CHAPTER

7

Mean Value Theorems

In this chapter we shall discuss some useful theorems of differential calculus such as Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Taylor's theorem and Maclaurin's theorem including series expansion of a function.

7.1 ROLLE'S THEOREM

Statement: Let $f(x)$ be a function defined on the closed interval $[a, b]$ satisfying the following conditions:

(i) $f(x)$ is continuous in the closed interval $[a, b]$,

(ii) $f(x)$ is derivable in the open interval (a, b), and

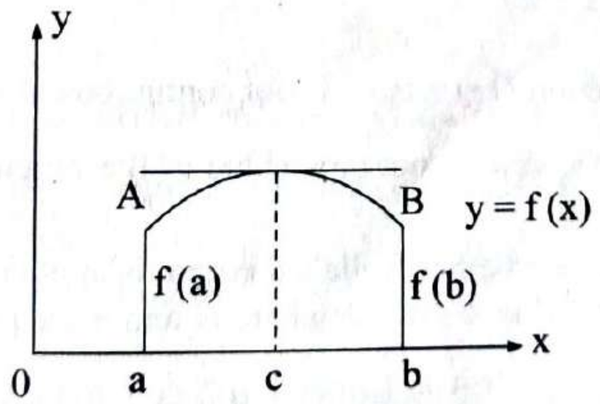
(iii) $f(a) = f(b)$.

Then there exists at least one value c where $a < c < b$ such that $f'(c) = 0$.

Proof: Outside the scope of this text.

Geometric Interpretation of Rolle's theorem: Since the function $f(x)$ is continuous on the closed interval $[a, b]$ and is derivable on the open interval (a, b), the continuous curve has a unique tangent at each point on it except at the end points A and B.

The third condition implies that the curve $y = f(x)$ has the same ordinate at A and B.



The slope of the tangent at $x = c$ is $f'(c) = \tan\theta$.

Now $f'(c) = 0$ implies $\theta = 0$ *i.e.*, the tangent to the curve $y = f(x)$ at $x = c$ is parallel to the x-axis.

Geometrically, the Rolle's theorem states that there exists at least one point c on the graph between A and B such that the tangent at c is parallel to the x-axis.

Example 1: Does $f(x) = |x|$ satisfy Rolle's conditions on $[-1, 1]$? Justify your answer.

Solution: Here $f(x) = \begin{cases} -x \text{ for } -1 \le x < 0 \\ x \quad \text{for } 0 \le x < 1 \end{cases}$

(*i*) $f(x)$ is continuous on $[-1, 1]$

(*ii*) Now $f'(0-0) = \lim\limits_{h \to 0-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0-} \dfrac{-h-0}{h} = -1$

and $f'(0+0) = \lim\limits_{h \to 0+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0+} \dfrac{h-0}{h} = 1$

Since $f'(0-0) \neq f'(0+0)$, then $f(x)$ is not derivable at $x = 0$. So $f(x)$ is not derivable on $(-1, 1)$

(*iii*) $f(1) = f(-1) = 1$

Since $f'(x)$ does not exist at $x = 0$, Rolle's theorem is not applicable to $f(x) = |x|$.

Example 2: Does $f(x) = x^2\sqrt{a^2 - x^2}$ satisfy Rolle's conditions in $[0, a]$? Justify your answer.

Solution: Here (*i*) $f(x)$ is continuous on $[0, a]$

(*ii*) $f'(x) = 2x\sqrt{a^2 - x^2} - \dfrac{x^2(2x)}{2\sqrt{a^2 - x^2}}$ exists in $(0, a)$

(*iii*) $f(0) = f(a) = 0$

Hence, all the conditions of Rolle's theorem are satisfied by $f(x)$.

Example 3: Examine whether Rolle's theorem is applicable to the function $f(x) = \tan x$ in $0 \le x \le \pi$.

Solution: The function $f(x) = \tan x$ is not continuous at $x = \dfrac{\pi}{2}$.

Hence, Rolle's theorem is not applicable to the function $f(x) = \tan x$ in $0 \le x \le \pi$.

Example 4: Examine whether Rolle's theorem is applicable to the function $f(x) = (x-a)^m (x-b)^n$ in $a \le x \le b$ where m and n are positive integers.

Solution: (*i*) Here $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (x-a)^m (x-b)^n = 0 \cdot (a-b)^n = 0$

and $f(a) = (a-a)^m (a-b)^n = 0$

$\therefore$ $f(x)$ is continuous at $x = a$.

Similarly, $\lim_{x \to b} f(x) = 0$ and $f(b) = 0$, hence $f(x)$ is continuous at $x = b$.

Further, $f(x)$ being a polynomial in x is continuous at every point between a and b.

Hence, $f(x)$ is continuous in $[a, b]$.

(*ii*) $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$ exists in (a, b).

(*iii*) $f(a) = f(b) = 0$.

Hence, all the conditions of Rolle's theorem are satisfied by $f(x)$.

Example 5: Does $f(x) = \frac{1}{x} + \frac{1}{1-x}$ satisfy Rolle's conditions in [0, 1]? Justify your answer.

Solution: Here (*i*) $f(x)$ is continuous in $0 < x < 1$ (not in $0 \le x \le 1$).

(*ii*) $f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2}$ exists in $0 < x < 1$.

(*iii*) $f(0)$ and $f(1)$ both are undefined.

Hence, Rolle's theorem is not applicable to the given function.

Example 6: Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$ in $1 \le x \le 3$.

Solution: Here (*i*) $f(x)$ is continuous in [1, 3]

(*ii*) $f'(x) = 3x^2 - 12x + 11$ exists in (1, 3)

(*iii*) $f(1) = f(3) = 0$.

Hence, all the conditions of Rolle's theorem are satisfied. Therefore, there exists a point $c(1 < c < 3)$ such that $f'(c) = 0$

$$\Rightarrow 3c^2 - 12c + 11 = 0 \Rightarrow c = \frac{12 \pm \sqrt{144-132}}{6} = \frac{12 \pm \sqrt{12}}{6}$$

$$\Rightarrow \qquad c = \frac{2 \pm \sqrt{3}}{3} = 2 \pm \frac{1}{\sqrt{3}}.$$

7.2 LAGRANGE'S MEAN VALUE THEOREM

Statement: If $f(x)$ be a function defined on the closed interval $[a, b]$ satisfying the following conditions:

(*i*) $f(x)$ is continuous in $[a, b]$

(*ii*) $f(x)$ is derivable in (a, b)

then there exists at least one value c where $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

The h-θ form of Lagrange's mean value theorem: Let $b = a + h$, then any point between a and b can be written as $a + \theta h$ where $0 < \theta < 1$. Then the above theorem take the following form:

If $f(x)$ be a function defined on the closed interval $[a, a + h]$ satisfying the following conditions:

(*i*) $f(x)$ is continuous in $[a, a + h]$

(*ii*) $f(x)$ is derivable in $(a, a + h)$

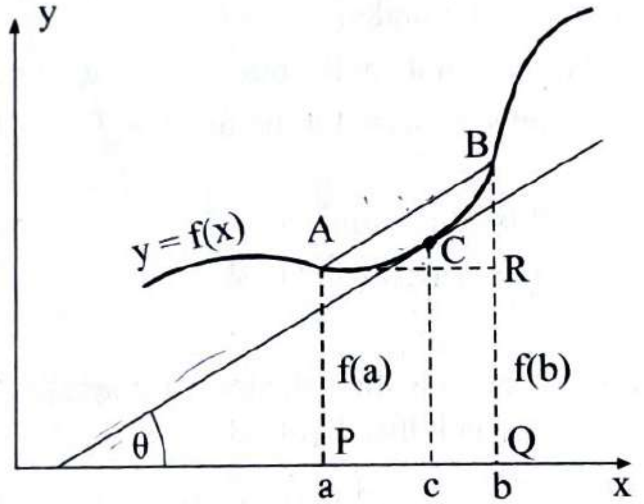
then there exists θ, where $0 < \theta < 1$ such that $f(a+h) = f(a) + h f'(a + \theta h)$.

Note: If $a = 0$ and $h = x$, we get the MVT in the form:

$$f(x) = f(0) + h f'(\theta h) \text{ where } 0 < \theta < 1$$

Geometric interpretation:

From the graph ACB in $[a, b]$, $\dfrac{f(b)-f(a)}{b-a} = \dfrac{BR}{PQ} = \dfrac{BR}{AR} = \tan \angle \text{BAR}$



The slope of the tangent to the curve $y = f(x)$ at $x = c$ is $f'(c) = \tan\theta$

From Lagrange MVT, we get

$\dfrac{f(b)-f(a)}{b-a} = f'(c)$ *i.e.*, $\tan \angle \text{BAR} = \tan\theta$ *i.e.*, the tangent to the curve $y = f(x)$ at $x = c$ is parallel to the chord AB.

Example 1: Verify Lagrange's MVT for $f(x) = x^2 + 3x + 2$ in $1 \le x \le 2$. Find c if the theorem is applicable.

Solution: Here (*i*) $f(x)$ is continuous in $[1, 2]$

(*ii*) $f'(x) = 2x + 3$ exists in $(1, 2)$

Hence, Lagrange's MVT is applicable.

∴ There exists a point c in $(1, 2)$ such that $\dfrac{f(2)-f(1)}{2-1} = f'(c)$

or $\quad \dfrac{12-6}{1} = 2c + 3$

or $\quad 6 = 2c + 3$ or $c = 3/2$.

Example 2: Let $f(x)$ be a function defined in $[-1, 1]$ as

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Does the mean value theorem hold for $f(x)$ in $[-1, 1]$?

Solution: Clearly (*i*) The function $f(x)$ is continuous in $[-1, 1]$.

(*ii*) $f'(x) = \sin\dfrac{1}{x} - \dfrac{x}{x^2}\cos\left(\dfrac{1}{x}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\left(\dfrac{1}{x}\right)$

for $\quad x \neq 0$

and $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin\frac{1}{h} - 0}{h} = \lim\limits_{h\to 0} \sin\dfrac{1}{h}$

which does not exist.

Hence, $f(x)$ is not derivable at $x = 0$.

$\therefore$ Lagrange's MVT is not applicable to the given function in $[-1, 1]$.

Example 3: In the mean value theorem $f(h) = f(0) + h f'(\theta h)$ $(0 < \theta < 1)$, show that the limiting value of θ as $h \to 0$ is $\dfrac{1}{2}$ when $f(x) = \cos x$.

Solution: Here $f(x) = \cos x$, then $f'(x) = -\sin x$ and $f(0) = 1$. From the given relation $f(h) = f(0) + h f'(\theta h)$, we get $\cos h = 1 + h(-\sin \theta h)$

or $\quad h \sin \theta h = 1 - \cos h = 2\sin^2\dfrac{h}{2} \qquad$ or $\sin \theta h = \dfrac{2\sin^2\frac{h}{2}}{h}$

or $\quad \theta\dfrac{\sin\theta h}{\theta h} = \dfrac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2 2} \qquad \therefore \lim\limits_{h\to 0}\theta \lim\limits_{h\to 0}\dfrac{\sin\theta h}{\theta h} = \dfrac{1}{2}\lim\limits_{h\to 0}\dfrac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}$

$\lim\limits_{h\to 0} \theta \cdot 1 = \dfrac{1}{2} \cdot 1 \quad$ or $\quad \lim\limits_{h\to 0} \theta = \dfrac{1}{2}$

Example 4: Apply Lagrange MVT to prove that the chord on the parabola $y = x^2 + 2ax + b$ joining the points at $x = \alpha$ and $x = \beta$ is parallel to its tangent at the point $x = \dfrac{1}{2}(\alpha + \beta)$.

Solution: Here $f(x) = x^2 + 2ax + b$

$\therefore \quad f(\alpha) = \alpha^2 + 2\alpha a + b$ and $f(\beta) = \beta^2 + 2\beta a + b$

$\therefore \quad f'(x) = 2x + 2a$

Let c be any point in (α, β), then by Lagrange's MVT, we get

$$\frac{f(\alpha) - f(\beta)}{\alpha - \beta} = f'(c)$$

or $$f'(c) = \frac{\alpha^2 + 2\alpha a + b - \beta^2 - 2\beta a - b}{\alpha - \beta} = \frac{(\alpha^2 - \beta^2) + 2(\alpha - \beta)a}{\alpha - \beta}$$

or $2c + 2a = \alpha + \beta + 2a \qquad \therefore \quad c = \dfrac{\alpha + \beta}{2}$

Hence, the chord joining the points $x = \alpha$ and $x = \beta$ is parallel to the tangent at the point $\dfrac{\alpha + \beta}{2}$.

Example 5: Examine the validity of Lagrange's MVT for the function $f(x) = 4 - (6 - x)^{2/3}$ in [5, 7].

Solution: Here $f(x)$ is continuous in [5, 7] and $f'(x) = \dfrac{2}{3}\dfrac{1}{(6-x)^{1/3}}$. But $f'(6)$ not defined.

Hence, Lagrange's MVT is not applicable to the given function.

Example 6: Use MVT to find an approximation to $\sqrt[3]{28}$.

Solution: Let $f(x) = x^{1/3}$ be defined in [27, 28]

$$\therefore \quad f'(x) = \frac{1}{3}x^{-2/3}$$

Applying MVT to $f(x) = x^{1/3}$ in [27, 28], we get

$$\frac{f(28) - f(27)}{28 - 27} = f'(c) \text{ where } 27 < c < 28$$

or $$f(28) = f(27) + f'(c) \text{ or } (28)^{1/3} = (27)^{1/3} + \frac{1}{3}c^{-2/3}$$

or $$(28)^{1/3} = 3 + \frac{1}{3} \cdot \frac{1}{c^{2/3}}$$

$$\left(\because c > 27 \Rightarrow \frac{1}{c} < \frac{1}{27} \Rightarrow \frac{1}{c^{2/3}} > \frac{1}{(27)^{2/3}} = \frac{1}{9}\right)$$

$$< 3 + \frac{1}{3} \cdot \frac{1}{9} = 3 + \frac{1}{27} = 3\frac{1}{27}.$$

Example 7: Using MVT, prove that $\sqrt{101}$ lies between 10 and 10.5.

Solution : Let $f(x) = x^{1/2}$ be defined in [100, 101]

$$\therefore \quad f'(x) = \frac{1}{2}x^{-1/2}$$

Applying Lagrange's MVT to $f(x)$ in [100, 101], we get $\frac{f(101)-f(100)}{101-100} = f'(c)$ where $100 < c < 101$.

or $f(101) = f(100) + f'(c)$

or $\sqrt{101} = \sqrt{100} + \frac{1}{2}c^{-1/2} = 10 + \frac{1}{2}c^{-1/2}$

$$< 10 + \frac{1}{2}\cdot\frac{1}{10}$$

$$\left[\because c > 100 \Rightarrow \frac{1}{c} < \frac{1}{100} \Rightarrow \frac{1}{c^{1/2}} < \frac{1}{\sqrt{100}} = \frac{1}{10}\right]$$

$$= 10 + \frac{1}{20} = 10 + 0.05 = 10.05$$

Again, $100 < 101 \Rightarrow \sqrt{100} < \sqrt{101} \Rightarrow 10 < \sqrt{101}$

$\therefore \quad 10 < \sqrt{101} < 10.05.$

Example 8: Show that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

Solution: Let $f(x)$ be defined in the closed interval $[0, x]$. Then applying Lagrange's MVT to $f(x)$ on $[0, x]$, we get $f(x) = f(0) + x f'(\theta x)$ where $0 < \theta < 1$.

Let $\quad f(x) = \log(1 + x)$. Then $f'(x) = \frac{x}{1+x}$

$$\therefore \log(1+x) = 0 + x\,\frac{1}{1+\theta x} = \frac{x}{1+\theta x} \quad (0 < \theta < 1) \qquad \text{... (1)}$$

Since $0 < \theta < 1$ and $x > 0$, then $0 < \theta x < x$

$\therefore 1 < 1 + \theta x < 1 + x$

$$\frac{1}{1+x} < \frac{1}{1+\theta x} < 1$$

$$\frac{x}{1+x} < \frac{x}{1+\theta x} < x \quad (\because x > 0)$$

$$\frac{x}{1+x} < \log(1+x) < x. \qquad \text{(by (1))}$$

Example 9: If $f(x) = \begin{vmatrix} \sin x & \sin\alpha & \sin\beta \\ \cos x & \cos\alpha & \cos\beta \\ \tan x & \tan\alpha & \tan\beta \end{vmatrix}, 0 < \alpha < \beta < \frac{\pi}{2}$

Show that $f'(\zeta) = 0$ where $\alpha < \zeta < \beta$.

Solution: Now $f(\alpha) = \begin{vmatrix} \sin\alpha & \sin\alpha & \sin\beta \\ \cos\alpha & \cos\alpha & \cos\beta \\ \tan\alpha & \tan\alpha & \tan\beta \end{vmatrix} = 0$ ($\because$ two columns are identical)

and $f(\beta) = \begin{vmatrix} \sin\beta & \sin\alpha & \sin\beta \\ \cos\beta & \cos\alpha & \cos\beta \\ \tan\beta & \tan\alpha & \tan\beta \end{vmatrix} = 0$ ($\because$ two columns are identical)

$\therefore \quad f(\alpha) = f(\beta) = 0$

For $0 < x < \frac{\pi}{2}$, $f(x)$ is continuous and differentiable.

$\because \quad 0 < \alpha < \beta < \frac{\pi}{2}$, then $f(x)$ is continuous and differentiable in $\alpha \le x \le \beta$.

Then by Rolle's theorem, there exists a point ζ in $\alpha < \zeta < \beta$ such that $f'(\zeta) = 0$.

Example 10: Find the value of θ in the mean value theorem

$$f(x + h) = f(x) + h f'(x + \theta h)$$

where (*i*) $f(x) = e^x$ (*ii*) $f(x) = \log x$.

Solution: (*i*) Here $f(x) = e^x$ and $f'(x) = e^x$

By the MVT, we get $f(x + h) = f(x) + h f'(x + \theta h)$ $(0 < \theta < 1)$

or $e^{x+h} = e^x + h\, e^{x+\theta h}$

or $\quad e^h = 1 + he^{\theta h}$ or $e^{\theta h} = \dfrac{e^h - 1}{h}$

$\therefore \quad \theta h = \log\left(\dfrac{e^h - 1}{h}\right), \; \theta = \dfrac{1}{h}\log\left(\dfrac{e^h - 1}{h}\right)$

(*ii*) Here $f(x) = \log x$. Then $f'(x) = \dfrac{1}{x}$

By the MVT, we get $f(x + h) = f(x) + h f'(x + \theta h)$ $(0 < \theta < 1)$

or $\log(x + h) = \log x + \dfrac{h}{x + \theta h}$

or $\quad \dfrac{h}{x + \theta h} = \log(x + h) - \log x = \log\left(\dfrac{x + h}{x}\right) = \log\left(1 + \dfrac{h}{x}\right)$

or $$\frac{x+\theta h}{h} = \frac{1}{\log\left(1+\frac{h}{x}\right)}$$

or $$\frac{x}{h}+\theta = \frac{1}{\log\left(1+\frac{h}{x}\right)}$$

or $$\theta = \frac{1}{\log\left(1+\frac{h}{x}\right)} - \frac{x}{h}.$$

7.3 CAUCHY'S MEAN VALUE THEOREM

Statement: If $f(x)$ and $g(x)$ be the two functions defined in the closed interval $[a, b]$ such that:

(*i*) $f(x)$ and $g(x)$ both are continuous in $a \le x \le b$;

(*ii*) $f(x)$ and $g(x)$ both are derivable in $a < x < b$,

(*iii*) $g'(x) \neq 0$ for any value of x in $a < x < b$,

then there exists at least one value c where $a < c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

The proof is outside the scope of this text.

Note: Let $b = a + h$, then $c = a + \theta h$ where $0 < \theta < 1$

$\therefore$ Cauchy's mean value theorem takes the following form:

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, 0 < \theta < 1$$

Deduction of Lagrange's Mean Value Theorem from Cauchy's Mean Value Theorem: Consider $f(x)$ given in Lagrange's theorem with $g(x) = x$.

Clearly (*i*) $f(x)$ and $g(x)$ both are continuous in $a \le x \le b$,

(*ii*) $f(x)$ and $g(x)$ both are derivable in $a < x < b$,

(*iii*) $g'(x) = 1 \neq 0$ for any value of x in $a < x < b$

(Recall by the hypothesis of Lagrange's theorem, $f(x)$ is continuous on $[a, b]$ and derivable on (a, b))

Then there exists at least one value c where $a < c < b$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$ which is the Lagrange Mean Value Theorem.

Example 1: Applying Cauchy's MVT to the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in the closed interval $[a, b]$, then prove that $c(a < c < b)$ is the arithmetic mean of a and b.

Solution: From Cauchy's MVT, we get

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \text{ where } a < c < b$$

or $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \quad \text{or} \quad \frac{e^b - e^a}{\frac{(e^a - e^b)}{e^a e^b}} = -e^{2c}$$

or $$-e^a e^b = -e^{2c} \quad \text{or} \quad e^{a+b} = e^{2c}$$

or $$a + b = 2c \quad \text{or} \quad c = \frac{a+b}{2}$$

Hence, c is the arithmetic mean of a and b.

Example 2: In Cauchy's mean value theorem, (*i*) if $f(x) = \sin x$ and $g(x) = \cos x$ (*ii*) if $f(x) = e^x$ and $g(x) = e^{-x}$, then show that θ is independent of both x and h, and is equal to $\frac{1}{2}$.

Solution: (*i*) Applying Cauchy's MVT to the functions $f(x)$ and $g(x)$ in $[x, x + h]$, we get

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)} \text{ where } 0 < \theta < 1$$

or $$\frac{\sin(x+h)-\sin x}{\cos(x+h)-\cos x} = \frac{\cos(x+\theta h)}{-\sin(x+\theta h)}$$

or $$\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}} = \frac{\cos(x+\theta h)}{-\sin(x+\theta h)}$$

or $\sin(x+\theta h)\cos\left(x+\frac{h}{2}\right) - \cos(x+\theta h)\sin\left(x+\frac{h}{2}\right) = 0$

or $\sin\left\{(x+\theta h) - \left(x+\frac{h}{2}\right)\right\} = 0$

or $$\sin\left(\theta h - \frac{h}{2}\right) = 0 \text{ or } \theta h - \frac{h}{2} = 0 \text{ or } \theta = \frac{1}{2}.$$

Hence, θ is independent of both x and h, and is equal to $\frac{1}{2}$.

(*ii*) Applying Cauchy's MVT to the functions $f(x)$ and $g(x)$ in $[x, x+h]$, we get

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, 0<\theta<1$$

or $$\frac{e^{x+h}-e^{x}}{e^{-(x+h)}-e^{-x}} = \frac{e^{(x+\theta h)}}{-e^{-(x+\theta h)}}$$

or $$\frac{e^{x}(e^{h}-1)}{e^{-x}(e^{-h}-1)} = -e^{2(x+\theta h)}$$

or $$e^{2x}\frac{(e^{h}-1)}{(1-e^{h})}e^{h} = -e^{2x}e^{2\theta h}$$

or $$e^{h} = e^{2\theta h} \text{ or } h = 2\theta h \text{ or } \theta = \frac{1}{2}$$

Hence, θ is independent of both x and h, and is equal to $\frac{1}{2}$.

Example 3: If $f(x) = x^2$ and $g(x) = x$, then find the value of c in terms of a and b in Cauchy's mean value theorem.

Solution: Applying Cauchy's MVT to the functions $f(x)$ and $g(x)$ in $[a, b]$,

we get $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{b^2-a^2}{b-a} = \frac{2c}{1} \Rightarrow \frac{(b+a)(b-a)}{b-a} = 2c$$

or $$c = \frac{a+b}{2}.$$

7.4 GENERALIZED MEAN VALUE THEOREM

We begin with Taylor's theorem which is known as generalised mean value theorem.

1. Taylor's theorem with Lagrange's form of Remainder

Statement : Let $f(x)$ be a function defined on the closed interval $[a, a+h]$ such that

(*i*) $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) nth derivative $f^{(n)}$ exists in $(a, a+h)$, then there exists at least one number θ where $0 < \theta < 1$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \dots + \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) + \frac{h^n}{\lfloor n}f^n(a+\theta h)$$

[Finite form]

The Lagrange's form of Remainder after n terms is denoted by R_n where

$$R_n = \frac{h^n}{\lfloor n} f^n(a+\theta h) \text{ where } 0 < \theta < 1.$$

Proof: Outside the scope of this text.

2. Taylor's theorem with Cauchy's form of Remainder

Statement: Let $f(x)$ be a function defined on the closed interval $[a, a+h]$ such that

(*i*) $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[a, a+h]$ and

(*ii*) nth derivative $f^{(n)}$ exists in $(a, a+h)$.

Then there exists at least one number θ where $0 < \theta < 1$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \dots$$

$$+ \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-1}}{\lfloor n-1}f^n(a+\theta h)$$

[finite form]

The Cauchy's Remainder after n terms is denoted by R_n where

$$R_n = \frac{h^n(1-\theta)^{n-1}}{\lfloor n-1}f^n(a+\theta h) \text{ where } 0 < \theta < 1$$

Proof : Outside the scope of this text.

3. Maclaurin's Theorem

Let $f(x)$ be a function defined in the closed interval $[0, x]$ such that

(*i*) $(n-1)$th derivative $f^{(n-1)}$ is continuous in $[0, x]$ and

(*ii*) nth derivative $f^{(n)}$ exists in $(0, x)$, then there exists at least one number θ where $0 < \theta < 1$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \dots + \frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0) + R_n$$

where $R_n = \dfrac{x^n}{\lfloor n}f^n(\theta x)\ (0 < \theta < 1)$ [Lagrange's form]

$= \dfrac{x^n(1-\theta)^{n-1}}{\lfloor n-1}f^n(\theta x),\ (0 < \theta < 1)$ [Cauchy's form]

Some Useful Limits: The following limits are useful for the derivation of many results of this chapter.

(*i*) $\lim_{n\to\infty} nx^n = 0$ for $|x| < 1$ (*ii*) $\lim_{n\to\infty} \frac{x^n}{n} = \begin{cases} 0 & \text{for } |x| \le 1 \\ \infty & \text{for } x > 1 \end{cases}$

(*iii*) $\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0$ for all values of x

(*iv*) $\lim_{n\to\infty} \frac{m(m-1)(m-2)\dots(m-n+1)}{\lfloor n} x^n = 0$ for $|x| < 1$.

7.5 SERIES FOR EXPONENTIAL, TRIGONOMETRIC AND LOGARITHM FUNCTIONS

1. Taylor's Series: Let $f(x), f'(x), f''(x), \dots, f^{(n)}(x)$ exist finitely however, large 'n' may be in any interval $(x - \delta, x + \delta)$ enclosing the point x and let $R_n \to 0$ as $n \to \infty$. Then Taylor's series of finite form can be extended to an infinite series of the form $f(x + h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \dots + \frac{h^n}{\lfloor n} f^{(n)}(x) + \dots$ to ∞ for $|h| < \delta$

2. Maclaurin's Infinite Series: Let $f(x), f'(x), f''(x), \dots, f^n(x)$ exist finitely however, large n may be in any interval $(-\delta, \delta)$ and let $R_n \to 0$ as $n \to \infty$. Then Maclaurin's series of finite form can be extended to an infinite series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots + \frac{x^n}{\lfloor n} f^n(0) + \dots \text{ to } \infty \text{ where } |x| < \delta.$$

7.6 EXPANSIONS OF FUNCTIONS IN INFINITE SERIES

We consider the expansions of the following functions:

1. e^x, 2. $\sin x$, 3. $\cos x$, 4. $\log(1 + x)$, 5. $(1 + x)^m$

(1) Let $f(x) = e^x$, then $f^n(x) = e^x$

$\therefore f^n(0) = e^0 = 1$.

Thus $f^n(0)$ exists and is finite for all 'n' (when n is large)

$$\text{Now, } R_n = \frac{x^n}{\lfloor n} f^n(\theta x),\ 0 < \theta < 1$$

$$= \frac{x^n}{\lfloor n} e^{\theta x},\ 0 < \theta < 1$$

Since $e^{\theta x}$ is finite for any given value of x and $\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0$,

$\therefore \quad R_n \to 0$ as $n \to \infty$

So, we may expand e^x in an infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots \text{ to } \infty.$$

Now $f(0) = f'(0) = f''(0) = \ldots = [e^x]_{x=0} = 1$

Hence, $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$ to ∞

Condition of validity: All values of x in $\mathbb{R}$.

(2) Let $\quad f(x) = \sin x$

Then $f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$

$\therefore f(x)$ possesses derivatives of all orders and for all values of x in $\mathbb{R}$.

Now, $\quad R_n = \frac{x^n}{\lfloor n} f^n(\theta x),\ 0 < \theta < 1$

$$= \frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right),\ 0 < \theta < 1$$

Since, $\sin\left(\frac{n\pi}{2} + \theta x\right)$ is finite $\left[-1 \le \sin\left(\frac{n\pi}{2} + \theta x\right) \le 1\right]$ for all values of x and n and $\frac{x^n}{\lfloor n} \to 0$ as $n \to \infty$, $\quad \therefore R_n \to 0$ as $n \to \infty$

So, we can expand $\sin x$ in an infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots \text{ to } \infty.$$

Now $f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1$

$f''(0) = -\sin 0 = 0, f'''(0) = -\cos 0 = -1$

$f^{iv}(0) = \sin 0 = 0, f^{v}(0) = \cos 0 = 1$ and so on.

Hence, $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} - \ldots$ to ∞.

Condition of validity: All values of x in $\mathbb{R}$.

(3) Let $\quad f(x) = \cos x$. Then $f^n(x) = \cos\left(\frac{n\pi}{2} + x\right)$

$\therefore f(x)$ possesses derivative of every order for all values of x (for large n)

Now. $R_n = \frac{x^n}{\lfloor n} f^n(\theta x),\ 0 < \theta < 1$

$$= \frac{x^n}{\lfloor n} \cos\left(\frac{n\pi}{2} + \theta x\right),\ 0 < \theta < 1$$

Since $\cos\left(\frac{n\pi}{2} + \theta x\right)$ is finite $\left[-1 \le \cos\left(\frac{n\pi}{2} + \theta x\right) \le 1\right]$ for all values of x and n and $\frac{x^n}{\lfloor n} \to 0$ as $n \to \infty$. Therefore, $R_n \to 0$ as $n \to \infty$.

So, we can expand $\cos x$ in an infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots \text{ to } \infty$$

Now $f(0) = \cos 0 = 1, f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1, f'''(0) = \sin 0 = 0$

$f^{iv}(0) = \cos 0 = 1, f^{v}(0) = -\sin 0 = 0$ and so on

Hence $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \frac{x^8}{\lfloor 8} - \ldots$ to ∞

Condition of validity:. All values of x in $\mathbb{R}$.

(4) Let $f(x) = \log(1 + x)$.

Then $f^n(x) = \dfrac{(-)^{n-1}\lfloor (n-1)}{(1+x)^n}$

which exists for all n when $x > -1$, *i.e.*, when $(1 + x) > 0$, otherwise [*i.e.*, when $(1 + x) \le 0$] $(1 + x)^n$ may tend to 0 if $|1 + x| < 1$ or $(1 + x)^n$ may oscillate and tend to $\pm\infty$ if $|1 + x| \geq 1$.

Case I: We take Lagrange's form of remainder:

$$R_n = \frac{x^n}{\lfloor n} f^n(\theta x) = \frac{x^n}{\lfloor n} \frac{(-1)^{n-1}\lfloor n-1}{(1+\theta x)^n} = (-1)^{n-1} \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

Now $\frac{1}{n} \to 0$ as $n \to \infty$

But $\left(\frac{x}{1+\theta x}\right)^n \to 0$ as $n \to \infty$ only if $0 \le x \le 1$, because in this case $\frac{x}{1+\theta x}$ is positive and less than 1.

$\therefore R_n \to 0$ as $n \to \infty$ under the condition that $0 \le x \le 1$.

Case II: Let $-1 < x < 0$

In this case $\frac{x}{1+\theta x}$ may not be numerically less than 1 and therefore, $\left(\frac{x}{1+\theta x}\right)^n$ may not tend to zero as $n \to \infty$. Thus, Lagrange's form of Remainder fails to tend to zero.

We, therefore, take Cauchy's form of remainder:

$$R_n = \frac{x^n (1-\theta)^{n-1}}{\lfloor n-1} f^n(\theta x) = \frac{x^n (1-\theta)^{n-1}}{\lfloor n-1} \frac{(-1)^{n-1} \lfloor n-1}{(1+\theta x)^n}$$

$$= (-1)^{n-1} (1-\theta)^{n-1} \left(\frac{x}{1+\theta x}\right)^n = (-1)^{n-1} \frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

Now, $\frac{1-\theta}{1+\theta x}$ is positive and less than 1 for $-1 < x < 0$ and $0 < \theta < 1$. So $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$. Also, $x^n \to 0$ as $n \to \infty$ for $-1 < x < 0$ and $\frac{1}{1+\theta x}$ is a finite quantity for $-1 < x < 0$ and $0 < \theta < 1$

$$\therefore \qquad R_n \to 0 \text{ as } n \to \infty$$

Therefore, in view of case I and case II, log $(1 + x)$ can be expanded in an infinite Maclaurin's series if $-1 < x \le 1$.

Now $f^n(x) = \frac{(-1)^{n-1} \lfloor n-1}{(1+x)^n}$, $\qquad f^n(0) = (-1)^{n-1} \lfloor n-1$

$\therefore \quad f(0) = \log(1+0) = 0, \qquad f'(0) = (-1)^{1-1} \lfloor 1-1 = 1$

$f''(0) = (-1)^{2-1} \lfloor 2-1 = -1, f'''(0) = (-1)^{3-1} \lfloor 3-1 = \lfloor 2$

$f^{iv}(0) = (-1)^{4-1} \lfloor 4-1 = -\lfloor 3$

and so, on.

Hence, $\log(1+x) = 0 + (x \cdot 1) + \frac{x^2}{\lfloor 2} \times (-1) + \left(\frac{x^3}{\lfloor 3} \times \lfloor 2\right) + \frac{x^4}{\lfloor 4} \times (-\lfloor 3) + \ldots$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ to } \infty$$

Condition of validity: $-1 < x \le 1$.

(5) Let $f(x) = (1 + x)^m$, m be any real number.

When m is a positive integer, $f^n(x) = 0$ for $n > m$ and any value of x. The expansion terminates after $(m + 1)$ th terms. The binomial expansion is valid for all x, because the series is finite.

We consider the case when m is not a positive integer, *i.e.*, m is negative or a fraction.

$\therefore f^n(x) = m(m-1) \ldots (m-n+1)(1+x)^{m-n}$ for $x > -1$ *i.e.*, $1 + x = 0$

We take Cauchy's form of remainder

$$R_n = \frac{x^n (1-\theta)^{n-1}}{\lfloor n-1}f^n(\theta x),\ 0 < \theta < 1$$

$$= \frac{m(m-1)\ldots(m-n+1)}{\lfloor n-1} x^n (1-\theta)^{n-1}(1+\theta x)^{m-n}$$

$$= \frac{m(m-1)\ldots(m-n+1)}{\lfloor n-1} x^n (1-\theta)^{n-1}\frac{(1+\theta x)^{m-1}}{(1+\theta x)^{n-1}}$$

$$= \frac{m(m-1)\ldots(m-n+1)}{\lfloor n-1} x^n (1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

Let $|x| < 1$. Also we have $0 < \theta < 1$.

$\therefore \dfrac{1-\theta}{1+\theta x}$ is positive and less than 1.

$\therefore \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Again $(1 + \theta x)^{m-1}$ is finite for any given x whether $(m - 1)$ is positive or negative.

Now, if $|x| < 1$, then

$$\frac{m(m-1)\ldots(m-n+1)}{\lfloor(n-1)} x^n \to 0 \text{ as } n \to \infty$$

Hence, $R_n \to 0$ as $n \to \infty$ and so for $|x| < 1$ Maclaurin's infinite expansion of $(1+x)^m$ is valid where m is a fraction or a negative integer.

$\therefore f(x) = (1+x)^m$

$\therefore f(0) = (1+0)^m = 1^m = 1,\quad f'(0) = [m(1+x)^{m-1}]_{x=0} = m$

$f''(0) = [m(m-1)(1+x)^{m-2}]_{x=0} = m(m-1)$

$f'''(0) = [m(m-1)(m-2)(1+x)^{m-3}]_{x=0} = m(m-1)(m-2)$ and so on

$$\therefore (1+x)^m = 1 + mx + \frac{x^2}{\lfloor 2} m(m-1) + \frac{x^3}{\lfloor 3} m(m-1)(m-2) + \ldots \text{ to } \infty$$

$$= 1 + mx + \frac{m(m-1)}{\lfloor 2} x^2 + \frac{m(m-1)(m-2)}{\lfloor 3} x^3 + \ldots \text{ to } \infty$$

Condition of validity: $|x| < 1$.

7.7 MISCELLANEOUS EXAMPLES

Example 1: Apply Maclaurin's theorem to $f(x) = (1 + x)^4$ to deduce that $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

Solution: Now $f(x) = (1 + x)^4$, then $f'(x) = 4(1 + x)^3, f''(x) = 12(1 + x)^2$, $f'''(x) = 24(1 + x), f^{iv}(x) = 24$

Since $f(x)$ possesses derivative of order four for all values of x, then by the Maclaurin's theorem with Lagrange's form of remainder after four terms.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{iv}(\theta x) \text{ where } 0 < \theta < 1$$

Now $f(0) = 1, f'(0) = 4, f''(0) = 12, f'''(0) = 24$ and $f^{iv}(\theta x) = 24$

$$\therefore \quad f(x) = 1 + 4x + \frac{x^2}{2}.12 + \frac{x^3}{6} \cdot 24 + \frac{x^4}{24} \cdot 24$$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4.$$

Example 2: Show that

$$a^x = 1 + x \log a + \frac{x^2}{\lfloor 2} (\log a)^2 + \ldots + \frac{x^{n-1}}{\lfloor n-1} (\log a)^{n-1} + \frac{x^n}{\lfloor n} a^{\theta x} (\log a)^n, \; 0 < \theta < 1.$$

Solution: Let $f(x) = a^x = e^{kx}$ where $k = \log_e a = \log a$

$\therefore \quad f^n(x) = k^n e^{kx}$ and $f^n(0) = k^n$

$\therefore \quad f^n(0)$ exists and is finite for all n.

By the Maclaurin's theorem with Lagrange's form of remainder after n terms, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{x^n}{\lfloor n} f^n(\theta x)$$

$$= 1 + x(\log a) + \frac{x^2}{\lfloor 2} (\log a)^2 + \ldots + \frac{x^{n-1}}{\lfloor n-1} (\log a)^{n-1} + \frac{x^n}{\lfloor n} e^{k\theta x} (\log a)^n$$

$$= 1 + x(\log a) + \frac{x^2}{\lfloor 2} (\log a)^2 + \ldots + \frac{x^{n-1}}{\lfloor n-1} (\log a)^{n-1} + \frac{x^n}{\lfloor n} a^{\theta x} (\log a)^n$$

where $0 < \theta < 1$.

Example 3: Expand the following functions in powers of x in an infinite series with conditions of their validity:

(*i*) $\tan^{-1}x$ (*ii*) $e^{ax}\sin bx$ (*iii*) $\frac{1}{1+x}$ (*iv*) $\frac{1}{1+x^2}$.

Solution: (*i*) Let $f(x) = \tan^{-1}x$. Then $f^n(x) = \frac{(-1)^{n-1}\lfloor n-1}{\sqrt{(1+x^2)^n}}\sin(n\cot^{-1}x)$

$\therefore\quad f^n(0) = (-1)^{n-1}\ \lfloor n-1 \sin\frac{n\pi}{2} = 0$ when n is even

and $f'(0) = 1,\ f'''(0) = (-1)^2\, 2 = \lfloor 2$

$f^v(0) = \lfloor 4 = 24...$

$\therefore f(x)$ possesses derivative of every order for all values of x.

$$\text{Now, } R_n = \frac{x^n}{\lfloor n} f^n(\theta x) \text{ where } 0 < \theta < 1$$

$$= \frac{(-1)^{n-1}\lfloor n-1}{\lfloor n}\left(\frac{x}{\sqrt{1+\theta^2x^2}}\right)^n \sin(n\cot^{-1}\theta x)$$

$$= \frac{(-1)^{n-1}}{n}\left(\frac{x}{\sqrt{1+\theta^2x^2}}\right)^n \sin(n\cot^{-1}\theta x)$$

Since $\sin(n\cot^{-1}\theta x)$ is finite $[-1 \le \sin(n\cot^{-1}\theta x) \le 1]$ for all values of x and n and $|x| < \sqrt{1+\theta^2x^2}$ when $|x| < 1$, then $R_n \to 0$ as $n \to \infty$. So, we can expand $\tan^{-1}x$ in infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots \text{ to } \infty.$$

$$\text{or}\quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \text{ to } \infty$$

Condition of validity: $-1 \le x \le 1$.

(*ii*) Let $f(x) = e^{ax}\sin bx$. Then $f^n(x) = r^n e^{ax}\sin(bx + n\phi)$ where $r^2 = a^2 + b^2$

and $\tan\phi = \frac{b}{a}$ *i.e.*, $\phi = \tan^{-1}\frac{b}{a}$

$\therefore f^n(0) = r^n \sin n\phi$ and $f(0) = 0$

$\therefore f(x)$ possesses derivative of every order for all values of x.

Now, $R_n = \frac{x^n}{\lfloor n} f^n(\theta x)$ where $0 < \theta < 1$

$$= \frac{x^n}{\lfloor n} r^n e^{a\theta x} \sin(bx\theta + n\phi) = \frac{(xr)^n}{\lfloor n} e^{a\theta x} \sin(bx\theta + n\phi)$$

Since $\sin(bx\theta + n\phi)$ is finite $[-1 \le \sin(bx\theta + n\phi) \le 1]$ for all values of x and n and $\frac{(xr)^n}{\lfloor n} \to 0$ as $n \to \infty$ for all x, then $R_n \to 0$ as $n \to \infty$. So, we can expand $f(x)$ in infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots \text{ to } \infty$$

$$= x\, r \sin\phi + \frac{x^2}{\lfloor 2} r^2 \sin 2\phi + \dots \text{ to } \infty$$

where $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$

Condition of validity: all values of x.

(*iii*) Let $f(x) = \frac{1}{1+x}$. Then $f^n(x) = \frac{(-1)^n \lfloor n}{(1+x)^{n+1}}$

$\therefore\ f^n(0) = (-1)^n \lfloor n$

$\therefore\ f(x)$ possesses derivative of every order for all values of x.

Now, $R_n = \frac{x^n}{\lfloor n} f^n(\theta x)$ where $0 < \theta < 1$

$$= \frac{x^n}{\lfloor n} \frac{(-1)^n \lfloor n}{(1+\theta x)^{n+1}} = \frac{x^n (-1)^n}{(1+\theta x)^{n+1}} = \frac{(-1)^n}{1+\theta x}\left(\frac{x}{1+\theta x}\right)^n$$

Since $x < 1 + \theta x$ for $0 \le x < 1$ and $\frac{1}{1+\theta x}$ is bounded, then $\left(\frac{x}{1+\theta x}\right)^n \to$ 0 as $n \to \infty$, and hence, $R_n \to 0$ as $n \to \infty$.

For $x < 0$, we consider the Cauchy's form of remainder

$$R_n = \frac{x^n}{\lfloor n-1} \frac{(1-\theta)^{n-1}}{1} f^n(x\theta)$$

$$= \frac{x^n}{\lfloor n-1}(1-\theta)^{n-1} \frac{(-1)^n \lfloor n}{(1+x\theta)^{n+1}} = (-1)^n \frac{n x^n}{(1+x\theta)^2}\left(\frac{1-\theta}{1+x\theta}\right)^{n-1}$$

Now, when $|x| < (1-\theta) < 1+\theta x, \left(\frac{1-\theta}{1+x\theta}\right)^{n-1} \to 0$ as $n \to \infty$ and

$nx^n \to 0$ as $n \to \infty$ and $\frac{1}{(1+\theta x)^2}$ is bounded, then $R_n \to 0$ as $n \to \infty$.

So, we can expand $f(x)$ in infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots \text{ to } \infty$$

$$= 1 - x + \frac{x^2}{\lfloor 2} \lfloor 2 + \frac{x^3}{\lfloor 3}(-1) \lfloor 3 + \ldots \text{ to } \infty$$

$$= 1 - x + x^2 - x^3 + \ldots \text{ to } \infty$$

Condition of validity : $-1 < x < 1$

(iv) Let $f(x) = \frac{1}{1+x^2}$, then $f^n(x) = \frac{(-1)^n \lfloor n}{\left(\sqrt{1+x^2}\right)^{n+1}} \sin\{(n+1)\cot^{-1}x\}$

$\therefore f^n(0) = (-1)^n \lfloor n \sin\left\{(n+1)\frac{\pi}{2}\right\}, f(0) = 1, f'(0) = 0, f''(0) = -\lfloor 2 \ldots.$

$\therefore f(x)$ possesses derivative of every order for all x.

Now, $R_n = \frac{x^n}{\lfloor n} f^n(\theta x)$ where $0 < \theta < 1$

$$= \frac{x^n}{\lfloor n} \frac{(-1)^n \lfloor n}{\left(\sqrt{1+x^2\theta^2}\right)^{n+1}} \sin\{(n+1)\cot^{-1}\theta x\}$$

$$= \frac{(-1)^n}{\sqrt{1+\theta^2 x^2}} \left(\frac{x}{\sqrt{1+\theta^2 x^2}}\right)^n \sin\{(n+1)\cot^{-1} x\theta\}$$

Since $\sin\{(n+1)\cot^{-1}\theta x\}$ is finite $[-1 \le \sin\{(n+1)\cot^{-1}\theta x\} \le 1]$,

$|x| < \sqrt{1+\theta^2 x^2}$ for $|x| < 1$, $\frac{1}{\sqrt{1+\theta^2 x^2}}$ is bounded and $\left(\frac{x}{\sqrt{1+\theta^2 x^2}}\right)^n \to 0$

as $n \to \infty$, then $R_n \to 0$ as $n \to \infty$

So, we can expand $f(x)$ in an infinite Maclaurin's series of the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \infty$$

$$= 1 - x^2 + x^4 - \dots \text{ to } \infty$$

Condition of validity : $-1 < x < 1$

Example 4: Find the value of ζ in the Mean Value Theorem

$f(b) - f(a) = (b - a) f'(\zeta)$

(*i*) if $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \dfrac{1}{2}$

(*ii*) if $f(x) = Ax^2 + Bx + C$ in (a, b).

Solution: (*i*) Now $f(x) = x(x-1)(x-2) = x[x^2 - 2x - x + 2]$

$$= x^3 - 3x^2 + 2x$$

$\therefore f'(x) = 3x^2 - 6x + 2$, $\therefore f'(\zeta) = 3\zeta^2 - 6\zeta + 2$

$$f(0) = 0,\ f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right) = +\frac{3}{8}$$

Now, $f\left(\dfrac{1}{2}\right) - f(0) = \left(\dfrac{1}{2} - 0\right) f'(\zeta)$

$$= \frac{1}{2}(3\zeta^2 - 6\zeta + 2)$$

or $$\frac{3}{8} - 0 = \frac{1}{2}(3\zeta^2 - 6\zeta + 2)$$

or $$\zeta^2 - 2\zeta + \frac{5}{12} = 0$$

$\therefore \zeta = 1 \pm \sqrt{\dfrac{7}{12}}$, since $0 < \zeta < \dfrac{1}{2}$, the + sign is to be rejected, and so $\zeta = 1 - \sqrt{\dfrac{7}{12}}$

(*ii*) Here $f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$, $f'(\zeta) = 2A\zeta + B$

Now $f(b) - f(a) = (b - a) f'(\zeta)$ gives

$Ab^2 + Bb + C - Aa^2 - Ba - C = (b - a)[2A\zeta + B]$

or $A(b^2 - a^2) + B(b - a) = (b - a)[2A\zeta + B]$

or $A(b + a) + \cancel{B} = 2A\zeta + \cancel{B}$ $(\because b - a \neq 0)$

or $A(b + a) = 2A\zeta$ or $\zeta = \dfrac{b + a}{2}$.

Example 5: In the Mean Value Theorem

$$f(h) = f(0) + h f'(\theta h)\ (0 < \theta < 1)$$

show that the limiting value of θ as $h \to 0+$ is $\frac{1}{2}$ or $\frac{1}{\sqrt{3}}$ according as $f(x) =$ $\cos x$ or $\sin x$.

Solution: Let $f(x) = \cos x$, then $f'(x) = -\sin x$

$\because \quad f(h) = f(0) + h f'(\theta h)$

$\therefore \quad \cos h = 1 + h(-\sin \theta h)$ or $h \sin \theta h = 1 - \cos h = 2\sin^2 \frac{h}{2}$

$$\therefore \quad \theta \frac{\sin \theta h}{h\theta} = \frac{1}{2}\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2$$

Now, when $h \to 0$, θh and $\frac{h}{2} \to 0$

$$\therefore \quad \lim_{h \to 0+} \theta \frac{\sin \theta h}{h\theta} = \frac{1}{2} \lim_{h \to 0+} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \text{ or } \lim_{h \to 0} \theta = \frac{1}{2}$$

Again, $f(h) = f(0) + h f'(\theta h)$ $(0 < \theta < 1)$ ... (1)

Applying the MVT of 3rd order to $f(h)$, we get

$$f(h) = f(0) + h f'(0) + \frac{h^2}{\lfloor 2} f''(0) + \frac{h^3}{\lfloor 3} f'''(\theta' h) \text{ where } 0 < \theta' < 1 \quad \text{... (2)}$$

Equating (1) and (2), we get

$$f'(\theta h) = f'(0) + \frac{h}{\lfloor 2} f''(0) + \frac{h^2}{\lfloor 3} f'''(\theta' h) \quad \text{... (3)}$$

Again, applying the mean value theorem (MVT) of 2nd order to $f'(\theta h)$, we get

$$f'(\theta h) = f'(0) + \theta h f''(0) + \frac{\theta^2 h^2}{\lfloor 2} f'''(\theta'' \theta h), \; 0 < \theta'' < 1 \quad \text{... (4)}$$

From (3) and (4), we get

$$\theta h f''(0) + \frac{\theta^2 h^2}{\lfloor 2} f'''(\theta'' \theta h) = \frac{h}{\lfloor 2} f''(0) + \frac{h^2}{\lfloor 3} f'''(\theta' h) \quad \text{... (5)}$$

when $f(x) = \sin x$, $f'(0) = 1$, $f''(0) = 0$, $f'''(x) = -\cos x$, then (5) becomes

$$\frac{-\theta^2 h^2}{\lfloor 2} \cos(\theta'' \theta h) = -\frac{h^2}{\lfloor 3} \cos(\theta' h)$$

$$\therefore \quad \theta^2 = \frac{\lfloor 2 \cos(\theta' h)}{\lfloor 3 \cos(\theta''\theta h)} = \frac{2\cos(\theta' h)}{2\cdot 3\cos(\theta''\theta h)} = \frac{\cos(\theta' h)}{3\cos(\theta''\theta h)}$$

when $h \to 0+$, $\lim_{h\to 0} \theta^2 \to \frac{1}{3}$, $\lim_{h\to 0} \theta \to \frac{1}{\sqrt{3}}$

Example 6: Show that $\sin x > x - \frac{x^3}{6}$ if $0 < x < \frac{\pi}{2}$.

Solution: Using Maclaurin's series, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(\theta x),\ 0 < \theta < 1 \quad \text{... (1)}$$

Since $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x$

$\therefore \quad f(0) = 0\, f'(0) = 1, f''(0) = 0$

$\therefore$ From (1), we get $\sin x = x - \frac{x^3}{6}\cos\theta x$

Since $0 < \theta < 1$ and $0 < x < \frac{\pi}{2}$, then $0 < \cos\theta x < 1$

$\therefore \quad \frac{x^3}{6}\cos\theta x < \left(\frac{x^3}{6}\right)$ or $\left(-\frac{x^3}{6}\right)\cos\theta x > -\frac{x^3}{6}$

or $x - \frac{x^3}{6}\cos\theta x > x - \frac{x^3}{6}$

or $\sin x > x - \frac{x^3}{6}$ $\left(\because \sin x = x - \frac{x^3}{6}\cos\theta x\right)$

Example 7: In the Mean Value Theorem $f(x + h) = f(x) + h f'(x + \theta h)$ if $f(x) = a + bx + cm^x$, then show that θ is independent of x.

Solution: Hence $f(x) = a + bx + cm^x$

$\therefore \quad f'(x) = b + c(\log m) m^x$

By Mean Value Theorem

$$f(x + h) = f(x) + h f'(x + \theta h), \text{ we get}$$

$$a + b(x + h) + cm^{x+h} = a + bx + cm^x + h(b + c(\log m) m^{x+\theta h})$$

or $\quad cm^x \cdot m^h = cm^x + c(\log m) hm^x \cdot m^{\theta h}$

or $\quad m^h = 1 + h(\log m) m^{\theta h}$ or $m^{\theta h} = \frac{m^h - 1}{h\log m}$

Taking logarithm, we get $\theta h \log m = \log\left(\frac{m^h - 1}{h \log m}\right) = A$ (say)

$\therefore\ \theta = \frac{A}{h \log m}$ which is independent of x.

Example 8: If $f(h) = f(0) + hf'(0) + \frac{h^2}{\underline{|2}} f''(\theta h), 0 < \theta < 1$, find θ when $h = 7$ and $f(x) = \frac{1}{1+x}$.

Solution: Here $f(x) = \frac{1}{1+x}$, $f'(x) = \frac{(-1)}{(1+x)^2}$, $f''(x) = \frac{2}{(1+x)^3}$

Then the given relation becomes

$$\frac{1}{1+h} = 1 - h + \frac{h^2}{\underline{|2}} \frac{2}{(1+\theta h)^3}$$

When $h = 7$, this becomes

$$\frac{1}{8} = 1 - 7 + \frac{49}{\not{2}} \frac{\not{2}}{(1+7\theta)^3}$$

or $$\frac{1}{8} + 6 = 49 \frac{1}{(1+7\theta)^3}$$

or $$\frac{49}{8} = \frac{49}{(1+7\theta)^3}$$

or $$(1 + 7\theta)^3 = 8 = 2^3 \Rightarrow 1 + 7\theta = 2 \text{ or } 7\theta = 1, \theta = \frac{1}{7}$$

$\therefore$ $$\theta = \frac{1}{7}.$$

Example 9: From the relation $f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(\theta x), 0 < \theta < 1$, show that $\log(1 + x) > x - \frac{x^2}{2}$ if $x > 0$ and $\cos x > 1 - \frac{x^2}{2}$ if $0 < x < \frac{\pi}{2}$.

Solution: When $f(x) = \log(1 + x)$, then $f(0) = \log 1 = 0$

$$f'(x) = \frac{1}{1+x},\ f'(0) = 1, f''(x) = \frac{(-1)}{(1+x)^2}$$

Hence, by the given relation, $\log(1+x) = 0 + x - \frac{x^2}{\underline{|2}} \frac{1}{(1+\theta x)^2}$

Since $x > 0, \theta > 0$, then $(1+\theta x)^2 > 1$

$$\therefore \quad \frac{x^2}{(1+\theta x)^2} < x^2 \text{ or } \frac{-x^2}{2(1+\theta x)^2} > \frac{-x^2}{2} \quad \left[\because x - \frac{x^2}{2(1+\theta x)^2} > x - \frac{x^2}{2}\right]$$

$$\therefore \quad \log(1+x) > x - \frac{x^2}{2}.$$

When $f(x) = \cos x$, we have $f(0) = 1$, $f'(x) = -\sin x$, $f'(0) = 0$, $f''(x) = -\cos x$
Hence, by the given relation, we get

$$\cos x = 1 - \frac{1}{2} x^2 \cos \theta x$$

For $0 < \theta < 1$ and $0 < x < \frac{\pi}{2}$, $0 < \cos \theta x < 1$

$$\therefore \quad \frac{x^2}{2} \cos \theta x < \frac{x^2}{2} \text{ or } \left(\frac{-x^2}{2}\right) \cos \theta x > \left(\frac{-x^2}{2}\right)$$

$$\text{or} \quad 1 - \frac{x^2}{2} \cos\theta x > 1 - \frac{x^2}{2} \quad \text{or} \quad \cos x > 1 - \frac{x^2}{2}.$$

Example 10: Using MVT, prove that $0 < \frac{1}{x} \log \frac{e^x - 1}{x} < 1$.

Solution: Let $f(x) = e^x$, $f'(x) = e^x$

Using Lagrange's MVT, we get

$f(x) = f(0) + x f'(\theta x)$ where $0 < \theta < 1$

$\therefore \quad e^x = 1 + x e^{\theta x}$ ... (1)

Since $0 < \theta < 1$, $\therefore 0 < \theta x < x$

$\therefore \quad e^0 < e^{\theta x} < e^x$ or $1 < e^{\theta x} < e^x$

or $\quad x < x e^{\theta x} < x e^x$ or $x < e^x - 1 < x e^x$ [From (1)]

$$\text{or} \quad 1 < \frac{e^x - 1}{x} < e^x$$

Taking log, we get, $0 < \log \frac{e^x - 1}{x} < x$

$$\text{or} \quad 0 < \frac{1}{x} \log \frac{e^x - 1}{x} < 1.$$ Hence, the result.

Example 11: Show that $a^x < x^a$ if $x > a \ge e$.

Solution: Let $f(x) = x \log a - a \log x,\ f'(x) = \log a - \dfrac{a}{x}$

Since $e \le a < x$, $\therefore \log a \ge 1$ and $\dfrac{a}{x} < 1$

$$\therefore \quad \log a \ge 1 \text{ and } -\frac{a}{x} > -1 \text{ or } \log a - \frac{a}{x} > 0$$

$$\therefore \quad f'(x) > 0 \text{ for } x > a \ge e$$

$\therefore f(x)$ is an increasing function and hence

$$f(x) > f(a) \text{ for } x > a \ge e$$

$$\therefore x \log a - a \log x > a \log a - a \log a = 0$$

$$\text{or} \quad x \log a > a \log x \text{ or } \log a^x > \log x^a$$

$$\therefore \quad a^x > x^a \text{ for } x > a \ge e. \text{ Hence, the result.}$$

Example 12: Use MVT to prove $\dfrac{x}{\sqrt{1-x^2}} \ge \sin^{-1} x \ge x$ if $0 \le x < 1$. When does the equality hold?

Solution: Let $f(x) = \sin^{-1}x,\ f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

Using Lagrange's MVT, we get

$$f(x) = f(0) + x f'(\theta x) \text{ where } 0 < \theta < 1$$

$$\text{or} \quad \sin^{-1}x = \frac{x}{\sqrt{1-\theta^2 x^2}},\ 0 < \theta < 1 \qquad \text{... (1)}$$

Since $0 < \theta < 1$, $\therefore 0 < \theta^2 < 1$

or $\quad 0 < x^2\theta^2 < x^2$ ($\because 0 \le x < 1$) or $0 \ge -\theta^2x^2 > -x^2$

or $\quad 1 \ge 1 - \theta^2x^2 > 1 - x^2$

$$\text{or} \quad 1 \ge \sqrt{1-\theta^2x^2} > \sqrt{1-x^2} \qquad \left[\begin{array}{l}\because 0<\theta<1 \text{ and } 0 \le x < 1\\ \therefore 0 \le \theta x < 1\\ \therefore 1-\theta^2x^2 > 0 \text{ and } 1-x^2>0\end{array}\right]$$

$$\text{or} \quad 1 \le \frac{1}{\sqrt{1-\theta^2x^2}} < \frac{1}{\sqrt{1-x^2}}$$

$$\text{or} \quad x \le \frac{x}{\sqrt{1-\theta^2x^2}} < \frac{x}{\sqrt{1-x^2}} \qquad [\because x \ge 0]$$

or $\quad x \le \sin^{-1}x < \dfrac{x}{\sqrt{1-x^2}}$ [From (1)]

Hence, the result and the sign of equality holds if $x = 0$.

Example 13: Show that $(x+h)^{3/2} = x^{3/2} + \frac{3}{2}x^{1/2}h + \frac{3\cdot 1}{2\cdot 2}\frac{h^2}{\lfloor 2}\frac{1}{\sqrt{x+\theta h}}$, $0 < \theta < 1$. Find θ when $x = 0$.

Solution: We know from the mean value theorem of 2nd order

$$f(x+h) = f(x) + h f'(x) + \frac{1}{2}h^2 f''(x+\theta h) \quad \text{... (1)}$$

Let us take $f(x) = x^{3/2}$, then from (1), we get

$$(x+h)^{3/2} = x^{3/2} + \frac{3}{2}x^{1/2}h + \frac{3}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{x+\theta h}}\frac{h^2}{\lfloor 2} \quad \text{... (2)}$$

When $x = 0$, we get from (2),

$$h^{3/2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{h^2}{\lfloor 2}\frac{1}{\sqrt{\theta h}}$$

or $\quad \dfrac{h^{3/2}}{h^2} = \dfrac{3}{8}\dfrac{1}{\sqrt{\theta h}} \qquad$ or $\quad \dfrac{1}{\sqrt{h}} = \dfrac{3}{8}\dfrac{1}{\sqrt{\theta h}}$

or $\quad \sqrt{\theta} = \dfrac{3}{8} \qquad$ or $\quad \theta = \dfrac{9}{64}$

Example 14: Prove that $\sin 46^\circ \sim \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$. Is this estimate high or low?

Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$

By Lagrange's MVT in $[a, a+h]$, we get

$$f(a+h) = f(a) + h f'(a+\theta h) \text{ where } 0 < \theta < 1.$$

Now, putting $a = 45^\circ$ and $h = 1^\circ$, we get $f(46^\circ) = f(45^\circ) + 1^\circ \cos(45^\circ + \theta\cdot 1^\circ)$

or $\quad \sin 46^\circ = \sin 45^\circ + \left(\dfrac{\pi}{180}\right)\cos(45^\circ + \theta^\circ) \quad$ ... (1)

$$\left(\because 1^\circ = \frac{\pi}{180}\right)$$

$$\simeq \sin 45^\circ + \left(\frac{\pi}{180}\right)\cos 45^\circ \quad [\because \theta^\circ \text{ is small}]$$

$\because \quad \sin 46^\circ \simeq \dfrac{1}{\sqrt{2}}\left(1+\dfrac{\pi}{180}\right) = \dfrac{\sqrt{2}}{2}\left(1+\dfrac{\pi}{180}\right)$

For small θ^o, $\cos(45^o + \theta^o) < \cos 45^o$. Hence, from (1), exact value of $\sin 46^o <$ approximate value of $\sin 46^o$.

$\therefore$ The estimate is high.

EXERCISES

1. Verify the Rolle's theorem in the given interval for the following functions:
   (i) $f(x) = x^2, -1 \le x \le 1$ (ii) $f(x) = 2 + (x-1)^{2/3}, 0 \le x \le 2$
   (iii) $f(x) = \sin x \cos x, 0 \le x \le \pi/2$.
   (iv) $f(x) = x^5 - 5x^4 + 6x^3 + 3x^2 - 9x, -1 \le x \le 3$.
2. Verify the Lagrange's Mean Value Theorem in the given interval for the following functions:
   (i) $f(x) = \begin{cases} x\cos\dfrac{1}{x} & \text{for } x \ne 0, -1 \le x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$
   (ii) $f(x) = x(x-1)(x-3); 0 \le x \le 4$ (iii) $f(x) = 4 - (7-x)^{2/3}, 4 \le x \le 9$
   (iv) $f(x) = |x|, -1 \le x \le 1$ (v) $f(x) = \dfrac{1}{x}, -1 \le x \le 1$
   (vi) $f(x) = (x-1)(x-2)(x-3), 0 \le x \le 4$.
3. Find the value of c in the Mean Value Theorem $f(b) - f(a) = f(b-a)f'(c)$
   (i) If $f(x) = x^2, a = 1, b = 2$ (ii) If $f(x) = \sqrt{x}, a = 4, b = 9$
   (iii) If $f(x) = (4x - 5x^2 + x^3)/(1-x), a = -2, b = 0$.
4. In the Mean Value Theorem $f(a+h) = f(a) + hf'(a + \theta h)$ if $a = 1, h = 3$ and $f(x) = \sqrt{x}$, find θ.
5. Using Mean Value Theorem prove that:
   (i) $x < \log\dfrac{1}{1-x} < \dfrac{x}{1-x}$ if $0 < x < 1$
   (ii) $1 + \dfrac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1 + \dfrac{x}{2}$ if $-1 < x < 0$
   (iii) $\dfrac{2x}{\pi} \le \sin x \le x$ if $0 \le x \le \pi/2$
   (iv) $\dfrac{2x}{1-x^2} > \log\dfrac{1+x}{1-x} > 2x$ if $0 < x < 1$
   (v) $\dfrac{x}{1+x^2} < \tan^{-1}x < x$ if $x > 0$.

6. Verify the Cauchy's Mean Value Theorem for the following functions:

(*i*) $f(x) = 1/x^2$ and $g(x) = 1/x$ in $a \le x \le b$

(*ii*) $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ in $a \le x \le b$

7. In the Mean Value Theorem $f(a + h) - f(a) = h f'(a + \theta h)$, $0 < \theta < 1$ if $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ and $a = 0$, $h = 3$, show that θ has got two values and find them.

8. Obtain the expansions of the following functions with the remainder in Lagrange's form: ($0 < \theta < 1$)

(*i*) e^x (*ii*) $\sin x$ (*iii*) $e^x \cos x$ (*iv*) $\log (x + h)$.

9. If $f(h) = f(0) + h f'(0) + \frac{h^2}{\lfloor 2} f''(\theta h)$, $0 < \theta < 1$, then prove that $\theta = \frac{9}{25}$ when $h = 1$ and $f(x) = (1 - x)^{5/2}$.

10. State the conditions under which a function $f(x)$ can be expressed in the form $f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^n}{\lfloor n} f^n(0) + \ldots + \infty$.

11. Verify the following series:

(*i*) $\log x = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \ldots$; for $0 < x \le 2$

(*ii*) $e^x \sin x = x + x^2 + \frac{x^3}{3} + \frac{x^5}{30} - \ldots$; for all x

(*iii*) $e^x \log (1 + x) = x + \frac{x^2}{\lfloor 2} + \frac{2x^3}{\lfloor 3} + \frac{9x^5}{\lfloor 5} - \ldots$

(*iv*) $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \ldots$

Answers

1. (*i*) Yes (*ii*) No (*iii*) Yes (*iv*) Yes
2. (*i*) No (*ii*) Yes (*iii*) No (*iv*) No (*v*) No (*vi*) Yes
3. (*i*) 1.5 (*ii*) 6.25 (*iii*) –1
4. $\frac{5}{12}$ 6. (*i*) Yes (*ii*) Yes 7. $\theta = \frac{1}{6}(3 \pm \sqrt{3})$

8. (i) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots + \frac{x^n}{\lfloor n} e^{\theta x}$

(ii) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots + \frac{x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + \theta x\right)$

(iii) $e^x \cos x = 1 + \frac{x}{\lfloor 1} \sqrt{2} \cos\left(1 \cdot \frac{\pi}{4}\right) + \frac{x^2}{\lfloor 2} (\sqrt{2})^2 \cos\left(2 \cdot \frac{\pi}{4}\right) + \ldots$

$$+ \frac{x^n}{\lfloor n} (\sqrt{2})^n e^{\theta x} \cos\left(\theta x + \frac{n\pi}{4}\right)$$

(iv) $\log(x + h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \ldots + (-1)^{n-1} \frac{h^n}{\lfloor n (x + \theta h)^n}$.