Linguistic variables and hedges:

- **Linguistic Variables**: Linguistic variables are the **input or output** variables of the system whose values are words from a natural language, instead of numerical values. The fuzzy set theory is rooted in linguistic variables.
- A linguistic variable is a fuzzy variable. For example:
  - The statement "**John is tall**" implies that the ***linguistic variable*** **John** takes the ***linguistic Value*** **tall**.
- The range of possible values of a linguistic variable represents the **universe of discourse** of that variable.
  - For example, the universe of discourse of the linguistic variable **speed** might be the range between **0 and 220 km/h** and may include such fuzzy subsets as **very slow, slow, medium, fast, and very fast**.
- A linguistic variable carries with it the concept of **fuzzy set qualifiers**, called **hedges**.
- **Hedges** are terms that modify the shape of fuzzy sets. They include adverbs such as **very**, **somewhat**, **quite**, **more or less** and **slightly**.

Hedges are useful as **operations**, but they can also break down continuums into fuzzy intervals:

- For example, the following hedges could be used to describe temperature: very cold, moderately cold, slightly cold, neutral, slightly hot, moderately hot and very hot.

Types of operators

1. Equality
2. Complement
3. Intersection
4. Union
5. Algebraic product
6. Multiplication of fuzzy set with crisp number
7. Power of fuzzy set
8. Algebraic sum

10. Bounded sum
11. Bounded difference
12. Cartesian product
13. Composition

1. Equal fuzzy sets

Two fuzzy sets A(x) and B(x) are said to be equal, if $\mu_A(x) = \mu_B(x)$ for all **x** ∈ X. It is expressed as follows

$$A(x) = B(x), \text{ if } \mu_A(x) = \mu_B(x)$$

Note: Two fuzzy sets A(x) and B(x) are said to be unequal, if $\mu_A(x) \neq \mu_B(x)$ for at least **x** ∈ X.

Example:

$A(x) = \{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x) = \{(x_1,0.1),(x_2,0.5),(x_3,0.3),(x_4,0.6)\}$

As $\mu_A(x) \neq \mu_B(x)$ for different $x \in X$, $A(x) \neq B(x)$

Complement

- Crisp Sets: Who does not belong to the set?
- Fuzzy Sets: How much do elements not belong to the set?

- The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.

- If A is the fuzzy set, its complement $\neg A$ can be found as follows:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

- **Example**: given a fuzzy set of tall men
  Tall men **(0/180, 0.25/182.5, 0.5/185, 0.75/187.5, 1/190)**
  - the fuzzy set of NOT tall men will be:
  NOT tall men **(1/180, 0.75/182.5, 0.5/185, 0.25/187.5, 0/190)**

Definition: **(Intersection)** The membership function $\mu_{C(x)}$ of the set $C = A \cap B$ is defined as $\mu_{C(x)} = \min \{\mu_{A(x)}, \mu_{B(x)}\}$, $x \in X$.

Definition: **(Union)** The membership function $\mu_{C(x)}$ of $C = A \cup B$ is defined as $\mu_{C(x)} = \max\{\mu_{A(x)}, \mu_{B(x)}\}$, $x \in X$.

Definition: **(Complement)** Membership function of the complement of a fuzzy set A, $\mu_{A'(x)}$ is defined as $\mu_{A'(x)} = [1 - \mu_{A(x)}]$, $x \in X$.

Example: Let X = { 1,2,3,4,5,6,7}

A = { (3, 0.7), (5, 1), (6, 0.8) } and B = {(3, 0.9), (4, 1), (6, 0.6) }

$A \cap B$ = { (3, 0.7), (6, 0.6) }

$A \cup B$ = { (3, 0.9), (4, 1), (5, 1), (6, 0.8) }

A' = { (1, 1), (2, 1), (3, 0.3), (4, 1), (6, 0.2), (7, 1) }

5. Algebraic product of fuzzy sets

The Algebraic product of two fuzzy sets A(x) and B(x) for all **x** ∈ X, is denoted by **A(x).B(x)** and defined as follows

$$A(x).B(x) = \{(x, \mu_A(x).\mu_B(x)), x \in X\}$$

Example

$A(x) = \{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$

$B(x) = \{(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)\}$

$A(x).B(x) = \{(x_1,0.05),(x_2,0.14),(x_3,0.24),(x_4,0.36)\}$

6. Multiplication of fuzzy sets by a crisp number

The product of fuzzy set A(x) and a crisp number 'd' is expressed as follows

$$A(x).\mathbf{d} = \{(x, d . \mu_A(x)), x \in X\}$$

Example

Let us consider a fuzzy set A(x) such that

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$

$d = 0.2$

then $d.A(x) = \{(x_1,0.02),(x_2,0.04),(x_3,0.06),(x_4,0.08)\}$

7. Power of a fuzzy set

The p-th power of a fuzzy set A(x) yields another fuzzy set $A^p(x)$, whose membership value can be determined as follows

$$\mu_{A^p}(x) = \{\mu_A(x)\}^P, \mathbf{x} \in X \}$$

Example:

Let us consider a fuzzy set A(x)

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$

P=2

Then $A^2(x) = \{(x_1,0.01),(x_2,0.04),(x_3,0.09),(x_4,0.16)\}$

8. Algebraic sum of two fuzzy sets

The Algebraic sum of two fuzzy sets A(x) and B(x) for all $\mathbf{x} \in X$, is denoted by **A(x)+B(x)** and defined as follows

$$A(x)+B(x)= \{(x,\mu_{A+B}(x), x \in X \}$$

Where $\mu_{A+B}(x) = \mu_A(x)+\mu_B(x) - \mu_A(x).\mu_B(x)$

Example:

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x)=\{(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)\}$

Now $(x)+B(x) = \{(x_1,0.55),(x_2,0.76),(x_3,0.86),(x_4,0.94)\}$

9. Bounded sum of two fuzzy sets

The bounded sum of two fuzzy sets A(x) and B(x) for all $\mathbf{x} \in X$, is denoted by $A(x) \oplus \boldsymbol{B}(\boldsymbol{x})$ and defined as follows

$$A(x) \oplus \boldsymbol{B}(\boldsymbol{x}) = \{(x,\mu_{A \oplus B}(x), \mathbf{x} \in X \}$$

Where $\mu_{A \oplus B}(x)= \min\{1,\mu_A(x)+\mu_B(x)\}$

Example:

$A(x)=\{(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)\}$
$B(x)=\{(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)\}$

$A(x) \oplus B(x)$ = {(x_1,0.6),(x_2,0.9),(x_3,1.0),(x_4,1.0)}

11. Bounded Difference of two fuzzy sets

The bounded difference of two fuzzy sets A(x) and B(x) for all **x** ∈ X, is denoted by **A(x)** ⊖ **B(x)** and defined as follows

$$A(x) \ominus B(x) = \{(x, \mu_{A \ominus B}(x), x \in X\}$$

Where $\mu_{A \ominus B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$

Example:

A(x)={(x_1,0.1),(x_2,0.2),(x_3,0.3),(x_4,0.4)}
B(x)={(x_1,0.5),(x_2,0.7),(x_3,0.8),(x_4,0.9)}

$A(x) \ominus B(x)$ = {(x_1,0),(x_2,0),(x_3,0.1),(x_4,0.3)}

12. Cartesian product of two fuzzy sets

Let us consider two fuzzy sets A(x) and B(y) defined on the Universal sets X and Y, respectively.The Cartesian product of fuzzy sets A(x) and B(y), is denoted by **A(x)** X **B(x), such that x** ∈ X, y∈ Y. It is determined, so that the following conditions satisfy

$$\mu_{(AXB)}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$$

Example:

A(x)={(x_1,0.2),(x_2,0.3),(x_3,0.5),(x_4,0.6)}
B(y)={(y_1,0.8),(y_2,0.6),(y_3,0.3)}

$\min\{\mu_A(x_1),\mu_B(y_1)\} = \min\{0.2,0.8\} = 0.2$ $\min\{\mu_A(x_1),\mu_B(y_2)\} = \min\{0.2,0.6\} = 0.2$
$\min\{\mu_A(x_1),\mu_B(y_3)\} = \min\{0.2,0.3\} = 0.2$

$\min\{\mu_A(x_2),\mu_B(y_1)\} = \min\{0.3,0.8\} = 0.3$ $\min\{\mu_A(x_2),\mu_B(y_2)\} = \min\{0.3,0.6\} = 0.3$
$\min\{\mu_A(x_2),\mu_B(y_3)\} = \min\{0.3,0.3\} = 0.3$

$\min\{\mu_A(x_3),\mu_B(y_1)\} = \min\{0.5,0.8\} = 0.5$ $\min\{\mu_A(x_3),\mu_B(y_2)\} = \min\{0.5,0.6\} = 0.5$
$\min\{\mu_A(x_3),\mu_B(y_3)\} = \min\{0.5,0.3\} = 0.3$

$\min\{\mu_A(x_4),\mu_B(y_1)\} = \min\{0.6,0.8\} = 0.6$ $\min\{\mu_A(x_4),\mu_B(y_2)\} = \min\{0.6,0.6\} = 0.6$
$\min\{\mu_A(x_4),\mu_B(y_3)\} = \min\{0.6,0.3\} = 0.3$

$$\mathbf{A\ X\ B} = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix}$$