin z\, dz$$

$$\because |\sin z| = \begin{cases} \sin z \text{ for } \frac{\pi}{4} \le z < \pi \\ -\sin z \text{ for } \pi < z \le \frac{5\pi}{4} \end{cases}$$

$$= \sqrt{2}\,[-\cos z]_{\pi/4}^{\pi} - \sqrt{2}\,[-\cos z]_{\pi}^{5\pi/4}$$

$$= \sqrt{2}\left[1 + \frac{1}{\sqrt{2}}\right] + \sqrt{2}\left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \sqrt{2} + 1 - 1 + \sqrt{2} = 2\sqrt{2}$$

(ii) $$I = \int_{-1}^{3/2} |x \sin \pi x|\, dx$$

$$= \int_{-1}^{1} |x \sin \pi x|\, dx + \int_{1}^{3/2} x\,|\sin \pi x|\, dx$$

$$= \int_{-1}^{1} x \sin \pi x\, dx + \int_{1}^{3/2} (-x \sin \pi x)\, dx$$

$$\because |\sin \pi x| = \begin{cases} \sin \pi x \text{ for } -1 < x \le 1 \\ -\sin \pi x \text{ for } 1 < x \le \frac{3}{2} \end{cases}$$

$$= 2\int_{0}^{1} x \sin \pi x\, dx - \int_{1}^{3/2} x \sin \pi x\, dx$$

Now $\int x \sin \pi x\, dx = -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi}\int \cos \pi x\, dx$

$$= -\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x + c \quad (c = \text{arbitrary constant})$$

$$\therefore \quad I = 2\left[-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right]_0^1$$

$$-\left[-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right]_1^{3/2}$$

$$= 2\left[-\frac{1}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi - 0\right]$$

$$-\left[-\frac{3}{2\pi}\cos\frac{3\pi}{2} + \frac{1}{\pi^2}\sin\frac{3\pi}{2} + \frac{1}{\pi}\cos\pi - \frac{1}{\pi^2}\sin\pi\right]$$

$$= 2\left[-\frac{1}{\pi}(-1) + 0\right] - \left[-\frac{3}{2\pi}.0 + \frac{1}{\pi^2}(-1) + \frac{1}{\pi}(-1) - 0\right]$$

$$= \frac{2}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}. = \frac{3}{\pi} + \frac{1}{\pi^2}$$

EXERCISES

1. Evaluate

(*i*) $\int\sqrt{\frac{x}{a-x}}\,dx$ (*ii*) $\int\frac{dx}{x+\sqrt{x^2-1}}$

(*iii*) $\int\frac{dx}{(1+x)\sqrt{1+2x-x^2}}$.

2. Evaluate

(*i*) $\int(x-1)\sqrt{x^2-x+1}\,dx$ (*ii*) $\int\sqrt{(x-\alpha)(\beta-x)}\,dx$

(*iii*) $\int\frac{dx}{(1-x)^{3/2}(1+x)^{1/2}}$.

3. Show that

(*i*) $\int_0^3\frac{dx}{(x+2)\sqrt{x+1}} = 2\tan^{-1}2 - \frac{\pi}{2}$

(*ii*) $\int_0^{2a}\sqrt{2ax-x^2}\,dx = \frac{\pi a^2}{2}$ (*iii*) $\int_0^{\pi^2/4}\sin\sqrt{x}\,dx = 2.$

4. Evaluate

(i) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{3/2}}dx$ (ii) $\int \frac{e^{-x}dx}{e^x+2e^{-x}+3}$

(iii) $\int \frac{dx}{1+x^2+x^4}$

5. Show that

(i) $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{1}{ab}\tan^{-1}\frac{b}{a}, a, b > 0$

(ii) $\int_0^{\pi/2} \frac{dx}{\sin x+\cos x} = \sqrt{2}\,\ln(\sqrt{2}+1)$

(iii) $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}$ if $a > b$

(iv) $\int_0^{\pi/2} \frac{\sin x\cos x}{a^2\sin^2 x+b^2\cos^2 x}dx = \begin{cases} \frac{1}{a^2-b^2}\ln\frac{a}{b} & \text{if } a^2 \neq b^2 \\ \frac{1}{2a^2} & \text{if } a = \pm b \end{cases}$

(v) $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = 2\sqrt{\tan x}+c.$

Answers

1. (i) $-\sqrt{ax-x^2}+\frac{a}{2}\sin^{-1}\frac{2x-a}{a}+c$

(ii) $\frac{1}{2}x(x-\sqrt{x^2-1})-\ln(x+\sqrt{x^2-1})+c$

(iii) $-\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x}{x+1}\right)+c$

2. (i) $\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{8}(2x-1)\sqrt{x^2-x+1}$

$-\frac{3}{16}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+1}\right)+c$

(ii) $\frac{1}{4}\left\{(2x-\alpha-\beta)\sqrt{(x-\alpha)(x-\beta)}+\frac{(\beta-\alpha)^2}{2}\sin^{-1}\frac{2x-\alpha-\beta}{\beta-\alpha}+c\right.$

(*iii*) $\sqrt{\dfrac{1-x}{1+x}}$.

4. (*i*) $e^{m\tan^{-1}x} \cdot m - \dfrac{\cos(\tan^{-1}x)+\sin(\tan^{-1}x)}{1+m^2} + c$

(*ii*) $\dfrac{1}{2}\ln(2e^{-x}+1) - \ln(e^{-x}+1) + c$

(*iii*) $\dfrac{1}{4}\ln\dfrac{1+x+x^2}{1-x+x^2} + \dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{\sqrt{3}x}{1-x^2}\right) + c$