CHAPTER

6

Indeterminate Forms: L'Hospital's Rule

6.1 INTRODUCTION

Let $f(x)$ and $g(x)$ be two functions then the ratio $\dfrac{f(x)}{g(x)}$ is undefined at $x = a$ if $g(a) = 0$. If $g(a) = f(a) = 0$, then the ratio $f(x)/g(x)$ has the indeterminate form $\dfrac{0}{0}$ at $x = a$ but $f(x)/g(x)$ may have a definite value as $x \to a$. Other indeterminate forms are ∞/∞, $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0 and 1^∞. We now discuss some of these forms below.

6.2 THE FORM $\left(\dfrac{0}{0}\right)$; L' HOSPITAL'S RULE

If $f(x)$ and $g(x)$ be two functions such that

(*i*) both are continuous in $a \le x \le a + h$

(*ii*) both are derivable in $a < x \le a + h$

and (*iii*) $f(a) = g(a) = 0$,

then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \text{ provided } g'(a) \neq 0$$

This rule admits of the following generalization: If $f(x)$ and $g(x)$ be two functions such that

(*i*) $f^{(n-1)}(x)$ and $g^{(n-1)}(x)$ are continuous at $x = a$

(*ii*) $f^{(n)}(x)$ and $g^{(n)}(x)$ both exist at $x = a$

and (*iii*) $f(a) = f^{1}(a) = \ldots = f^{(n-1)}(a) = 0$ and $g(a) = g'(a) = \ldots = g^{n-1}(a) = 0$,

then
$$\lim_{x \to a} \frac{f(a)}{g(a)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)} \text{ provided } g^{(n)}(a) \neq 0$$

The proofs of the both are outside the scope of this text.

log1 = 0

Example 1: Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

Solution: Here $\lim\limits_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos x}{1 - \cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} \frac{e^x - \cos x\, e^{\sin x}}{1 - \cos x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos^2 x + \sin x\, e^{\sin x}}{\sin x} \qquad \text{[By L'Hospital rule)}$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} \sin x}{\sin x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{\sin x} \cos^3 x + e^{\sin x}\, 2\cos x \sin x + e^{\sin x} \cos x + e^{\sin x} \sin x \cos x}{\cos x}$$

[By L'Hospital rule]

$$= \frac{1 - 1 + 0 + 1 + 0}{1} = \frac{1}{1} = 1$$

Example 2: Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$.

Solution: Here $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x} \qquad \text{[By L'Hospital rule]} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x}{\sin x} \qquad \text{[By L'Hospital rule]} \left(\text{again } \frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\sec^4 x + 4\sec^2 x \tan^2 x}{\cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{2 + 0}{1} = 2.$$

Example 3: Evaluate $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$.

Solution: Here $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x} \quad \text{[By L'Hospital rule]} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} = \frac{-\sin x - \sin x - x\cos x + \left(\frac{1}{1+x}\right)^2}{2} \quad \text{[By L'Hospital rule]}$$

$$= \frac{-0-0-0+1}{2} = \frac{1}{2}$$

Example 4: Evaluate $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

Solution: Here $\lim_{x\to 0} \frac{x-\sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{+\sin x}{6x} \quad \text{[By L'Hospital rule]} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6} \quad \text{[By L'Hospital rule]}$$

Example 5: Evaluate $\lim_{x\to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$.

Solution: Here $\lim_{x\to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{e^x + \cos x}{\frac{1}{1+x}} \quad \text{[By L'Hospital rule]} = \frac{1+1}{\frac{1}{1+0}} = \frac{2}{1} = 2$$

6.3 THE FORM $\frac{\infty}{\infty}$

Let $f(x)$ and $g(x)$ be the two functions such that (*i*) both are continuous in $a \le x \le a+h$,

(*ii*) both are derivable in $a < x \le a+h$,

(*iii*) $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$,

and (*iv*) $\lim_{x \to a} \dfrac{f'(x)}{g'(x)} = l < \infty$, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = l$.

Example 1: Evaluate $\lim_{x \to \infty} \dfrac{2x^2 - 4}{x^2 + 2x + 3}$.

Solution: Here $\lim_{x \to \infty} \dfrac{2x^2 - 4}{x^2 + 2x + 3}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty} \frac{4x}{2x+2} \quad \left(\frac{\infty}{\infty} \text{ form}\right).$$

$$= \lim_{x \to \infty} \frac{4}{2} = 2.$$

Example 2: Evaluate $\lim_{x \to \infty} \dfrac{\log x}{x}$.

log ∞ = ∞

Solution: Here $\lim_{x \to \infty} \dfrac{\log x}{x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x \to \infty} \frac{1/x}{1} = 0$$

$\frac{1}{\infty} = 0$

Example 3: Evaluate $\lim_{x \to 0} \dfrac{\log \sin x}{\cot x}$.

Solution: Here $\lim_{x \to 0} \dfrac{\log \sin x}{\cot x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right) = \lim_{x \to 0} \dfrac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

log0 = undefined

$$= \lim_{x \to 0} \frac{\cos x}{-\operatorname{cosec} x} = \lim_{x \to 0} -\frac{1}{2}\, 2 \sin x \cos x$$

$$= \lim_{x \to 0} \left(\frac{1}{2}\right) \sin 2x = 0.$$

Example 4: Evaluate $\lim_{x \to \infty} \dfrac{x^n}{e^x}$ $(n > 0)$.

Solution: Here $\lim_{x\to\infty} \frac{x^n}{e^x}$ $\left(\frac{\infty}{\infty}\text{ form}\right) = \lim_{x\to\infty} \frac{n\,x^{n-1}}{e^x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$\lim_{x\to\infty} \frac{n(n-1)\,x^{n-2}}{e^x} \qquad \left(\text{Again } \frac{\infty}{\infty}\text{ form}\right) = \ldots =$$

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2)\ldots 3.2.1}{e^x} = \frac{\lfloor n}{\infty} = 0.$$

6.4 THE FORM (0 . ∞)

The form $0 \,.\, \infty$ can be reduced to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Let $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)\,.\,g(x)$ $(0\,.\,\infty$ form) can be written as $\lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ $\left(\frac{0}{0}\text{ form}\right)$ or $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

Example 1: Evaluate $\lim_{x\to 0} \cot x \log\left(\frac{1+x}{1-x}\right)$.

Solution: Here $\lim_{x\to 0} \cot x \log\left(\frac{1+x}{1-x}\right)$ $\qquad (0\,.\,\infty \text{ form})$

$$= \lim_{x\to 0} \frac{\log(1+x) - \log(1-x)}{\tan x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x} + \frac{1}{1-x}}{\sec^2 x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{\frac{1}{1+0} + \frac{1}{1-0}}{1} = \frac{1+1}{1} = 2.$$

Example 2: Evaluate $\lim_{x\to 0} x \log \tan x$.

Solution: Here $\lim_{x\to 0} x \log \tan x$ $\qquad (0\,.\,\infty \text{ form})$

$$= \lim_{x\to 0} \frac{\log \tan x}{1/x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sec^2 x/\tan x}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{-x^2 \sec^2 x}{\tan x} = \lim_{x\to 0} \frac{-x^2}{\cos^2 x \frac{\sin x}{\cos x}}$$

$$= \lim_{x\to 0} \frac{-2x^2}{\sin 2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-4x}{2\cos 2x} \qquad \text{[By L'Hospital rule]} = \frac{-4.0}{2} = 0$$

Example 3: Evaluate $\lim_{x\to 0} x^2 \log (x^2)$.

Solution: Here $\lim_{x\to 0} x^2 \log (x^2)$ $(0 . \infty \text{ form})$

$$= \lim_{x\to 0} \frac{\log (x^2)}{\frac{1}{x^2}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{x^2} 2x}{-\frac{2}{x^3}} = \lim_{x\to 0} -x^2 = 0.$$

Example 4: Evaluate $\lim_{x\to 0} \sin x \log (x^2)$.

Solution: Here $\lim_{x\to 0} \sin x \log (x^2)$ $(0 . \infty \text{ form})$

$$= \lim_{x\to 0} \frac{\log x^2}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{2x}{x^2}}{-\operatorname{cosce} x . \cot x} = \lim_{x\to 0} \frac{-2\sin^2 x}{x \cos x} = \lim_{x\to 0} \frac{-2 \sin x \tan x}{x}$$

$$\left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\cos x \tan x - 2 \sin x \sec^2 x}{1} \qquad \text{(By L'Hospital rule)}$$

$$= \frac{0-0}{1} = 0.$$

6.5 THE FORM ($\infty - \infty$)

The form $\infty - \infty$ can be reduced to $\frac{0}{0}$ form, then we apply the L'Hospital rule.

Let $f(x)$ and $g(x) \to \infty$ as $x \to a$, then $\lim_{x \to a} \{f(x) - g(x)\}$ ($\infty - \infty$ form) can be

written as $\lim_{x \to a} \dfrac{\left[\dfrac{1}{g(x)} - \dfrac{1}{f(x)}\right]}{\dfrac{1}{f(x)} \cdot \dfrac{1}{g(x)}}$ $\left(\frac{0}{0} \text{ form}\right)$

Example 1: Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution: Here $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \frac{\sin x - x}{x \sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{\sin x + x \cos x} \quad \text{[By L'Hospital rule]} \left(\text{again } \frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x}{\cos x - x \sin x + \cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{0}{1 - 0 + 1} = \frac{0}{2} = 0.$$

Example 2: Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

Solution: Here $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x - 2x}{2x \sin^2 x + 2 \sin x \cos x\, x^2} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} \frac{\sin 2x - 2x}{2x \sin^2 x + \sin 2x\, x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 4x \sin x \cos x + 2\cos 2x\, x^2 + 2x \sin 2x}$$

[By L'Hospital rule] $\left(\text{again } \frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{-4\sin 2x}{2\sin 2x + 4[\sin 2x + 2x\cos 2x] + [4x\cos 2x - 4x^2 \sin 2x]}$$

[By L'Hospital rule]

$$= \lim_{x \to 0} \frac{-4\sin 2x}{6\sin 2x + 12x\cos 2x - 4x^2 \sin 2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-8\cos 2x}{12\cos 2x + 12\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 2x}$$

[By L'Hospital rule]

$$= \frac{-8}{12 + 12 - 0 - 0 - 0} = \frac{-8}{24} = \frac{-1}{3}.$$

Example 3: Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solution: Here $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x}. \qquad \text{[By L'Hospital rule]} = 0$$

Example 4: Evaluate $\lim_{x \to 1} \left\{\frac{x}{x-1} - \frac{1}{\log x}\right\}$.

Solution: Here $\lim_{x \to 1} \left\{\frac{x}{x-1} - \frac{1}{\log x}\right\}$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 1} \frac{x \log x - (x-1)}{(x-1)\log x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{\frac{x}{x} + \log x - 1}{\log x + \frac{x-1}{x}} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 1} \frac{x \log x}{x \log x + x - 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{\frac{x}{x} + \log x}{\frac{x}{x} + \log x + 1} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 1} \frac{1 + \log x}{2 + \log x} = \frac{1}{2}.$$

Example 5: Evaluate $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{\log(1+x)}{x^2}\right\}$

Solution: Here $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{\log(1+x)}{x^2}\right\}$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \frac{x - \log(1+x)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} \frac{x}{2x(1+x)} = \lim_{x \to 0} \frac{1}{2(1+x)} = \frac{1}{2}.$$

Example 6: Evaluate $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{2}{x(e^x + 1)}\right\}$.

Solution: Here $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{2}{x(e^x + 1)}\right\}$ $(\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \frac{(e^x + 1) - 2}{x(e^x + 1)} = \lim_{x \to 0} \frac{e^x - 1}{x(e^x + 1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x}{(e^x + 1) + xe^x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{1}{1+1} = \frac{1}{2}.$$

6.6 THE FORMS 0^0, $1^{\pm\infty}$ AND ∞^0

For the three exponential forms 0^0, $1^{\pm\infty}$ and ∞^0, we find the limits of the functions of the form $y = \{f(x)\}^{g(x)}$ as $x \to a$ [That is when (*i*) both $f(x)$ and $g(x) \to 0$ as $x \to a$, (*ii*) $f(x) \to 1$ and $g(x) \to \pm\infty$ as $x \to a$, (*iii*) $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$] by taking their logarithms and reducing it to the form $0 . \infty$.

Example 1: Evaluate $\lim\limits_{x\to 0} (\sin x)^{2\tan x}$.

Solution: Let $y = (\sin x)^{2\tan x}$, then $\log y = 2\tan x \log \sin x$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} 2\tan x \log \sin x \qquad (0 . \infty \text{ form})$$

$$= \lim_{x\to 0} \frac{2\log \sin x}{\cot x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\dfrac{\cos x}{\sin x}}{(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x\to 0} -2\sin x \cos x = \lim_{x\to 0} -\sin 2x = 0$$

Since $\lim\limits_{x\to 0} \log y = \log\left(\lim\limits_{x\to 0} y\right)$, then $\log\left(\lim\limits_{x\to 0} y\right) = 0 = \log 1$

$$\therefore \quad \lim_{x\to 0} y = 1$$

or $\lim\limits_{x\to 0} (\sin x)^{2\tan x} = 1$.

Example 2: Evaluate: $\lim\limits_{x\to \frac{\pi}{2}} (\cos x)^{\frac{\pi}{2} - x}$

Solution: Let $y = (\cos x)^{\frac{\pi}{2} - x}$, then $\log y = \left(\frac{\pi}{2} - x\right) \log \cos x$

$$\therefore \quad \lim_{x\to \frac{\pi}{2}} \log y = \lim_{x\to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \log \cos x \qquad (0 . \infty \text{ form})$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\log \cos x}{\dfrac{1}{\frac{\pi}{2} - x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\dfrac{\sin x}{\cos x}}{\left(\dfrac{1}{\frac{\pi}{2}-x}\right)^2} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x\left(\frac{\pi}{2}-x\right)^2}{\cos x} = \frac{0}{1} = 0$$

Since $\lim_{x\to\frac{\pi}{2}} \log y = \log\left(\lim_{x\to\frac{\pi}{2}} y\right)$, then $\log\left(\lim_{x\to\frac{\pi}{2}} y\right) = 0 = \log 1$

$$\lim_{x\to\frac{\pi}{2}} 4 = 1 \text{ or } \lim_{x\to\frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x} - 1$$

Example 3: Evaluate $\lim_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$.

Solution: Let $y = \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$, then $\log y = \dfrac{1}{x} \log\left(\dfrac{\tan x}{x}\right)$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{x}{\tan x}\,\dfrac{x\sec^2 x - \tan x}{x^2}\right)}{1} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \frac{2x - \sin 2x}{x \sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x} \quad \text{[By L'Hospital rule]} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{4\sin 2x}{2\cos 2x + 2\cos 2x - 4x\sin 2x} \quad \text{[By L'Hospital rule]}$$

$$= \frac{0}{2+2-0} = \frac{0}{4} = 0$$

Since $\lim\limits_{x \to 0} \log y = \log\left(\lim\limits_{x \to 0} y\right)$, then $\log\left(\lim\limits_{x \to 0} y\right) = 0 = \log 1$

$$\therefore \quad \lim_{x \to 0} y = 1 \Rightarrow \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1.$$

Example 4: Evaluate $\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

Solution: Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$. Then $\log y = \frac{1}{x} \log\left(\frac{\sin x}{x}\right)$

$$\therefore \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{1}{x} \log\left(\frac{\sin x}{x}\right) = \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\left[\left(\frac{x}{\sin x}\right) \frac{x \cos x - \sin x}{x^2}\right]}{1} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x \sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} \frac{-x \sin x}{\sin x + x \cos x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{0 - 0}{1 + 1 - 0} = \frac{0}{2} = 0$$

Since $\lim\limits_{x \to 0} \log y = \log\left(\lim\limits_{x \to 0} y\right)$, then $\log\left(\lim\limits_{x \to 0} y\right) = 0 = \log 1$

$$\therefore \quad \lim_{x \to 0} y = 1 \Rightarrow \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1.$$

Example 5: Evaluate $\lim\limits_{x \to 0} \left(1 + \frac{1}{x}\right)^{x}$.

log 1 = 0 , log 0 = ∞ , log ∞ = ∞

Solution: Let $y = \left(1+\frac{1}{x}\right)^x$, then $\log y = x \log\left(1+\frac{1}{x}\right) = \frac{\log(x+1) - \log x}{\frac{1}{x}}$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(x+1) - \log(x)}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{x+1} - \frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}\left(-\frac{x^2}{1+x} + x\right) = 0$$

Since $\lim_{x\to 0} \log y = \log\left(\lim_{x\to 0} y\right)$, then $\log\left(\lim_{x\to 0} y\right) = 0 = \log 1$

$$\therefore \quad \lim_{x\to 0} y = 1 \Rightarrow \lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = 1.$$

Example 6: Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$.

Solution: Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$, then $\log y = \tan x \log\left(\frac{1}{x^2}\right) = \frac{-\log x^2}{\cot x}$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{-2\log x}{\cot x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= -2 \lim_{x\to 0} \frac{\frac{1}{x}}{(-\operatorname{cosec}^2 x)} = 2 \lim_{x\to 0} \frac{\sin^2 x}{x}$$

$$= 2 \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot x = 2 \,.\, 1 \,.\, 1 \,.\, 0 = 0$$

Since $\lim_{x\to 0} \log y = \log\left(\lim_{x\to 0} y\right)$, $\log\left(\lim_{x\to 0} y\right) = 0 = \log 1$

$$\therefore \quad \lim_{x\to 0} y = 1 \Rightarrow \lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x} = 1.$$

Example 7: Find the values of a, b and c such that

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2, \text{ applying L'Hospital rule.}$$

Solution: Since the given function has finite limit as $x \to 0$, and denominator $\to 0$ as $x \to 0$, the numerator should also $\to 0$ as $x \to 0$ which gives

$$a - b + c = 0 \qquad ...(1)$$

$$\therefore \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x} \qquad \text{[By L'Hospital rule]} = 2$$

Again, this function has finite limit 2 as $x \to 0$ and the denominator $\to 0$ as $x \to 0$, then the numerator should also $\to 0$ as $x \to 0$ which gives

$$a - c = 0 \qquad ...(2)$$

$$\therefore \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x a + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{a+b+c}{2} = 2 \qquad (\because \text{ given limit is 2})$$

$\therefore \ a + b + c = 4$ ...(3)

From (1) and (2), we get $a + c = b \Rightarrow b = 2a$ ($\because a = c$) ...(4)

From (2), (3) and (4), we get $a + 2a + a = 4 \Rightarrow a = 1$

$\therefore \ b = 2$ and $c = 1$

Hence, $a = 1$, $b = 2$ and $c = 1$.

Example 8: Find the values of a and b such that

$\lim_{x \to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3} = \frac{1}{3}$ (Assume that L'Hospital's rule is applicable)

Solution: Here $\lim_{x \to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{(1 - a\cos x) + ax\sin x + b\cos x}{3x^2} \qquad \text{[By L'Hospital rule]}$$

Since this function has finite limit $\frac{1}{3}$ as $x \to 0$ and denominator $\to 0$ as $x \to 0$, numerator should also $\to 0$ as $x \to 0$ which gives

$$1 - a + b = 0 \qquad ...(1)$$

$$\therefore \quad \lim_{x \to 0} \frac{(1 - a\cos x) + ax\sin x + b\cos x}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{a\sin x + a\sin x + ax\cos x - b\sin x}{6x}$$

[By L'Hospital rule] $\left(\text{again } \frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{2a\cos x + a\cos x - ax\sin x - b\cos x}{6} \qquad \text{[By L'Hospital rule]}$$

$$= \frac{3a - b}{6} = \frac{1}{3} \qquad \left(\because \text{Given limit is } \frac{1}{3}\right)$$

or $3a - b = 2$ ...(2)

From (1) and (2), we get $1 - a + 3a - 2 = 0$ or $a = 1/2$

$$\therefore \; b = a - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$a = \frac{1}{2} \text{ and } b = -\frac{1}{2}.$$

6.7 MISCELLANEOUS EXAMPLES

Example 1: Find the following limit:

(*i*) $\lim_{x \to 0} \frac{e^{2x} - 1}{\log(1 + x)}$ (*ii*) $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{\tan^3 x}$

(*iii*) $\lim_{x \to 0} \frac{x - \sin^{-1} x}{\sin^3 x}$ (*iv*) $\lim_{x \to 0} \frac{\sin\log(1 + x)}{\log(1 + \sin x)}$.

Solution: (*i*) $\lim_{x \to 0} \frac{e^{2x} - 1}{\log(1 + x)}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{2e^{2x}}{\frac{1}{1 + x}} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to 0} 2(1 + x)e^{2x} = 2(1 + 0)e^0 = 2$$

(*ii*) $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{\tan^3 x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{3\tan^2 x\sec^2 x} \qquad \text{[By L'Hospital rule]} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6\tan x\sec^4 x + 6\tan^3 x\sec x}$$

$$= \lim_{x\to 0} \frac{-2\sin x + 8\sin x\cos x}{6\frac{\sin x}{\cos x}\sec^4 x + 6\tan^2 x\frac{\sin x}{\cos x}\sec x}$$

$$= \lim_{x\to 0} \frac{-2 + 8\cos x}{6\sec^5 x + 6\tan^2 x\sec^2 x} = \frac{-2+8}{6+0} = \frac{6}{6} = 1$$

(*iii*) $$\lim_{x\to 0} \frac{x - \sin^{-1} x}{\sin^3 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1 - \frac{1}{\sqrt{1-x^2}}}{3\sin^2 x\cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \frac{1 - (1-x^2)^{-\frac{1}{2}}}{3\sin^2 x\cos x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-x(1-x^2)^{-\frac{3}{2}}}{2.3.\sin x\cos^2 x - 3\sin^3 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

[By L'Hospital rule]

$$= \lim_{x\to 0} \frac{-(1-x^2)^{-\frac{3}{2}} - 3x^2(1-x^2)^{-\frac{5}{2}}}{6\cos^3 x - 12\cos x\sin^2 x - 9\sin^2 x\cos x}$$

$$= \frac{-1-0}{6.1 - 12.0 - 9.0} = \frac{-1}{6}$$

(*iv*) $$\lim_{x\to 0} \frac{\sin\log(1+x)}{\log(1+\sin x)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos\log(1+x)}{(1+x).\frac{1}{1+\sin x}.\cos x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \frac{(1+\sin x)\cos\log(1+x)}{(1+x)\cos x} = \frac{(1+0)\cos 0}{(1+0)\cos 0} = 1.$$

Example 2: Show that $\lim_{x\to 0} \log_{\tan^2 x} \tan^2 2x = 1$

Solution: Since $\log_n m = \dfrac{\log_e m}{\log_e n}$, the required limit

$$= \lim_{x\to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)} = \lim_{x\to 0} \frac{2\log(\tan 2x)}{2\log(\tan x)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x}{\tan 2x} \times \frac{\tan x}{\sec^2 x} = \lim_{x\to 0} \frac{\sec^2 2x}{\sec^2 x} \cdot \frac{\dfrac{\tan x}{x}}{\dfrac{\tan 2x}{2x}}$$

[By L'Hospital rule]

$$= 1 \times 1 \left(\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= 1$$

Example 3: Show that $\lim_{x\to 0} x \log \sin^2 x = 0$

Solution: Now $\lim_{x\to 0} x \log \sin^2 x$ $\qquad (0 \times \infty \text{ form})$

$$= \lim_{x\to 0} \frac{\log \sin^2 x}{\dfrac{1}{x}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\dfrac{2\sin x \cos x}{\sin^2 x}}{-\dfrac{1}{x^2}} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \left\{-2\frac{\cos x}{\sin x} \times x^2\right\} = \lim_{x\to 0} \left\{x \cos x \frac{1}{\dfrac{\sin x}{x}} (-2)\right\}$$

$$= (-2)(0 \,.\, 1)\frac{1}{1} = 0.$$

Example 4. Show that (*i*) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$ (*ii*) $\lim_{x\to 0} (\cot^2 x)^{\sin x} = 1$

(*iii*) $\lim_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$ (*iv*) $\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$.

Solution: (*i*) Let $y = (\sin x)^{\tan x}$ $\therefore$ $\log y = \tan x \log \sin x$

$$\therefore \lim_{x \to \frac{\pi}{2}} \log y = \lim_{x \to \frac{\pi}{2}} \tan x \log \sin x$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{\cot x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{-\text{cosec}^2 x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x \to \frac{\pi}{2}} \sin x \cos x = 0$$

$$\because \quad \lim_{x \to \frac{\pi}{2}} \log y = \log\left(\lim_{x \to \frac{\pi}{2}} y\right)$$

$$\therefore \quad \log\left(\lim_{x \to \frac{\pi}{2}} y\right) = 0 = \log 1$$

or $$\lim_{x \to \frac{\pi}{2}} y = 1$$

or $$\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

(*ii*) Let $y = (\cot^2 x)^{\sin x}$ $\therefore$ $\log y = \sin x \log \cot^2 x$

$$\therefore \lim_{x \to 0} \log y = \lim_{x \to 0} \sin x \log \cot^2 x$$

$$= \lim_{x \to 0} \frac{\log \cot^2 x}{\text{cosec}\, x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{2}{\cot x}(-\text{cosec}^2 x)}{-\text{cosec}\, x \cot x} = \lim_{x \to 0} \frac{2\,\text{cosec}\, x}{\cot^2 x}$$

$$= \lim_{x \to 0} \frac{2 \sin x}{\cos^2 x} = 0$$

Since $\lim_{x \to 0} \log y = \log\left(\lim_{x \to 0} y\right)$, we have

$$\log\left(\lim_{x\to 0} y\right) = 0 = \log 1$$

or $\quad \lim_{x\to 0} y = 1$ or $\lim_{x\to 0} (\cot^2 x)^{\sin x} = 1$

(*iii*) Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $\therefore \log y = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$

$$\therefore \quad \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \left[\frac{0}{0} \text{ Form}, \because \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

[By L'Hospital rule]

$$= \lim_{x\to 0} \frac{\frac{x}{\tan x} \times \frac{x\sec^2 x - \tan x}{x^2}}{2x} = \lim_{x\to 0} \frac{x - \frac{1}{2}\sin 2x}{x^2 \sin 2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1 - \cos 2x}{2x^2 \cos 2x + 2x \sin 2x} \quad \text{[By L'Hospital rule]} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\sin 2x}{-4x^2 \sin 2x + 4x\cos 2x + 2\sin 2x + 4x\cos 2x}$$

[By L'Hospital rule] $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{4\cos 2x}{-8x^2 \cos 2x - 8x\sin 2x + 8\cos 2x - 16x\sin 2x + 4\cos 2x}$$

[By L'Hospital rule]

$$= \frac{4}{8+4} = \frac{1}{3}$$

$$\therefore \quad \lim_{x\to 0} \log y = \log\left(\lim_{x\to 0} y\right), \therefore \log\left(\lim_{x\to 0} y\right) = \frac{1}{3}$$

$$\therefore \quad \lim_{x\to 0} y = \lim_{x\to 0} \left(\frac{\tan}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$$

(*iv*) Let $y = (\cos x)^{\cot^2 x}$, $\log y = \cot^2 x \log \cos x$

$$\therefore \quad \lim_{x\to 0} \log y = \lim_{x\to 0} \cot^2 x \log \cos x$$

$$= \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\tan x}{2 \tan x \sec^2 x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \frac{-1}{2 \sec^2 x} = -\frac{1}{2}$$

$$\because \quad \lim_{x\to 0} \log y = \log\left(\lim_{x\to 0} y\right)$$

$$\therefore \quad \log\left(\lim_{x\to 0} y\right) = -\frac{1}{2} \text{ or } \lim_{x\to 0} y = e^{-\frac{1}{2}}$$

or $\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$.

Example 5: If $\lim_{x\to 0} \dfrac{a \sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a, and the limit.

Solution: $$\lim_{x\to 0} \frac{a \sin x - \sin 2x}{\tan^3 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{a \cos x - 2 \cos 2x}{3 \tan^2 x \sec^2 x} \qquad \text{[By L'Hospital rule]}$$

When $x \to 0$, the denominator $3 \tan^2 x \sec^2 x \to 0$; since the given limit is finite, the numerator $(a \cos x - 2 \cos 2x)$ should tend to zero as $x \to 0$.

$$\therefore \quad a - 2 = 0, \text{ i.e., } a = 2$$

Putting $a = 2$, the given limit becomes

$$= \lim_{x\to 0} \frac{2 \cos x - 2 \cos 2x}{3 \tan^2 x (1 + \tan^2 x)}$$

$$= \lim_{x\to 0} \frac{2 \cos x - 2 \cos 2x}{3 \tan^2 x + 3 \tan^4 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2 \sin x + 4 \sin 2x}{6 \tan x \sec^2 x + 12 \tan^3 x \sec^2 x} \qquad \text{[By L'Hospital rule]}$$

$$= \lim_{x\to 0} \frac{-\sin x + 2 \sin 2x}{(1 + \tan^2 x)(3 \tan x + 6 \tan^3 x)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x + 4\cos 2x}{3\sec^2 x + \text{terms involving product of } \tan x \text{ and its higher power}}$$

$$= \frac{-1+4}{3} = 1.$$

Example 6: Find the values of a and b in order that

$$\lim_{\theta \to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1.$$

Solution:
$$\lim_{\theta \to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{\theta \to 0} \frac{(1 + a\cos\theta) - a\theta\sin\theta - b\cos\theta}{3\theta^2} \qquad \text{...(A)}$$

When $\theta \to 0$, the denominator $3\theta^2 \to 0$, since the given limit is finite, then numerator $(1 + a\cos\theta) - a\theta\sin\theta - b\cos\theta$ should be zero as $\theta \to 0$, we get

$$1 + a - 0 - b = 0 \text{ or } 1 + a - b = 0 \qquad \text{...(1)}$$

and hence by L'Hospital rule, then (A) becomes

$$= \lim_{\theta \to 0} \frac{-a\sin\theta - a\sin\theta - a\theta\cos\theta + b\sin\theta}{6\theta} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{\theta \to 0} \frac{-2a\cos\theta - a\cos\theta + a\theta\sin\theta + b\cos\theta}{6}$$

[By L'Hospital rule]

$$= \frac{-3a+b}{6}$$

$$\because \lim_{\theta \to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1 \therefore \frac{-3a+b}{6} = 1$$

or
$$-3a + b = 6 \qquad \text{...(2)}$$

From (1) and (2), $-3a + 1 + a = 6, \Rightarrow a = -\dfrac{5}{2}$

and
$$b = 1 + a = 1 - \frac{5}{2} = \frac{-3}{2}$$

$$\therefore \quad a = \frac{-5}{2} \text{ and } b = \frac{-3}{2}.$$

EXERCISES

Evaluate the following limits:

1. $\lim_{x\to 0} \dfrac{x - \sin x \cos x}{x^3}$

2. $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2\cos x}{x \sin x}$

3. $\lim_{x\to 1} \operatorname{cosec}(\pi x) \log x$

4. $\lim_{x\to 2} \left(\dfrac{4}{x^2 - 4} - \dfrac{1}{x - 2}\right)$

5. $\lim_{x\to 0} x^{2\sin x}$

6. $\lim_{x\to 1-} (1 - x^2)^{\frac{1}{\log(1-x)}}$

7. $\lim_{x\to\infty} \left(1 + \dfrac{1}{x^2}\right)^x$

8. $\lim_{x\to 0} x^{2x}$

9. $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$

10. $\lim_{x\to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right)$

11. $\lim_{x\to\infty} \left\{x - x^2 \log\left(1 + \dfrac{1}{x}\right)\right\}$

12. $\lim_{x\to 1} x^{\frac{1}{1-x}}$

13. $\lim_{x\to 0} \dfrac{\log \sin 2x}{\log \sin x}$

14. $\lim_{x\to 0} \dfrac{\log \sin x}{\cot x}$

15. $\lim_{x\to 0} (1 + \sin x)^{\cot x}$

16. $\lim_{x\to 0} \dfrac{x\cos x - \log(1 + x)}{x^2}$

17. $\lim_{x\to 0} \left(\dfrac{1}{x} - \cot x\right)$

18. $\lim_{x\to 0+} \dfrac{\log x - \cot \dfrac{\pi x}{2}}{\cot \pi x}$

19. $\lim_{x\to \frac{\pi}{2}} (\cos x)^{\cos^2 x}$

Answers

1. $\dfrac{2}{3}$ 2. 2 3. $\left(-\dfrac{1}{\pi}\right)$ 4. $\left(-\dfrac{1}{4}\right)$ 5. 1 6. e

7. 1 8. 1 9. 0 10. $-\dfrac{1}{3}$ 11. $\dfrac{1}{2}$ 12. $\dfrac{1}{e}$

13. 1 14. 0 15. 1 16. $\dfrac{1}{2}$ 17. 0 18. 2

19. 1