CHAPTER

1

Curvature, Evolute and Involute

In this case we give a definite numerical measure of bending of a curve at any point on it. We define below the concept of curvature and its associate radius of curvature, evolute and involutes.

Definition: The **curvature** of a curve C at a point P on a curve is defined as $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$ where the tangent at P on C make an angle ψ with the positive x-axis and the tangent at Q at on arc distance (PQ) Δs on the curve makes an angle $\psi + \Delta\psi$ with the positive x-axis.

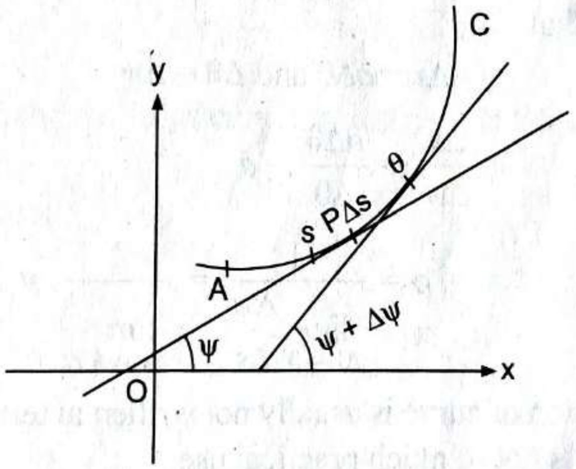

Fig. 1

This is usually denoted by κ (kappa).

Thus,
$$\text{curvature} = \kappa = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}.$$

The quantity $\Delta\psi$ is called the **total curvature** of the arc length PQ and the ratio $\frac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ.

The inverse of curvature is called the **radius of curvature** which is denoted by ρ(rho) and defined by

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

Theorem: The radius of curvature at every point of a circle of radius a is a.

Proof: Let P be any arbitrary point on the circle and Q be a point at an arc-distance Δs from P and $\Delta\theta$ be the angle made by the normals at P and Q at the centre of the circle with radius a.

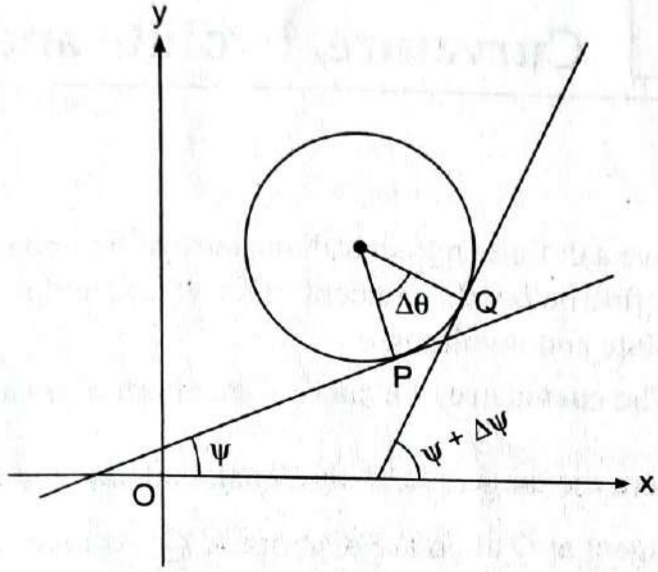

Fig. 2

Then we know that

$$\Delta s = a\Delta\theta \text{ and } \Delta\theta = \Delta\psi$$

$$\therefore \qquad \frac{\Delta s}{\Delta\psi} = \frac{a\Delta\theta}{\Delta\theta} = a$$

Hence
$$\rho = \frac{1}{\lim\limits_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}} = \frac{1}{\lim\limits_{\Delta s \to 0} \frac{1}{a}} = a$$

Note: As the equation of curve is usually not written in terms of s and ψ, then the above formula is not of much practical use.

Formula 1: For Cartesian equation (explicit function) $y = f(x)$ or $x = f(y)$,

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \text{ or } \frac{(1 + x_1^2)^{3/2}}{x_2}$$

where
$$y_1 = \frac{dy}{dx},\ x_1 = \frac{dx}{dy}$$

and
$$y_2 = \frac{d^2y}{dx^2},\ x_2 = \frac{d^2x}{dy^2}$$

For the function $y = f(x)$, $y_1 = \frac{dy}{dx} = \tan\psi$ where ψ is the angle made by the tangent at (x, y) of the curve with the positive x-axis

$$\therefore \quad y_2 = \frac{d^2y}{dx^2} = \sec^2\psi\frac{d\psi}{dx} = \sec^2\psi\frac{d\psi}{dx}\cdot\frac{ds}{dx}$$

$$\therefore \quad \frac{ds}{d\psi} = \frac{\sec^2\psi\dfrac{ds}{dx}}{y_2}$$

$$= \frac{(1+y_1^2)(1+y_1^2)^{1/2}}{y_2}$$

$$\left[\because\ ds^2 = dy^2 + dx^2 \quad \text{or} \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+y_1^2}\right]$$

$$= \frac{(1+y_1^2)^{3/2}}{y_2}$$

Similarly for the function $x = f(y)$,

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

Example 1: Find the radius of curvature at (x, y) on the curve

(i) $y = a\log\sec\left(\frac{x}{a}\right)$

(ii) $y = a\cosh\left(\frac{x}{a}\right)$

(iii) $y^2 = 4ax$

Solution: (i) Here $y = a\log\sec\left(\frac{x}{a}\right)$,

$$\therefore \quad y_1 = a\frac{1}{\sec\left(\frac{x}{a}\right)}\frac{\sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right)}{a}$$

or

$$y_1 = \tan\left(\frac{x}{a}\right)$$

$$\therefore \quad y_2 = \frac{1}{a}\sec^2\left(\frac{x}{a}\right)$$

Hence, the radius of curvature at (x, y) is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2\left(\frac{x}{a}\right)\right)^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)} = a\sec\left(\frac{x}{a}\right)$$

(ii) Here $y = a\cosh\left(\frac{x}{a}\right)$,

$\therefore$ $$y_1 = a\sinh\left(\frac{x}{a}\right)\frac{1}{a} = \sinh\left(\frac{x}{a}\right)$$

and $$y_2 = \left(\frac{1}{a}\right)\cosh\left(\frac{x}{a}\right)$$

Hence, the radius for curvature at (x, y) is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1+\sinh^2\left(\frac{x}{a}\right)\right]^{3/2}}{\left(\frac{1}{a}\right)\cosh\left(\frac{x}{a}\right)}$$

$$= \frac{a\cosh^3\left(\frac{x}{a}\right)}{\cosh\left(\frac{x}{a}\right)} = a\cosh^2\left(\frac{x}{a}\right) = \frac{y^2}{a}$$

(iii) Here $y^2 = 4ax$

or $$x = \frac{1}{4a}y^2,$$

then $$\frac{dx}{dy} = x_1 = \frac{y}{2a}$$

and $$x_2 = \frac{d^2x}{dy^2} = \frac{1}{2a}$$

Hence, the radius of curvature at (x, y) is given by

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2} = \frac{\left(1+\frac{y^2}{4a^2}\right)^{3/2}}{\frac{1}{2a}}$$

$$= 2a\,(4a^2 + y^2)^{3/2} \cdot \frac{1}{(2a)^3}$$

$$= \frac{1}{(2a)^2}(4a^2 + y^2)^{3/2}$$

$$= \frac{(4a^2 + 4ax)^{3/2}}{4a^2} = 2a\left(1 + \frac{x}{a}\right)^{3/2}$$

Formula 2: For the implicit Cartesian form $f(x, y) = 0$ possessing continuous 2nd order partial derivatives

$$\rho = -\frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

We known that

$$y_1 = \frac{dy}{dx} = -\frac{f_x}{f_y}$$

or $$f_x + f_y\frac{dy}{dx} = 0$$

Now differentiating with respect to x, we get

$$f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\frac{d^2y}{dx^2} = 0$$

or $$f_{xx} + 2f_{xy}\left(-\frac{f_x}{f_y}\right) + f_{yy}\left(-\frac{f_x}{f_y}\right)^2 + f_y\frac{d^2y}{dx^2} = 0$$

or $$f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2 + f_y^3\frac{d^2y}{dx^2} = 0$$

$\therefore$ $$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

Hence, $$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \dfrac{f_x^2}{f_y^2}\right)^{3/2}}{-\left(\dfrac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}\right)}$$

$$= -\frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

Example 2: Find the radius of curvature of the curves

(i) $\sqrt{x} + \sqrt{y} = 1$ at (1, 1)

(ii) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Solution: (i) Here $f(x, y) = \sqrt{x} + \sqrt{y} - 1,$

then
$$f_x = \frac{1}{2}x^{-1/2}, \quad f_y = \frac{1}{2}y^{-1/2}$$
$$f_{xx} = -\frac{1}{4}x^{-3/2}, \quad f_{yy} = -\frac{1}{4}y^{-3/2}, \quad f_{xy} = 0$$

$\therefore$ At (1, 1),
$$f_x = \frac{1}{2} = f_y, \quad f_{xx} = -\frac{1}{4} = f_{yy} \quad \text{and} \quad f_{xy} = 0$$

$\therefore$
$$\rho = -\frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$
$$= -\frac{\left(\frac{1}{4} + \frac{1}{4}\right)^{3/2}}{-\frac{1}{4}\left(\frac{1}{4}\right) - 2.0 + \left(-\frac{1}{4}\right)\frac{1}{4}}$$
$$= \frac{-\left(\frac{1}{2}\right)^{3/2}}{-\frac{1}{8}} = \frac{\left(\frac{1}{2}\right)^{3/2}}{\left(\frac{1}{2}\right)^3} = \left(\frac{1}{2}\right)^{3/2-3} = \left(\frac{1}{2}\right)^{-3/2}$$

(ii) Here $f(x, y) = x^3 + y^3 - 3axy,$

then
$$f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax$$
$$f_{xx} = 6x, f_{yy} = 6y$$

and
$$f_{xy} = -3a$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,
$$f_x = 3\cdot\frac{9a^2}{4} - 3a\cdot\frac{3a}{2} = \frac{9a^2}{2}\left(\frac{3}{2} - 1\right) = \frac{9a^2}{4}$$
$$f_y = 3\cdot\frac{9a^2}{4} - 3a\cdot\frac{3a}{2} = \frac{9a^2}{2}\left(\frac{3}{2} - 1\right) = \frac{9a^2}{4}$$
$$f_{xx} = 9a = f_{yy} \quad \text{and} \quad f_{xy} = -3a$$

Hence, the radius of curvature is given by
$$\rho = -\frac{(f_x^2 + f_y^2)^{3/2}}{(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2)}$$

$$= -\frac{\left[\frac{81}{16}a^4 + \frac{81}{16}a^4\right]^{3/2}}{\left[9a\cdot\frac{81}{16}a^4 - 2(-3a)\frac{9a^2}{4}\cdot\frac{9a^2}{4} + 9a\cdot\frac{81}{16}a^4\right]}$$

$$= -\frac{\left(\frac{81}{8}a^4\right)^{3/2}}{\frac{9\times 81a^5}{8} + \frac{81\times 3a^5}{8}}$$

$$= -\frac{\left(\frac{81}{8}a^4\right)^{3/2}}{\frac{12\times 81a^5}{8}} = -\frac{81\times 9a^6\times 8}{8\times 2^{3/2}\times 12\times 81\times a^5}$$

$$= -\frac{3a}{8\sqrt{2}}$$

Formula 3: For the parametric equation $x = \phi(t)$, $y = \psi(t)$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

where '' denotes the derivatives with respect to t.

We know that

$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{1}{\frac{dx}{dt}}$$

$$= \frac{x'y'' - y'x''}{[x']^2}\cdot\frac{1}{x'} = \frac{x'y'' - y'x''}{(x')^3}$$

Hence,

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{y'^2}{x'^2}\right)^{3/2}}{\frac{x'y'' - y'x''}{(x')^3}}$$

$$= \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

Example 3: Find the radius of curvature of the parametric equations

(i) $x = a\cos^3\theta,\ y = a\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$

(ii) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$ at θ

Solution: (i) Here $x = a\cos^3\theta,\ y = a\sin^3\theta$

$$\therefore \quad x' = -3a\cos^2\theta\sin\theta,$$

$$x'' = -3a\cos^3\theta + 6a\cos\theta\sin^2\theta$$

$$y' = 3a\sin^2\theta\cos\theta$$

$$y'' = -3a\sin^3\theta + 6a\sin\theta\cos^2\theta$$

$\therefore$ At $\theta = \dfrac{\pi}{4}$,

$$x' = -\frac{3a}{2^{3/2}} = -\frac{3a}{2\sqrt{2}}$$

$$x'' = -\frac{3a}{2\sqrt{2}} + \frac{6a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$$

$$y' = \frac{3a}{2\sqrt{2}}$$

and

$$y'' = \frac{3a}{2\sqrt{2}}$$

Hence, the radius of curvature at $\theta = \dfrac{\pi}{4}$ is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - y'x'')} = \frac{\left(\dfrac{9a^2}{8} + \dfrac{9a^2}{8}\right)^{3/2}}{-\dfrac{9a^2}{8} - \dfrac{9a^2}{8}}$$

$$= \frac{\left(\dfrac{3}{2}a\right)^3}{-\dfrac{9a^2}{4}} = -\frac{3}{2}a$$

(ii) Here $x = a(\theta + \sin\theta)$

and $$y = a(1 - \cos\theta),$$

then $$x' = a(a + \cos\theta),\ x'' = -a\sin\theta$$

and $$y' = -a\sin\theta,\ y'' = a\cos\theta$$

Hence, the radius of curvature at θ is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - y'x'')}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{[a(1+\cos\theta)(a\cos\theta) - (-a\sin\theta)](a\cos\theta)}$$

$$= \frac{[a^2 + 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]^{3/2}}{[a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]}$$

$$= \frac{[2a^2 + (1+\cos\theta)]^{3/2}}{a^2(1+\cos\theta)} = \frac{a^3 2^3 \cos^3\frac{\theta}{2}}{a^2 2\cos^2\frac{\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Formula 4: For the polar equation $r = f(\theta)$,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

where $r_1 = \frac{dr}{d\theta}, r_2 = \frac{d^2r}{d\theta^2}$

We have
$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\frac{d\theta}{d\psi} = \frac{\frac{ds}{d\theta}}{\frac{d\psi}{d\theta}}$$

Again
$$\psi = \theta + \phi = \theta + \tan^{-1}\left(\frac{r}{r_1}\right)$$

where $r_1 = \frac{dr}{d\theta}$ and $r\frac{d\theta}{dr} = \tan\phi$

$$\therefore \quad \frac{d\psi}{d\theta} = 1 + \frac{d}{d\theta}\left(\tan^{-1}\left(\frac{r}{r_1}\right)\right)$$

$$= 1 + \frac{1}{1+\frac{r^2}{r_1^2}}\left[\frac{r_1^2 - rr_2}{r_1^2}\right]$$

$$= 1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2} = \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}$$

Also
$$ds^2 = r^2 d\theta^2 + dr^2, \text{ then } \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

or
$$\frac{ds}{d\theta} = (r^2 + r_1^2)^{1/2}$$

Hence, $$\rho = \frac{\frac{ds}{d\theta}}{\frac{d\psi}{d\theta}} = \frac{(r^2 + r_1^2)^{1/2}}{\frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)}} = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

Note: For the polar equation of $u = f(\theta)$ where $u = \frac{1}{r}$,

$$\rho = \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)}$$

where $u_1 = \frac{du}{d\theta}$ and $u_2 = \frac{d^2u}{d\theta^2}$

Example 4: Find the radius of curvature of the polar equation

(i) $r = a(1 - \cos\theta)$ at θ

(ii) $r = \frac{l}{1 + e\cos\theta}$ at $\theta = \pi$

Solution: (i) Here $r = a(1 - \cos\theta)$,

so $r_1 = a\sin\theta$ and $r_2 = a\cos\theta$

Hence
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1 - \cos\theta)^2 + 2a^2\sin^2\theta - a(1 - \cos\theta)a\cos\theta}$$

$$= \frac{[a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]^{3/2}}{a^2[1 - 2\cos\theta + \cos^2\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$$

$$= \frac{a^3(2 - 2\cos\theta)^{3/2}}{3a^2(1 - \cos\theta)}$$

$$= \frac{a}{3}2\sqrt{2}\frac{(1 - \cos\theta)^{3/2}}{(1 - \cos\theta)} = \frac{2\sqrt{2}a}{3}(1 - \cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}a}{3}\sqrt{2}\sin\frac{\theta}{2} = \frac{4a}{3}\sin\left(\frac{\theta}{2}\right)$$

(ii) Here $r = \frac{l}{1 + e\cos\theta}$,

$\therefore$ $\frac{1}{r} = \frac{1}{l}(1 + e\cos\theta)$

or $$u = \frac{1}{l}(1 + e\cos\theta)$$

where $$u = \frac{1}{r}$$

$\therefore$ $$u_1 = \frac{du}{d\theta} = -\frac{e}{l}\sin\theta$$

and $$u_2 = \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

Hence, at $\theta = \pi$, $$u = \frac{1}{l}(1 + e\cos\pi) = \frac{1-e}{l}$$

$$u_1 = -\frac{e}{l}\sin\pi = 0$$

and $$u_2 = \frac{e}{l}$$

Hence, the radius of curvature at π is given by

$$\rho = \frac{(u^2 + u_1^2)}{u^3(u + u_2)} = \frac{\left[\left(\frac{1-e}{l}\right)^2 + 0\right]^{3/2}}{\left(\frac{1-e}{l}\right)^3\left[\frac{1-e}{l} + \frac{e}{l}\right]}$$

$$= \frac{\left(\frac{1-e}{l}\right)^3}{\left(\frac{1-e}{l}\right)^3\left(\frac{1}{l}\right)} = l$$

Formula 5: For the pedal equation $p = f(r)$,

$$\rho = r\frac{dr}{dp}$$

where p is the perpendicular distance from the origin to the curve.

We know $$p = r\sin\phi$$

$\therefore$ $$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{d\phi}{dr} \qquad \left[\because \sin\phi = r\frac{d\theta}{ds}, \cos\phi = \frac{dr}{ds}\right]$$

$$= r\left[\frac{d\theta}{ds} + \frac{d\phi}{ds}\right] = r\frac{d}{ds}(\theta + \phi) = r\frac{d\psi}{ds} = \frac{r}{\rho}$$

$$\therefore \qquad \rho = r\frac{dr}{dp}$$

Note: For the tangential polar form $p = f(\psi)$,

$$\rho = p + \frac{d^2p}{d\psi^2}$$

We have $$\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi} = \frac{dp}{dr}(\cos\phi)\rho$$

$$= \frac{dp}{dr}\cos\phi\, r\frac{dr}{dp} = r\cos\phi$$

$$\therefore \qquad p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2$$

Then differentiating with respect to to p, we get

$$2p + 2\frac{d}{d\psi}\left(\frac{dp}{d\psi}\right)^2\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

or $$p + \frac{d^2p}{d\psi^2}\frac{dp}{d\psi}\cdot\frac{d\psi}{dp} = r\frac{dr}{dp}$$

or $$p + \frac{d^2p}{d\psi^2} = \rho \qquad \left[\because \rho = r\frac{dr}{dp}\right]$$

Hence $$\rho = p + \frac{d^2p}{d\psi^2}$$

Example 5: Find the radius of curvature of the curves

(i) $\dfrac{a^2b^2}{p^2} = a^2 + b^2 - r^2$ at an arbitrary point

(ii) $p = a(1 + \sin\psi)$ at an arbitrary point,

Solution: (i) We have the equation as

$$\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$$

Differentiating with respect to p, then we get

$$\frac{-2a^2b^2}{p^3} = -2r\frac{dr}{dp} = -2\rho$$

$$\therefore \qquad \rho = \frac{a^2b^2}{p^3}$$

(ii) Here $\qquad p = a(1 + \sin\psi)$

$$\therefore \qquad \frac{dp}{d\psi} = a\cos\psi$$

and $\qquad \dfrac{d^2p}{d\psi^2} = -a\sin\psi$

Hence $\qquad \rho = p + \dfrac{d^2p}{d\psi^2} = a(1 + \sin\psi) - a\sin\psi = a$

The curvatures and radii of curvatures of curves passing through the origin can be found by the above formulae, but often it is convenient to derive them by special method known as Newton's method.

Newton Method: If a continuous curve passes through the origin, then its curvature ρ at the origin is given by

$$\rho = \begin{cases} \dfrac{1}{2}\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{2y} & \text{if the } x \text{ axis is tangent to the curve} \\ \dfrac{1}{2}\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{2x} & \text{if the } y \text{ axis is tangent to the curve} \\ \dfrac{1}{2}\sqrt{a^2+b^2}\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{ax+by} & \text{if the line } ax+by=0 \text{ is tangent to the curve} \end{cases}$$

Proof: Case 1: The x-axis is tangent to the curve at the origin O. Let $P(x, y)$ be a point on the curve near the origin and let OPB be the circle passing through P and tangent to the x-axis at O. First note that as P tends to O, the circle OPB tends to the circle of curvature and in the limiting position, the radius of the circle becomes the radius of curvature at P.

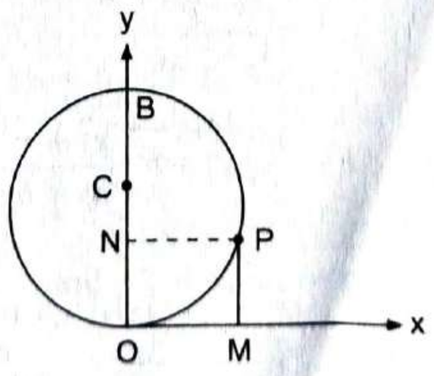

Fig. 3

Now, from the elementary geometry, we get

$$ON \cdot NB = NP^2$$

or
$$y(2r - y) = x^2$$

or
$$r = \frac{1}{2}\left(y + \frac{x^2}{y}\right)$$

Hence,
$$\rho = \lim_{P \to 0} r = \lim_{(x,y)\to(0,0)} \frac{1}{2}\left(y + \frac{x^2}{y}\right)$$

$$= \frac{1}{2} \lim_{(x,y)\to(0,0)} \frac{x^2}{y}$$

Case 2: The y-axis is tangent to the curve at the origin O. Similar to the case 1.

Case 3: The line $ax + by = 0$ is tangent to the curve at origin O. From the elementary geometry, we get

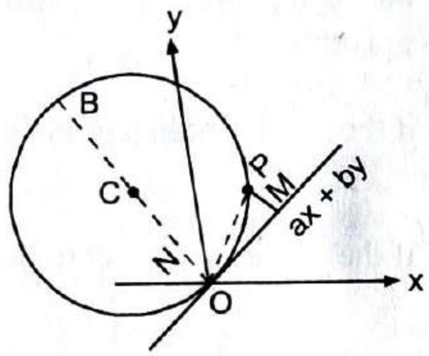

Fig. 4

$$ON.NB = NP^2$$

or
$$ON.(OB - ON) = NP^2$$

or
$$OB = \frac{NP^2 + ON^2}{ON} = \frac{OP^2}{PM}$$

or
$$2r = \frac{x^2 + y^2}{\frac{ax + by}{\sqrt{a^2 + b^2}}}$$

or
$$r = \frac{1}{2}\sqrt{a^2 + b^2}\,\frac{x^2 + y^2}{ax + by}$$

Hence
$$\rho = \lim_{P \to 0} r = \lim_{(x,y)\to(0,0)} \frac{1}{2}\sqrt{a^2 + b^2}\,\frac{x^2 + y^2}{ax + by}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2} \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{ax + by}$$

Note: The above derivations are also possible if the function describing the curve admits of Maclaurin's expression wherefrom y_1 and y_2 at (0, 0) can be easily found out.

Example 6: Find the radius of curvature at the origin of the curves

(i) $y - x = x^2 + 2xy + y^2$

(ii) $y^2 = x^2(a + x)/(a - x)$

Solution: (i) If we put $y = x$ in the given equation, then we get $x^2 = 0$ which gives two equal roots of x viz. 0 and 0. So the line $y = x$ touches the curve at O.

Hence
$$\rho = \frac{1}{2}\sqrt{1^2 + (-1)^2} \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{y - x}$$

$$= \frac{\sqrt{2}}{2} \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + 2xy + y^2}$$

$$= \frac{\sqrt{2}}{2} \frac{1+1}{1+2+1}$$

since $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x} = 1$ on $y - x = 0$

$$= \frac{\sqrt{2}}{4}$$

(ii) The tangents at the origin of the curve $y^2 = x^2(a + x)/(a - x)$ or $y^2(a - x) - x^2(a + x) = 0$ or $a(y + x)(y - x) = x^3 + xy^2$ are given by $y + x = 0, y - x = 0$. [If we put $y = x$ and $y = -x$ in the given equation, then we get three equal roots of x viz. 0, 0 and 0]

When $y - x = 0$ is the tangent to the curve.

$$\rho = \frac{1}{2}\sqrt{1^2 + (-1)^2} \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{y - x}$$

$$= \frac{\sqrt{2}}{2} \lim_{(x,y)\to(0,0)} \frac{a(x^2 + y^2)(y + x)}{x^3 + xy^2}$$

$$= \frac{a\sqrt{2}}{2} \lim_{(x,y)\to(0,0)} \frac{\left[1 + \left(\frac{y}{x}\right)^2\right]\left(1 + \frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}$$

$$= \frac{a\sqrt{2}}{2} \frac{[1+1][1+1]}{1+1}$$

since $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x} = 1$ on $y - x = 0$

$$= a\sqrt{2}$$

When $y + x = 0$ is tangent to the curve

$$\rho = \frac{1}{2}\sqrt{1^2 + 1^2} \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{y + x}$$

$$= \frac{\sqrt{2}}{2} \lim_{(x,y)\to(0,0)} \frac{a(x^2 + y^2)(y - x)}{x^3 + xy^2}$$

$$= \frac{a\sqrt{2}}{2} \lim_{(x,y)\to(0,0)} \frac{\left[1+\left(\frac{y}{x}\right)^2\right]\left[\frac{y}{x}-1\right]}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{a\sqrt{2}}{2} \frac{[1+1][-1-1]}{1+1}$$

since $\lim_{(x,y)\to(0,0)} \frac{y}{x} = -1$ on $y + x = 0$

$$= -a\sqrt{2}$$

Example 7: Find the radius of curvature of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the vertex.

Solution: The vertex of the cycloid is $\theta = 0$ i.e., $(x, y) = (0, 0)$

Now $$\frac{x^2}{2y} = \frac{a^2(\theta + \sin\theta)^2}{2a(1 - \cos\theta)} = \frac{a\theta^2\left(1 + \frac{\sin\theta}{\theta}\right)^2}{4\sin^2\frac{\theta}{2}}$$

$\therefore$ Hence $$\rho = \lim_{(x,y)\to(0,0)} \frac{x^2}{2y} = \lim_{\theta\to 0} \frac{a\left(1 + \frac{\sin\theta}{\theta}\right)^2}{\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2} = \frac{a(1+1)^2}{1} = 4a$$

Example 8: Find the radius of curvature at the origin of the curve $3x^2 + 4y^2 + 3x - y = 0$

Solution: Here $$3x^2 + 4y^2 + 3x - y = 0 \qquad ...(1)$$

Now, differentiating (1) with respect to x, we get

$$6x + 8yy_1 + 3 - y_1 = 0 \qquad ...(2)$$

or $$y_1(8y - 1) = -(3 + 6x)$$

or $$y_1 = \frac{6x + 3}{1 - 8y}$$

Again, differentiating (2), with respect to x, we get

$$6 + 8y_1^2 + 8yy_2 - y_2 = 0$$

or $$y_2(8y - 1) = -(6 + 8y_1^2)$$

or $$y_2 = \frac{6 + 8y_1^2}{1 - 8y}$$

At (0, 0), $$y_1 = 3 \text{ and } y_2 = \frac{6 + 72}{1} = 78 \ (\because\ y_1 = 3 \text{ at } (0, 0))$$

Hence, the radius of curvature at the origin is given by

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1+9)^{3/2}}{78} = \frac{10\sqrt{10}}{78}$$

Centre of Curvature: The centre of curvature at any point P of the curve is defined as the point of intersection (C) of the normal to the carver at P with the normal to the curve at a neighbouring point Q on the curve as Q tends to P along the curve.

The normals at the points P and Q to the curve is intersect at the point C and $\Delta\psi$ be the angle between them *i.e.* $\angle PCQ = \Delta\psi$, Δs be the arc length PQ, then we get from ΔCPQ

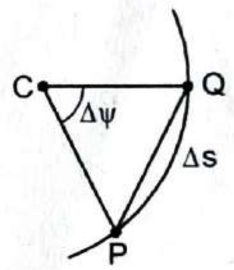

Fig. 5

$$\frac{CP}{PQ} = \frac{\sin \angle CQP}{\sin \Delta\psi}$$

or $$CP = (\sin \angle CQP)\frac{PQ}{\sin \Delta\psi}$$

$$= (\sin \angle CQP)\frac{PQ}{\Delta s}\cdot\frac{\Delta s}{\Delta\psi}\frac{\Delta\psi}{\sin \Delta\psi}$$

when $Q \to P$, the values of $\frac{PQ}{\Delta s} \to 1$, $\frac{\Delta\psi}{\sin \Delta\psi} \to 1$, $\angle CQP \to \frac{\pi}{2}$ and hence

$$CP = \frac{ds}{d\psi} = \rho.$$

The circle drawn with centre at the centre of curvature and radius as the radius of curvature at a point of the curve is called the circle of curvature at that point of the curve.

Definition: The locus of the centre of curvatures of a curve is called its **evolute** and the curve is then called an **involute** of the evolute.

Theorem: The centre of curvature $(\bar{x}, \bar{y})$ of a smooth curve $y = f(x)$ at an arbitrary point (x, y) is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2} (y_2 \neq 0)$$

Proof: Let $P(x, y)$ be any point on the curve $y = f(x)$. Let $C(\bar{x}, \bar{y})$ be the centre of curvature which lies on the normal at P. Let us draw the circle of radius ρ, the radius of curvature at P, which makes the angle ψ with the positive x-axis. Then clearly $\angle PCQ = \psi$

Therefore, $$\bar{x} = OT = OM - TM = OM - PQ$$
$$= x - \rho \sin \psi$$

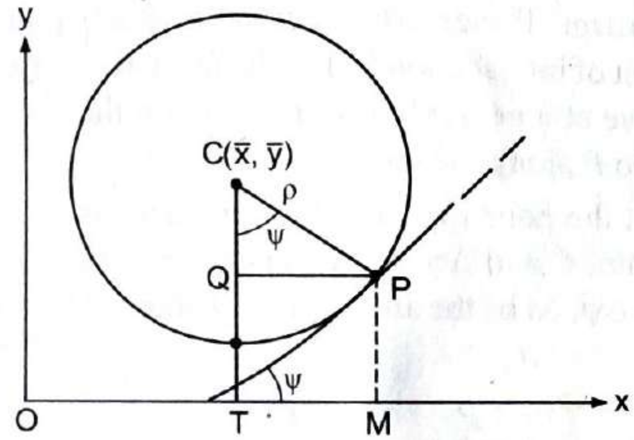

Fig. 6

and $$\bar{y} = CT = QT + QC = y + \rho \cos \psi$$

We know that $y_1 = \tan \psi$, then $\sin \psi = \frac{y_1}{\sqrt{1+y_1^2}}$ and $\cos \psi = \frac{1}{\sqrt{1+y_1^2}}$ and

we also know that

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Hence $$\bar{x} = x - \frac{(1+y_1^2)^{3/2}}{y_2} \frac{y_1}{\sqrt{1+y_1^2}} = x - \frac{y_1(1+y_1^2)}{y_2}$$

and $$\bar{y} = y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1+y_1^2)}} = y + \frac{(1+y_1^2)}{y_2}$$

Note: The equation of the evolute of the curve $y = f(x)$ therefore can be found out by eliminating x and y from the expressions of the centre of curvature and the equation of the curve.

Example 9: Find the centre of curvature of the parabola $y^2 = 4ax$ at (x, y).

Solution: We have $y^2 = 4ax$

$$\therefore \qquad y_1 = \frac{2a}{y} = \frac{2a}{\sqrt{4ax}} = \sqrt{\frac{a}{x}}$$

and
$$y_2 = -\frac{\sqrt{a}}{2x^{3/2}}$$

The centre of curvature $(\bar{x}, \bar{y})$ of the parabola is given by

$$\bar{x} = x - \frac{y_1(1 + y_1^2)}{y_2}$$

$$= x - \frac{\sqrt{\frac{a}{x}}\left(1 + \frac{a}{x}\right)}{-\frac{\sqrt{a}}{2x^{3/2}}} = x + 2(x + a) = 3x + a$$

and
$$\bar{y} = y + \frac{(1 + y_1^2)}{y_2} = y + \frac{\left(1 + \frac{a}{x}\right)}{-\frac{\sqrt{a}}{2x^{3/2}}}$$

$$= 2\sqrt{ax} - \frac{2x^{1/2}(a + x)}{\sqrt{a}} = \frac{2\sqrt{x}}{\sqrt{a}}(a - a - x)$$

$$= -\frac{2x^{3/2}}{\sqrt{a}}$$

Hence, the centre of curvature is $\left(3x + a, -\frac{2x^{3/2}}{\sqrt{a}}\right)$.

Example 10: Find the radius of curvature, centre of curvature and the equation of circle of curvature of the curve $x = e^y$ at $(1, 0)$.

Solution: Here $x = e^y$, then $y = \log x$ and $y_1 = \frac{1}{x}$, $y_2 = -\frac{1}{x^2}$

At $(1, 0)$, $\qquad y_1 = 1$ and $y_2 = -1$

Hence, the radius of curvature at $(1, 0)$ is given by

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + 1)^{3/2}}{-1} = -2\sqrt{2}$$

The centre of curvature $(\bar{x}, \bar{y})$ is given by $(3, -2)$ where

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 1 - \frac{1(1+1^2)}{-1} = 1 + 2 = 3$$

and
$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 0 + \frac{(1+1^2)}{-1} = -2$$

The equation of circle of curvature is given by

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

or
$$(x-3)^2 + (y+2)^2 = 8$$

or
$$x^2 - 6x + 9 + y^2 + 4y + 4 = 8$$

or
$$x^2 + y^2 - 6x + 4y + 5 = 0$$

Example 11: Find the radius of curvature, centre of curvature and the equation of the circle of curvature of $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at the point θ.

Solution: Here $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$\therefore$
$$\frac{dx}{d\theta} = a[-\sin\theta + \sin\theta + \theta\cos\theta] = a\theta\cos\theta$$

and
$$\frac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta] = a\theta\sin\theta$$

$\therefore$
$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\,\theta\sin\theta}{a\,\theta\cos\theta} = \tan\theta$$

and
$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx} = \sec^2\theta\frac{1}{a\theta\cos\theta} = \frac{1}{a\theta\cos^3\theta}$$

Hence, the radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta)^{3/2}}{\frac{1}{a\theta\cos^3\theta}}$$

$$= \sec^3\theta\, a\theta\cos^3\theta = a\theta$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\tan\theta\,(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$

$$= a[\cos\theta + \theta\sin\theta] - a\theta\cos^3\theta\tan\theta\sec^2\theta$$
$$= a[\cos\theta + \theta\sin\theta - \theta\sin\theta] = a\cos\theta$$

and
$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = y + \frac{(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$
$$= a[\sin\theta - \theta\cos\theta] + a\theta\cos^3\theta\sec^2\theta$$
$$= a[\sin\theta - \theta\cos\theta + \theta\cos\theta] = a\sin\theta$$

Hence, the centre of curvature is $(a\cos\theta, a\sin\theta)$.

The equation of circle of curvature is given by
$$(x - a\cos\theta)^2 + (y - a\sin\theta)^2 = a^2\theta^2$$
or
$$x^2 + y^2 - 2a(x\cos\theta + y\sin\theta) = a^2(\theta^2 - 1)$$

Example 12: Find the evolute of the parabola $y^2 = 4ax$

Solution: We have $y^2 = 4ax$

$\therefore$
$$y_1 = \sqrt{\frac{a}{x}}$$
and
$$y_2 = -\frac{\sqrt{a}}{2x^{3/2}}$$

Now, if $(\bar{x}, \bar{y})$ denotes the centre of curvature, we have
$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$
$$= x - \frac{\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)}{-\frac{\sqrt{a}}{2x^{3/2}}} = 3x + 2a$$

and
$$\bar{y} = y + \left(\frac{1+y_1^2}{y_2}\right) = y + \frac{\left(1+\frac{a}{x}\right)}{-\frac{\sqrt{a}}{2x^{3/2}}}$$
$$= -\frac{2}{\sqrt{a}}x^{3/2} \qquad [\because y = 2\sqrt{ax}]$$

From the above we get
$$\bar{y} = -\frac{2}{\sqrt{a}}\left(\frac{\bar{x}-2a}{3}\right)^{3/2}$$

Squaring and then changing $(\overline{x}, \overline{y})$ to current coordinates (x, y) we get

$$y^2 = \frac{4}{a}\left(\frac{x-2a}{3}\right)^3$$

or
$$27ay^2 = 4(x-2a)^3$$

which gives the equation of the evolute.

Example 13: Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution: We have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The parametric equation of the hyperbola is given by

$$x = a\cosh\theta,\ y = b\sinh\theta$$

$\therefore$
$$\frac{dx}{d\theta} = a\sinh\theta$$

and
$$\frac{dy}{d\theta} = b\cosh\theta$$

$\therefore$
$$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cosh\theta}{a\sinh\theta} = \left(\frac{b}{a}\right)\coth\theta$$

and
$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{b}{a}(-\text{cosech}^2\theta)\frac{1}{a\sinh\theta}$$

$$= -\frac{b}{a^2}\text{cosech}^3\theta$$

Let $(\overline{x}, \overline{y})$ be the centre of curvature, then we get

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cosh\theta - \frac{\left(\frac{b}{a}\right)\coth\theta\left(1+\frac{b^2}{a^2}\coth^2\theta\right)}{\left(-\frac{b}{a^2}\right)\text{cosech}^3\theta}$$

$$= a\cosh\theta + a\cosh\theta\sinh^2\theta\left(1+\frac{b^2}{a^2}\coth^2\theta\right)$$

$$= a\cosh\theta\left[1+\sinh^2\theta+\frac{b^2}{a^2}\cosh^2\theta\right][\because \cosh^2\theta - \sinh^2\theta = 1]$$

$$= a\cosh\theta\left[\cosh^2\theta+\frac{b^2}{a^2}\cosh^2\theta\right] = \left(\frac{a^2+b^2}{a}\right)\cosh^3\theta$$

or $\left(\frac{a\bar{x}}{a^2+b^2}\right) = \cosh^3\theta$ or $\left(\frac{a\bar{x}}{a^2+b^2}\right)^2 = \cosh^6\theta$ ...(1)

and $$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = b\sinh\theta + \frac{\left(1+\frac{b^2}{a^2}\coth^2\theta\right)}{\left(-\frac{b}{a^2}\right)\text{cosech}^3\theta}$$

$$= b\sinh\theta - \frac{a^2}{b}\sinh^3\theta\left(1+\frac{b^2}{a^2}\coth^2\theta\right)$$

$$= b\sinh\theta\left[1-\frac{a^2}{b^2}\sinh^2\theta - \cosh^2\theta\right]$$

$$= b\sinh\theta\left[1-\sinh^2\theta-\frac{a^2}{b^2}\sinh^2\theta\right]$$

$$= -\left(\frac{a^2+b^2}{b}\right)\sinh^3\theta \text{ or } \left(\frac{b\bar{y}}{a^2+b^2}\right) = -\sinh^3\theta$$

or $\left(\frac{b\bar{y}}{a^2+b^2}\right)^2 = \sinh^6\theta$ ...(2)

From (1) and (2), we get $\left(\frac{a\bar{x}}{a^2+b^2}\right)^{2/3} - \left(\frac{b\bar{y}}{a^2+b^2}\right)^{2/3} = \cosh^2\theta - \sinh^2\theta = 1$

Then changing $(\bar{x}, \bar{y})$ to current coordinate (x, y), we get $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$ which gives the equation of evolute.

Example 14: Find the evolute of the astroid $x^{3/2} + y^{2/3} = a^{2/3}$.

Solution: The parametric equation of the astroid is

$$x = a\cos^3\theta \text{ and } y = a\sin^3\theta$$

$$\therefore \quad dx = -3a\cos^2\theta\sin\theta\, d\theta$$

and $dy = 3a \sin^2\theta \cos\theta \, d\theta$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\text{and} \quad y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta \frac{1}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature, then we get

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos^3\theta - \frac{(-\tan\theta)(1+\tan^2\theta)}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$= a\cos^3\theta + 3a\cos^4\theta\sin\theta\frac{\sin\theta}{\cos\theta}\frac{1}{\cos^2\theta}$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta$$

$$= a(\cos^3\theta + 3\cos\theta\sin^2\theta)$$

$$\text{and} \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2} = a\sin^3\theta + \frac{(1+\tan^2\theta)}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$= a\sin^3\theta + 3a\cos^4\theta\sin\theta\sec^2\theta$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$= a(\sin^3\theta + 3\cos^2\theta\sin\theta)$$

$$\therefore \quad (\bar{x} + \bar{y}) = a[\cos^3\theta + 3\cos\theta\sin^2\theta + 3\cos^2\theta\sin\theta + \sin^3\theta]$$

$$= a(\cos\theta + \sin\theta)^3$$

$$\text{and} \quad (\bar{x} - \bar{y}) = a[\cos^3\theta + 3\cos\theta\sin^2\theta - 3\cos^2\theta\sin\theta - \sin^3\theta]$$

$$= a(\cos\theta - \sin\theta)^3$$

$$\therefore \quad (\bar{x} + \bar{y})^{2/3} + (\bar{x} - \bar{y})^{2/3} = a^{2/3}[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2]$$

$$= a^{2/3}[\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta]$$

$$= 2a^{2/3}$$

Then changing $(\bar{x}, \bar{y})$ to the current coordinates (x, y), we get $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$ which gives the equation of the evolute.

Example 15: Find the evolute of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$$

Solution: Here $x = a(\theta + \sin\theta)$ and $y = a(1 + \cos\theta)$

$\therefore$ $dx = a(1 + \cos\theta)d\theta$ and $dy = -a\sin\theta\, d\theta$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{-\sin\theta}{(1+\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= -\tan\frac{\theta}{2}$$

and
$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(-\tan\frac{\theta}{2}\right)\frac{d\theta}{dx}$$

$$= -\left(\sec^2\frac{\theta}{2}\right)\frac{1}{2}\frac{1}{a(1+\cos\theta)}$$

$$= -\left(\frac{1}{2a}\right)\frac{\sec^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\left(\frac{1}{4a}\right)\sec^4\frac{\theta}{2}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a(\theta+\sin\theta) - \frac{\left(-\tan\frac{\theta}{2}\right)\left(1+\tan^2\frac{\theta}{2}\right)}{\left(-\frac{1}{4a}\right)\sec^4\frac{\theta}{2}}$$

$$= a(\theta+\sin\theta) - 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a(\theta+\sin\theta) - 2a\sin\theta = a(\theta - \sin\theta)$$

and
$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= a(1+\cos\theta) + \frac{\left(1+\tan^2\frac{\theta}{2}\right)}{\left(-\frac{1}{4a}\right)\sec^4\frac{\theta}{2}}$$

$$= a(1+\cos\theta) - 4a\cos^2\frac{\theta}{2}$$

$$= a(1+\cos\theta) - 2a(1+\cos\theta) = -a(1+\cos\theta)$$

The equation of the evolute is given by

$$x = a(\theta - \sin\theta)$$
$$y = -a(1+\cos\theta)$$

EXERCISES

1. Find the radius of curvature of the following curves:

   (i) $y = x^2$ at $(1, 1)$ (ii) $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ at $(0, 0)$

   (iii) $y = e^{-x^2}$ at $(0, 1)$ (iv) $x = a\sec\phi, y = b\tan\phi$ at $\phi = 0$

   (v) $x = a\cos t, y = b\sin t$ at t

   (vi) $x = at^2, y = 2at$ at $t = 0$

   (vii) $r = ae^{\theta\cot\alpha}$ at the point $\theta = \frac{\pi}{2}$

   (viii) $r^m = a^m\cos m\theta$ at $\theta = \theta_0$

   (ix) $r = a(\theta + \sin\theta)$ at $\theta = 0$

   (x) $r^3 = a^2p$ at (p, r)

   (xi) $p^2(r^2 + a^2) = r^4$ at (p, r)

   (xii) $p^2 + a^2\cos 2\psi = 0$

2. Find the radius of curvature at the origin of the following curves:

   (i) $3x^2 + xy + y^2 - 4x = 0$ (ii) $x^3 + y^3 = 3axy$

   (iii) $x^3 + 5x^2 + 6x - y = 0$

3. Find the centre of curvatures of the following curves at the indicated points.

   (i) $y = \sin^2 x$ at $(0, 0)$ (ii) $x = at^2, y = 2at$ at $(0, 0)$

   (iii) $x^{2/3} + y^{2/3} = a^{2/3}$ at (x_0, y_0)

4. Find the radius of curvature, centre of curvature and equation of the circle of curvature of the following curves at the indicated points:

   (i) $x^2 = 4ay$ at (x, y)

   (ii) $2xy + x + y = 4$ at $(1, 1)$

5. Find the evolute of the following curves

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii) $xy=a^2$

(iii) $x^2=4ay$

(iv) $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$

(v) $x=a(\cos t+t\sin t), y=a(\sin t-t\cos t)$

Answers

1. (i) $\frac{5\sqrt{5}}{2}$ (ii) $\frac{y^2}{a}$ (iii) $\frac{1}{2}$ (iv) $\frac{b^2}{a}$

(v) $\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$ (vi) $2a$ (vii) $ae^{5\cot\alpha}\operatorname{cosec}\alpha$

(viii) $\frac{a^m}{(m+1)r_0^{m-1}}$ where $r_0^m=a^m\cos m\theta_0$ (ix) a

(ix) $\frac{a^2}{3r}$ (xi) $\frac{(r^2+a^2)^{3/2}}{r^2+2a^2}$ (xii) $\frac{a^4}{p^3}$

2. (i) 2 (ii) $\frac{3}{2}a, \frac{3}{2}a$ (iii) $\frac{37\sqrt{37}}{10}$

3. (i) $\left(0,\frac{1}{2}\right)$ (ii) $(2a, 0)$ (iii) $(x_0+3x_0^{1/3}y_0^{2/3}, y_0+3y_0^{1/3}x_0^{2/3})$

4. (i) $\frac{2(a+y)^{3/2}}{\sqrt{a}}, \left(-\frac{x^3}{4a^2}, 2a+\frac{3x^2}{4a}\right)$

(ii) $\frac{3\sqrt{2}}{2}, \left(\frac{5}{2},\frac{5}{2}\right), 4x^2+4y^2-20x-20y+32=0$

5. (i) $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$ [Hint: $x=a\cos\theta, y=b\sin\theta$]

(ii) $(x+y)^{2/3}-(x-y)^{2/3}=4a$

(iii) $(y-2a)^3=\frac{27}{4}ax^2$

(iv) $x=a(\theta+\sin\theta), y=-a(1-\cos\theta)$

(v) $x^2+y^2=a^2$